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On Strong Solvability of the Dirichlet Problem for a Class of no Uniformly Degenerated Elliptic Equations of Second Order

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Abstract. In this paper it has been considered the Dirichlet problem for a class of no uniformly degenerated elliptic equations of second order. For that, it has been proved main a prior estimate, which helps to prove a strong solvability and uniqueness result in the anisotropic weighted Sobolev spaces.
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1. Introduction

Let E_n be n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$, $n \geq 3$, $D-$ be a bounded domain lying in E_n, $\partial D-$ be the boundary of domain D, wherein $\partial D \in C^2 u$, $0 \in \bar{D}$. Let us consider in D the first boundary value problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{ij} + \sum_{i,j=1}^{n} b_i(x) u_i + c(x) u = f(x), x \in D, \tag{1}$$

$$u|_{\partial D} = 0, \tag{2}$$

with $\|a_{ij}(x)\| -$ be real and symmetric matrix having measurable elements defined in D, such that for any $x \in D$, $\zeta \in E_n$ it holds the condition

$$\gamma \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \leq \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2. \tag{3}$$

Where $\gamma \in (0,1]$ is a constant, $u_i = \frac{\partial u}{\partial x_i}, u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $\lambda_i(x) = g_i(\rho(x))$, $\rho(x) = \sum_{i=1}^{n} \omega(|x_i|)$, $g_i(t) = \left(\omega_i^{-1}(t)/t\right)^2, i = 1, ..., n$, $\omega_i(t)$ are positive and continues functions, monotony increasing on $[0, diamD]$, and $\omega_i(0) = 0, \omega_i^{-1}(t) -$ are the inverse functions to $\omega_i(t)$. The functions $\frac{\omega_i(t)}{t}$ are decreasing for $t > 0$ and the constants $\alpha, \beta, \eta \in (0, \infty)$ there exist such that

$$\alpha \omega_i(t) \leq \omega_i(\eta t) \leq \beta \omega_i(t), t \in (0, diamD). \tag{4}$$

Further more, we assume that $\lambda_i(x)$ −are positive and finite a.e. in D, such that the coefficients and right hand side terms in (1) are measurable functions in$D$.

Also the condition

$$h_{ij}(x) = \frac{a_{ij}(x)}{\sqrt{\lambda_i(x)\lambda_j(x)}} \in C\left(\bar{D}\right), \ \ i,j = 1,...,n. \tag{5}$$

will be assumed.

From the condition (5) it follows that there is a positive and continuous function $\omega(t)$ on $[0, diamD]$ such that $\omega(0) = 0$ and

$$|h_{ij}(x) - h_{ij}(y)| \le \omega(|x-y|), \quad x, y \in \bar{D}, \ \ i,j = 1,...,n. \tag{6}$$

Concerning the little term coefficients of operator L the following conditions

$$b_i(x) \in L_m(D), m = n+2; c(x) \in L_\mu(D), \mu = \frac{n+2}{2}, \ c(x) \le 0$$

$$\text{for a. e. } x \in D, \text{ and } f(x) \in L_q(D), q > \frac{n}{3} \text{ is assumed.} \tag{7}$$

Let $x^0 \in E_n$, $R > 0, K > 0,$ for $\prod_{R:K}\left(x^0\right)$ −being the parallelepiped $\left\{x : \left|x_i - x_i^0\right| < K \cdot \omega_i^{-1}(R)\right\}$, and $\Theta_{R:K}\left(x^0\right)$ is the ellipsoid $\left\{x : \sum_{i=1}^{n} \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2} < K^2\right\}$, $B_R\left(x^0\right)$ is a ball $\left\{x : \left|x - x^0\right| < R\right\}$.

Let $x' \in \partial\Theta_{R:1+r/2}(0)$, $\Theta_r = \Theta_r(x') = \Theta_{R:r}(x')$ and $\bar{\Theta}_{R:1+r}(0) \subset D$, where R−is an arbitrary fixed number in $(0,1]$, and $r \in \left(0, \frac{1}{2}\right]$ that will be specified latter.

We say that $u \in C_0^\infty(\Theta_r)$ if a compact set $K_u \subset Y'$ exists such that, $\sup pu(x) \subset K_u, u(x) \in C^\infty\left(\bar{\Theta}_r\right)$. We call the number

$$\varphi_{f:p}(\sigma) = \left(\sup_{\substack{E \subset D \\ mesE \le \sigma}} \int_E |\varphi(x)|^p \, dx\right)^{1/p},$$

$AC-$ modulo of continuity of the function $|\varphi|^p$ for $\varphi(x) \in L_p(D)$, $1 < p < \infty$.

