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Oscillatory Integral Operators in Morrey Spaces with Variable Exponent

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Abstract. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the conditions in terms of Calderón-Zygmund-type integral inequalities for oscillatory integral operators in the Morrey spaces with variable exponent.

Key Words and Phrases: Calderón-Zygmund-type integral inequalities for oscillatory integral operators, Morrey space with variable exponent.

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1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [18] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [8, 9, 10, 18].

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [7, 14, 20], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [3, 5, 6].

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [2] in the Euclidean setting in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [2]. P. Hästö in [12] used his new "local-to-global" approach to extend the result of [2] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [13].

In the case of constant p and λ , the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [19], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [4].

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A distribution kernel $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y)\in\Omega\times\Omega : x\neq y\}$ and satisfying the estimates

$$|K(x,y)|\le C|x-y|^{-n} \text{ for all } x\neq y,$$

$$|K(x,y)-K(x,z)|\le C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}},\ \ \sigma>0,\ \text{ if } |x-y|>2|y-z|,$$

$$|K(x,y)-K(\xi,y)|\le C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}},\ \ \sigma>0,\ \text{ if } |x-y|>2|x-\xi|$$

Calderón-Zygmund type singular operator and the oscillatory integral operator are defined by

$$Tf(x)=\int_{\Omega}K(x,y)f(y)dy, \tag{1}$$

$$Sf(x)=\int_{\Omega}e^{P(x,y)}K(x,y)f(y)dy, \tag{2}$$

where $P(x,y)$ is a real valued polynomial defined on $\Omega\times\Omega$. Lu and Zhang [17] used L^2 -boundedness of T to get L^p - boundedness of S with $1<p<\infty$.

Let

$$T^*f(x)=\sup_{\varepsilon>0}|T_{\varepsilon}f(x)|$$

be the maximal singular operator, where $T_{\varepsilon}f(x)$ is the usual truncation

$$T_{\varepsilon}f(x)=\int_{\{y\in\Omega:|x-y|\ge\varepsilon\}}K(x,y)f(y)dy.$$

We use the following notation: $\mathbb{R}^n$ is the n -dimensional Euclidean space, $\Omega\subset\mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E\subseteq\mathbb{R}^n$, $B(x,r)=\{y\in\mathbb{R}^n : |x-y|<r\}$, $\widetilde{B}(x,r)=B(x,r)\cap\Omega$, by c, C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent weighted Lebesgue and Morrey spaces

We refer to the book [5] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1<p_-\le p(x)\le p_+<\infty, \tag{3}$$

where $p_- := \operatorname{ess}\inf\limits_{x\in\Omega} p(x)$, $p_+ := \operatorname{ess}\sup\limits_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{f(x) : \left|\int_\Omega f(y)g(y)dy\right| < \infty \ \ for\ all \ \ g \in L^{p'(\cdot)}(\Omega)\right\} \tag{4}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}} \leq 1} \left|\int_\Omega f(y)g(y)dy\right| \tag{5}$$

see [15, Theorem 2.3], or [21, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [22], [15].

$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1, \infty)$;

$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \ \ |x-y| \leq \frac{1}{2} \ \ x, y \in \Omega, \tag{6}$$

where $A = A(p) > 0$ does not depend on x, y ;

$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \to \mathbb{R}^n$ satisfying the condition (6);

$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$;

for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_\infty}{\ln(2 + |x|)}, \ \ x \in \mathbb{R}^n. \tag{7}$$

where $p_\infty = \lim\limits_{x\to\infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [5], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq Cr^{\theta_p(x,r)}, \qquad x \in \Omega, \ p \in \mathbb{P}^{log}_\infty(\Omega), \tag{8}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1, \\ \frac{n}{p(\infty)}, & r \ge 1 \end{cases}$.

