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The Boundedness Weighted Hardy Operator in the Orlicz-Morrey Spaces

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Abstract. We prove the boundedness of the weighted Hardy operator in the locally Orlicz-Morrey spaces $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: Orlicz-Morrey space, Hardy operator.
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1. Introduction

Inequalities involving classical operators of harmonic analysis, such as maximal functions, fractional integrals and singular integrals of convolution type have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [1, 16, 17]. Generalizations of these results to Zygmund spaces are presented in [1]. As far as Orlicz spaces are concerned, we refer to the books [8, 9] and note that a characterization of Young functions A for which the Hardy-Littlewood maximal operator or the Hilbert and Riesz transforms are of weak or strong type in Orlicz space L_A is known (see for example [2, 8]). In paper [4] give necessary and sufficient conditions for the strong and weak boundedness of the Riesz potential operator I_α on Orlicz spaces. Also in paper [2] found necessary and sufficient conditions on general Young functions Φ and Ψ ensuring that this operator is of weak or strong type from L^Φ into L^Ψ. Our characterizations for the boundedness of the abovementioned operator are different from the ones in [2]. In paper [4] as an application of these results, we consider the boundedness of the commutators of Riesz potential operator $[b, I_\alpha]$ on Orlicz spaces when b belongs to the BMO and Lipschitz spaces, respectively.

The classical Morrey spaces were originally introduced by Morrey in [10] to study the local behavior of solutions to second order elliptic partial differential equations. We recall its definition as

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_p^{\rm loc}(\mathbb{R}^n) : \|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\}, \qquad (1.1)$$

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where $0 \le \lambda \le n$, $1 \le p < \infty$. Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. $M_{p,\lambda}(\mathbb{R}^n)$ was an expansion of $L_p(\mathbb{R}^n)$ in the sense that $M_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space. Morrey found that many properties of solutions to PDE can be attributed to the boundedness of some operators on Morrey spaces. Hardy operators, maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as its prototype, nowadays intimately connected with PDE, operator theory and other fields.

2. Some preliminaries on Orlicz and Orlicz-Morrey spaces

Definition 2.1. A function $\Phi : [0,+\infty] \to [0,+\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to+\infty} \Phi(r) = \Phi(+\infty) = +\infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \ge s$.

We say that $\Phi \in \Delta_2$, if for any $a > 1$, there exists a constant $C_a > 0$ such that $\Phi(at) \le C_a \Phi(t)$ for all $t > 0$.

Recall that a function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that

$$\omega(t) \le \Phi(t) \le c\omega(ct), \; t \in [0,+\infty).$$

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \qquad \text{for} \qquad 0 < r < +\infty \tag{2.1}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Definition 2.2. (Orlicz Space). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < +\infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. The space $L_\Phi^{\rm loc}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n))$ for all balls $B \subset \mathbb{R}^n$.

Note that, $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\right\},$$

see, for example, [14], Section 3, Theorem 10, so that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big) dx \le 1.$$

Definition 2.3. The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) = \{f \in L^1_{1\mathrm{oc}}(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1\right\},$$

where $m(f,\ t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$.

For Young functions Φ and Ψ, we write $\Phi \sim \Psi$ if there exists a constant $C \ge 1$ such that

$$\Phi(C^{-1}r) \le \Psi(r) \le \Phi(Cr) \qquad \text{for all } r \ge 0$$

If $\Phi \approx \Psi$, then $L_\Phi(\mathbb{R}^n) = L_\Psi(\mathbb{R}^n)$ with equivalent norms.

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = +\infty).$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < +\infty.$$

A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ +\infty & , \quad r = +\infty. \end{cases} \tag{2.2}$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1 + r)\log(1 + r) - r$.

Remark 2.4. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. Also, if Φ is a Young function, then $\Phi \in \nabla_2$ if and only if Φ^γ be quasiconvex for some $\gamma \in (0,1)$ (see, for example, [8], p. 15).

It is known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0. \tag{2.3}$$

Definition 2.5. (Orlicz-Morrey Space). For a Young function Φ and $0 \le \lambda \le n$, we denote by $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ the locally Orlicz-Morrey space, defined as the space of all functions $L^{loc}_{\Phi}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)} = \sup_{r>0} \Phi^{-1}(r^{-\lambda})\|f\chi_{B(0,r)}\|_{L_\Phi}.$$

Also by $WL^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ we denote the weak Orlicz-Morrey space of all functions $f \in WL^{\rm loc}_{\Phi}(\mathbb{R}^n)$ for which

$$\|f\|_{WL^{0,loc}_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}(r^{-\lambda})\,\|f\|_{WL_\Phi(B(x,r))} < \infty.$$

If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n) = L^{0,loc}_{p,\lambda}(\mathbb{R}^n)$. If $\lambda = 0$, then $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n) = L_\Phi(\mathbb{R}^n)$.