Denote by $W_{2,\lambda}^2(D)$ a Banach space of functions $u(x)$ in D, such that the norm

$$\|U\|_{w_{2,\lambda}^2(D)} = \left(\int_D \left(U^2 + \sum_{i=1}^n \lambda_i(x) U_i^2 + \sum_{i,j=1}^n \lambda_i(x)\lambda_j(x) U_{ij}^2\right) dx\right)^{1/2}$$

is finite. Let $\overset{o}{W}{}_{2,\lambda}^2(D)-$ be close of the functions class $u(x) \in C^\infty\left(\bar{D}\right)$, $u|_{\partial D} = 0$ on the norm $W_{2,\lambda}^2(D)$.

The aim of this paper is to prove the unique strong solvability of the problem (1), (2) in weighted Sobolev's spaces. Notice, the proof for an analogous result in the case of uniformly elliptic equations may be found in [1-3]. As to uniformly degenerated elliptic equations we refer to [4-5]. The elliptic equations having weak degeneration (logarithmic) the strong solvability and uniqueness results have been proved in [6]. We refer to [7-9], for a study of the strong solvability and uniqueness results of the first boundary value problem in the case of no uniformly degeneration of power function type degeneration in a fixed point. We refer also to the results in [10-12] on strong solvability.

2. Auxiliary integral estimates

Lemma 1. *Let $x \in \Theta_r$, then it holds the estimates*

$$C_1(n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2 \le \lambda_i(x) \le C_2(n)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, \quad i = 1, ..., n. \tag{8}$$

Proof. Always in the feature, by $C(.,.,.)$ we denote the different positive constants, the value of which depends on the content in the bracket.

Let $x \in \Theta_r$, then using the Minkowsky inequality it follows

$$\left(\sum_{i=1}^{n} \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)}\right)^{1/2} \le \left(\sum_{i=1}^{n} \frac{(x_i - x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} + \left(\sum_{i=1}^{n} \frac{(x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \le$$

$$\le r + 1 + \frac{r}{2} = 1 + \frac{3r}{2} \le 1 + \frac{3}{4} = \frac{7}{4}.$$

Therefore, for $i = 1, ..., n$

$$|x_i| < \frac{7}{4}\omega_i^{-1}(R).$$

From condition (4) we get

$$\rho(x) = \sum_{i=1}^{n} \omega_i(|x_i|) \le \beta n R.$$

On other hand,

$$\left(\sum_{i=1}^{n} \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \ge \left(\sum_{i=1}^{n} \frac{(x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} - \left(\sum_{i=1}^{n} \frac{(x_i - x_i')^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \ge$$

$$\le 1 + \frac{r}{2} - r = 1 - \frac{r}{2} = 1 - \frac{1}{4} = \frac{3}{4}.$$

Then it will be found an i_0, $1 \le i_0 \le n$ such that,

$$|x_i| \ge \frac{3}{4\sqrt{n}}\omega_i^{-1}(R)$$

therefore

$$\rho(x) = \sum_{i=1}^{n} \omega_i(|x_i|) \ge \sum_{i=1}^{n} \omega_i\left(\frac{3}{4\sqrt{n}}\omega_i^{-1}(R)\right) \ge \alpha n R$$

Hence

$$\left(\frac{\omega_i^{-1}(\alpha nR)}{\beta nR}\right)^2 \leq \lambda_i(x) \leq \left(\frac{\omega_i^{-1}(\beta nR)}{\alpha nR}\right)^2$$

that completes the proof of Lemma 1.

Let us to consider the operator with a constant coefficients

$$L_0 = \sum_{i=1}^{n} \lambda_i(x') \frac{\partial^2}{\partial x_i^2}, \quad \lambda_i(x') = const.$$

Lemma 2. *Let $u(x) \in C_0^\infty(\Theta_r)$. Then it holds an inequality*

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i(x) \lambda_j(x) u_{ij}^2 dx \leq C_3(n) \int_{\Theta_r} (L_0 u)^2 dx. \tag{9}$$