A locally integrable function $\omega : \Omega \to (0,\infty)$ is called a weight. We say that $\omega \in A_p(\Omega)$, $1 < p < \infty$, if there is a constant $C > 0$ such that

$$\left(\frac{1}{|\widetilde{B}(x,t)|}\int_{\widetilde{B}(x,t)} \omega(x)dx\right)\left(\frac{1}{|\widetilde{B}(x,t)|}\int_{\widetilde{B}(x,t)} \omega^{1-p'}(x)dx\right)^{p-1} \le C,$$

where $1/p + 1/p' = 1$. We say that $\omega \in A_1(\Omega)$ if there is a constant $C > 0$ such that $M\omega(x) \le C\omega(x)$ almost everywhere.

The extrapolation theorems (Lemma 1 and Lemma 2 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [3]. Here we use the form in [5], see Theorem 7.2.1 and Theorem 7.2.3 in [5].

Lemma 1. *([5]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\int_\Omega |f(x)|^{p_0}\omega(x)dx \le C_0 \int \Omega |g(x)|^{p_0}\omega(x)dx.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \le p_-$. If maximal operator is bounded on $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 2. *([5]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < q_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\left(\int_\Omega |f(x)|^{q_0}\omega(x)dx\right)^{\frac{1}{q_0}} \le C_0 \left(\int_\Omega |g(x)|^{p_0}\omega^{\frac{p_0}{q_0}}(x)dx\right)^{\frac{1}{p_0}}.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \le p_-$ and $\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{p_+}$, and define $q(x)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

If maximal operator is bounded on $L^{\left(\frac{q(\cdot)}{q_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{q(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Singular operators within the framework of the spaces with variable exponents were studied in [6]. From Theorem 4.8 and Remark 4.6 of [6] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω , but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 1. *([6, Theorem 4.8]) Let $\Omega \subset \mathbb{R}^n$ be a unbounded open set and $p \in \mathbb{P}^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0,n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively.

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [11].

Theorem 2. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3. Oscillatory integral operators in $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$

Lemma 3. *(see [16]). If K is a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$, then for any real polynomial $P(x,y)$ and $\omega \in A_p$ $(1 < p < \infty)$, there exists constants $C > 0$ independent of the coefficients of P such that*

$$\|Sf\|_{L^p_\omega(\Omega)} \le C\|f\|_{L^p_\omega(\Omega)}.$$

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.*

Proof. By the Lemma 1 and Lemma 3, we derive the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following local estimates are valid.

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}^{log}_\infty(\Omega)$ *and* $f \in L^{p(\cdot)}(\Omega)$ *. Then*

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}, \tag{1}$$

where C *does not depend on* f *,* $x \in \Omega$ *and* t *.*

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \quad t > 0, \tag{2}$$

and have

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By the Theorem 1 we obtain

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)},$$

so that

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

we get

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{3}$$

To estimate $\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|Sf_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|Sf_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy,$$

To estimate Sf_2 , we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{4}$$

To this end, we choose $\delta > 0$ and proceed as follows

$$\int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le \delta \int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^{\infty} s^{-\delta-1}ds$$
$$\le C\int_t^{\infty} s^{-n}\frac{ds}{s}\int_{\{y\in\Omega:2t\le|x-y|\le s\}}|f(y)|dy \le C\int_t^{\infty} s^{-n}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|\chi_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\Omega)}\frac{ds}{s}$$
$$\le C\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{5}$$

Hence by inequality (5), we get

$$\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$
$$= Ct^{\theta_p(x,t)}\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{6}$$

From (3) and (6) we arrive at (1).

Theorem 3. *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ *and* $0 \le \lambda(x) < n$ *. Then the singular integral operator* S *is bounded from the space* $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$ *to the space* $\mathcal{L}^{p(\cdot),\lambda}(\Omega)$ *.*

Proof. Let $f \in \mathcal{L}^{p(\cdot),\lambda}(\Omega)$. As usual, when estimating the norm

$$\|Sf\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}. \tag{7}$$

We estimate $\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}$ in (7) by means of Theorems 2, 2 and obtain

$$\|Sf\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)}$$
$$\le C\sup_{x\in\Omega,\, t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)}t^{\theta_p(x,t)}\int_t^{\infty} s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$
$$\le C\sup_{x\in\Omega,\, t>0} t^{\frac{\lambda(x)}{p(x)}-\theta_p(x,t)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathcal{L}^{p(\cdot),\lambda}(\Omega)}.$$

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