3. The weighted Hardy operator in the spaces $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$

We consider the following weighted Hardy operator

$$H_\beta f(x) = |x|^{\beta-n} \int_{|y|<|x|} \frac{f(y)}{|y|^\beta} dy.$$

Theorem 3.1. *Let Φ any Young function and $0 \le \lambda < n$. Suppose also that the function $\frac{r^{n-\beta}\Phi^{-1}(r^{-n})}{\Phi^{-1}(r^{-\lambda})}$ is almost increasing and*

$$\int_0^r \frac{t^{n-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt \le C \frac{r^n\Phi^{-1}(r^{-n})}{\Phi^{-1}(r^{-\lambda})}, \tag{3.1}$$

$$\int_0^\varepsilon \frac{s^{n-\beta-1}\Phi^{-1}(s^{-n})}{\Phi^{-1}(s^{-\lambda})} ds < \infty, \tag{3.2}$$

where $\varepsilon > 0$ and $r > 0$.

1) Then the operator H_β is bounded from $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ to $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ if and only if $\frac{\Phi^{-1}(|\cdot|^{-n})}{\Phi^{-1}(|\cdot|^{-\lambda})} \in L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$.

2) Then the operator H_β is bounded from $L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ to $WL^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$ if and only if $\frac{\Phi^{-1}(|\cdot|^{-n})}{\Phi^{-1}(|\cdot|^{-\lambda})} \in WL^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$.

Proof. 1) Let $f \in L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$. We have

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz = \sum_{k=0}^{\infty} \int\limits_{B_k(y)} \frac{|f(z)|}{|z|^\beta} dz, \tag{3.3}$$

where $B_k(y) = \left\{z : 2^{-k-1}r < |z| < 2^{-k}r\right\}$. Applying this in (3.3) and making use of the following Hölder's inequality

$$\|f\|_{L_1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}$$

we obtain

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le \sum_{k=0}^{\infty} \frac{2|B(0,2^{-k}r)|\Phi^{-1}\left(|B(0,2^{-k}r)|^{-1}\right)||f||_{L_\Phi(B(0,2^{-k}r))}}{(2^{-k}r)^\beta}$$

so that

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \sum_{k=0}^{\infty} \frac{(2^{-k}r)^{n-\beta}\Phi^{-1}\left((2^{-k}r)^{-n}\right)}{\Phi^{-1}\left((2^{-k}r)^{-\lambda}\right)}.$$

We have $\mathcal{A}(r) = \sum\limits_{k=0}^{\infty} \int\limits_{2^{-k-1}r}^{2^{-k}r} \frac{t^{n-\beta-1}\Phi^{-1}\left(t^{-n}\right)}{\Phi^{-1}\left(t^{-\lambda}\right)} dt$. Since the function $\frac{t^{n-\beta}\Phi^{-1}\left(t^{-n}\right)}{\Phi^{-1}\left(t^{-\lambda}\right)}$ is almost decreasing, we obtain

$$\mathcal{A}(r) \ge C\sum_{k=0}^{\infty} \frac{(2^{-k}r)^{n-\beta}\Phi^{-1}\left((2^{-k}r)^{-n}\right)}{\Phi^{-1}\left((2^{-k}r)^{-\lambda}\right)} \ge C,$$

Therefore

$$\sum_{k=0}^{\infty} \frac{(2^{-k}r)^{n-\beta}\Phi^{-1}\left((2^{-k}r)^{-n}\right)}{\Phi^{-1}\left((2^{-k}r)^{-\lambda}\right)} \le C\mathcal{A}(r)$$

or

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \int\limits_0^r \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt.$$

We proved that

$$|H_\beta f(x)| \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} |x|^{\beta-n} \int\limits_0^{|x|} \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt = C\|f\|_{L^{0,loc}_{\Phi,\lambda}} B(|x|). \tag{3.4}$$

Then by the (3.4), we have

$$\|H_\beta f\|_{L_\Phi(B(0,r))} \le C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \int\limits_0^{|\cdot|} \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt \right\|_{L_\Phi(B(0,r))}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \frac{|\cdot|^{n-\beta}\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{L_\Phi}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| \frac{\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{L_\Phi} \leq \frac{C}{\Phi^{-1}(r^{-\lambda})}\|f\|_{L^{0,loc}_{\Phi,\lambda}}.$$

2) Let $f \in L^{0,loc}_{\Phi,\lambda}(\mathbb{R}^n)$. The preceding division will prove to be the proper analogy. We have

$$\|H_\beta f\|_{L_{W\Phi}(B(0,r))} \leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \int\limits_0^{|\cdot|} \frac{t^{n-\beta-1}\Phi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda})} dt \right\|_{WL_\Phi(B(0,r))}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| |\cdot|^{\beta-n} \frac{|\cdot|^{n-\beta}\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{WL_\Phi}$$

$$\leq C\|f\|_{L^{0,loc}_{\Phi,\lambda}} \left\| \frac{\Phi^{-1}(|\cdot|^{-n})\chi_{B(0,r)}}{\Phi^{-1}(|\cdot|^{-\lambda})} \right\|_{WL_\Phi} \leq \frac{C}{\Phi^{-1}(r^{-\lambda})}\|f\|_{L^{0,loc}_{\Phi,\lambda}}.$$

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