Proof. Apply a change of coordinate variables $y_i = x_i/\omega_i^{-1}(R)$, $i = 1, ...n$. Let $\tilde{u}(y)$ and $\tilde{\Theta}_r-$ be the image of the function $u(x)$ and the ellipsoid Θ_r, respectively. It is clear that the operator L_0 will be transformed to

$$\tilde{L}_0 = \sum_{i=1}^{n} \lambda_i(x') \cdot \frac{1}{\left(\omega_i^{-1}(R)\right)^2} \frac{\partial^2}{\partial y_i^2} \tag{10}$$

According to Lemma 1 for any $\zeta \in E_n$ it holds

$$C_1(n) R^{-2} |\zeta|^2 \leq \sum_{i=1}^{n} \lambda_i(x') \frac{1}{\left(\omega_i^{-1}(R)\right)^2} \zeta_i^2 \leq C_2(n) R^{-2} |\zeta|^2, \tag{11}$$

i.e. $\tilde{L}_0$ is a uniformly elliptic operator in $\tilde{\Theta}_r$.

We have

$$\int_{\tilde{\Theta}_r} \left(\tilde{L}_0 \tilde{u}\right) dy = \int_{\tilde{\Theta}_r} \left(\sum_{i=1}^{n} \frac{\lambda_i(x')}{\left(\omega_i^{-1}(R)\right)^2} \tilde{u}_{ii}\right)^2 dy = \int_{\tilde{\Theta}_r} \sum_{i,j=1}^{n} \frac{\lambda_i(x') \lambda_j(x')}{\left(\omega_i^{-1}(R)\right)^2 \left(\omega_j^{-1}(R)\right)^2} \tilde{u}_{ii} \cdot \tilde{u}_{jj} =$$

$$= \sum_{i,j=1}^{n} \int_{\tilde{\Theta}_r} \frac{\lambda_i(x') \lambda_j(x')}{\left(\omega_i^{-1}(R)\right)^2 \left(\omega_j^{-1}(R)\right)^2} \tilde{u}_{ij}^2 dy \geq \frac{C_1^2}{R^4} \int_{\tilde{\Theta}_r} \sum_{i,j=1}^{n} \tilde{u}_{ij}^2 dy. \tag{12}$$

Coming back, the preceding variables$x$, we infer that

$$\frac{C_1^2}{R^4} \int_{\Theta_r} \sum_{i,j=1}^{n} [\omega_i^{-1}(R), \omega_j^{-1}(R)]^2 u_{ij}^2 dx \leq \int_{\Theta_r} (L_0 u)^2 dx.$$

Now it suffices to apply Lemma 1 in order to get the estimate (9) and to complete the proof of Lemma 2.

Corollary 1. *Let*

$$L_a = \sum_{i,j=1}^{n} a_{ij}\left(x'\right) \frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij}\left(x'\right) = const$$

Then for the function $u\left(x\right) \in C_0^{\infty}\left(\Theta_r\right)$ *satisfying condition* (3) *it holds an estimate*

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i\left(x\right) \lambda_j\left(x\right) u_{ij}^2 dx \leq C_4\left(\gamma, n\right) \int_{\Theta_r} \left(L_a u\right)^2 dx.$$

First, consider the operator without little terms

$$L' = \sum_{i,j=1}^{n} a_{ij}\left(x\right) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Lemma 3. *Let the conditions* (3) *and* (6) *be satisfied for the coefficients of operator* L'. *Then the estimate*

$$\int_{\Theta_r} \sum_{i,j=1}^{n} \lambda_i\left(x\right) \lambda_j\left(x\right) u_{ij}^2 dx \leq C_5\left(\gamma, n\right) \int_{\Theta_r} \left(L'u\right)^2 dx. \tag{13}$$

holds for a function $u\left(x\right) \in C_0^{\infty}\left(\Theta_r\right)$ *as* $r \leq r_0\left(L', n\right)$.

Proof. Assume that $r_0 \leq \frac{1}{2}$. We have

$$\left(L_a u\right)^2 \leq 2\left(L'u\right)^2 + 2\left(\left(L' - L_a\right) u\right)^2. \tag{14}$$

On other hand

$$\left(L' - L_a\right) u = \sum_{i,j=1}^{n} \left[\frac{a_{ij}\left(x\right)}{\sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)}} - \frac{a_{ij}\left(x'\right)}{\sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)}}\right] \sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)} \cdot u_{ij}\left(x\right).$$

Therefore

$$\left|\left(L' - L_a\right) u\right| \leq \sum_{i,j=1}^{n} \left| h_{ij}\left(x\right) - h_{ij}\left(x'\right) + \frac{a_{ij}\left(x'\right)}{\sqrt{\lambda_i\left(x'\right)\lambda_j\left(x'\right)}} - \frac{a_{ij}\left(x'\right)}{\sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)}} \right| \cdot$$

$$\cdot \sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)} \cdot \left|u_{ij}\right| \leq \sum_{i,j=1}^{n} \left| h_{ij}\left(x\right) - h_{ij}\left(x'\right) \left| \sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)} \right| u_{ij} \right| +$$

$$+ \sum_{i,j=1}^{n} \left|h_{ij}\left(x'\right)\right| \left| 1 - \sqrt{\frac{\lambda_i\left(x'\right)\lambda_j\left(x'\right)}{\lambda_i\left(x\right)\lambda_j\left(x\right)}} \right| \sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)} \left|u_{ij}\right| = j_1 + j_2. \tag{15}$$

Further, we have

$$j_1 \leq \omega\left(\left|x - x'\right|\right) \sum_{i,j=1}^{n} \sqrt{\lambda_i\left(x\right)\lambda_j\left(x\right)} \left|u_{ij}\right|. \tag{16}$$

for $x\in\Theta_r$ and $i=1,...,n$ $|x_i-x_i'|\le r\omega_i^{-1}(R)$.

Inserting $\zeta_i=\frac{\eta_i}{\sqrt{\lambda_i(x)}}$, $i=1,...,n$ in condition (3) for $x\in D$, we get

$$\gamma\,|\eta|^2\le\sum_{i,j}^{n}h_{ij}\,(x)\,\eta_i\eta_j\le\gamma^{-1}\,|\eta|^2\,,$$

where $\eta\in E_n$.

From this it follows that $\gamma\le h_{ii}\,(x)\le\gamma^{-1}$ and $2\gamma\le h_{ii}\,(x)+h_{jj}\,(x)+2h_{ij}\,(x)\le 2\gamma^{-1}$ for $i\ne j, x\in D, i,j=1,...,n$. Thus it follows that $|h_{ij}\,(x)|\le h_0\,(\gamma)\,,\quad i,j=1,...,n$ for $x\in D$.

Therefore

$$j_2\le h_0\sum_{i,j=1}^{n}\left|1-\sqrt{\frac{\lambda_i\,(x')\,\lambda_j\,(x')}{\lambda_i\,(x)\,\lambda_j\,(x)}}\right|\sqrt{\lambda_i\,(x)\,\lambda_j\,(x)}\,|u_{ij}|\,.$$

On other hand

$$\left|1-\sqrt{\frac{\lambda_i\,(x')\,\lambda_j\,(x')}{\lambda_i\,(x)\,\lambda_j\,(x)}}\right|\le\left|1-\sqrt{\frac{\lambda_j\,(x')}{\lambda_j\,(x)}}\right|+\sqrt{\frac{\lambda_j\,(x')}{\lambda_j\,(x)}}\left|1-\sqrt{\frac{\lambda_i\,(x')}{\lambda_i\,(x)}}\right|\le$$

$$\le\left|1-\sqrt{\frac{\lambda_j\,(x')}{\lambda_j\,(x)}}\right|+\sqrt{\frac{c_2}{c_1}}\left|1-\sqrt{\frac{\lambda_j\,(x')}{\lambda_j\,(x)}}\right|=\left(1+\sqrt{\frac{c_2}{c_1}}\right)K_i,$$

$$i=1,...,n.$$

Therefore

$$K_i=\left|1-\sqrt{\frac{\lambda_i\,(x')}{\lambda_i\,(x)}}\right|\le\frac{|\lambda_i\,(x)-\lambda_i\,(x')|}{\lambda_i\,(x)}\le C_6\,(n)\tag{17}$$

Thus

$$j_2\le C_7\,(n)\,h_0\sum_{i,j=1}^{n}\sqrt{\lambda_i\,(x)\,\lambda_j\,(x)}\,|u_{ij}|\le$$

$$\le C_7h_0n\left(\sum_{i,j=1}^{n}\lambda_i\,(x)\,\lambda_j\,(x)\,u_{ij}^2\right)^{1/2}.\tag{18}$$

Inserting conditions (16) and (18) in (15), we get

$$\left|\left(L'-L_a\right)u\right|^2\le n^2\left[\omega\left(r_0\sqrt{n}\right)+C_7h_0\right]^2\sum_{i,j}^{n}\lambda_i\,(x)\,\lambda_j\,(x)\,u_{ij}^2.\tag{19}$$

Now, from (14), (19) and Lemma 2 it follows that

$$\int_{\Theta_r}\sum_{i,j=1}^{n}\lambda_i\,(x)\,\lambda_j\,(x)\,u_{ij}^2dx\le 2C_4\int_{\Theta_r}\left(L'u\right)^2dx+$$

$$+2C_4n^2\left[\omega\left(r_0\sqrt{n}\right)+C_7h_0\right]^2\cdot\int_{\Theta_r}\sum_{i,j=1}^{n}\lambda_i\left(x\right)\lambda_j\left(x\right)u_{ij}^2dx.$$

Now it suffices to choose r_0 from the condition

$$\omega\left(r_0\sqrt{n}\right)+C_7h_0\leq\frac{1}{2n\sqrt{C_4}}$$

in order to get the estimate (13).

In the feature, we assume that $r=r_0$ no reminding about.

Lemma 4. *Let be satisfied all conditions of preceding Lemma. Then for a function* $u\left(x\right)\in C_0^{\infty}\left(\Theta_r\right)$ *it holds the estimate*

$$\left\|u\right\|_{W_{2,\lambda}^2\left(\Theta_r\right)}\leq C_8\left(\gamma,n\right)\left\|L'u\right\|_{L_2\left(\Theta_r\right)}\tag{20}$$

Proof. It suffices to show that for a function $u\left(x\right)\in C_0^{\infty}\left(\Theta_r\right)$ it is satisfied the inequalities

$$\int_{\Theta_r}u^2dx\leq C_9\left(n\right)\int_{\Theta_r}\sum_{i=1}^{n}\lambda_i\left(x\right)u_i^2dx,$$

$$\int_{\Theta_r}\sum_{i=1}^{n}\lambda_i\left(x\right)u_i^2dx\leq C_{10}\left(n\right)\int_{\Theta_r}\sum_{i,j=1}^{n}\lambda_i\left(x\right)\lambda_j\left(x\right)u_{ij}^2dx.$$

To proof this, apply the change of coordinate axes as that was carry out in Lemma 2. Let $\Theta_r^0=\left\{y:\left|y_i-y_i'\right|<r\right\},\ i=1,...,n$, where y' is image of the point x'. Continue the function $\tilde{u}\left(y\right)$ on $\bar{\Theta}_r^0\backslash\tilde{\Theta}_r$ inserting zero in it and denote it again as $\tilde{u}\left(y\right)$. Let $y''=\left(y_2,...,y_n\right),\ y_1\in\left(y_1'-r,y_1'+r\right)$. We have

$$\tilde{u}\left(y_1,y''\right)=\int_{y_1'-r}^{y_1}\tilde{u}_1\left(z,y''\right)dz,\quad\text{i.e.}$$

$$\tilde{u}^2\left(y_1,y''\right)=\left(\int_{y_1'-r}^{y_1}\tilde{u}_1\left(z,y''\right)dz\right)^2\leq\left(\int_{y_1'-r}^{y_1}1^2dz\right)\left(\int_{y_1'-r}^{y_1}\tilde{u}_1^2dz\right)=$$

$$=\left(y_1-y_1'+r\right)\int_{y_1'-r}^{y_1}\tilde{u}_1^2dz\leq 2r\int_{y_1'-r}^{y_1'+r}\tilde{u}_1^2dz.$$

After integration the last inequality over Θ_r^0, we get

$$\int_{\Theta_r^0}\tilde{u}^2dy\leq 4r^2\int_{\Theta_r^0}\tilde{u}_1^2dy\leq 4r^2\int_{\Theta_r^0}\sum_{i=1}^{n}\tilde{u}_i^2dy.$$

Therefore

$$\int_{\Theta_r^0}\tilde{u}^2dy\leq 4r^2\int_{\tilde{\Theta}_r}\sum_{i=1}^{n}\tilde{u}_i^2dy.$$

By the analogy, we can derive

$$\int_{\tilde{\Theta}_r} \sum_{i=1}^{n} \tilde{u}_i^2 dy \leq 4r^2 \int_{\tilde{\Theta}_r} \sum_{i,j=1}^{n} \tilde{u}_{ij}^2 dy.$$

Coming back to the first variables x, we get

$$\int_{\Theta_r} u^2 dx \leq 4r^2 \int_{\Theta_r} \sum_{i=1}^{n} \left[\omega_i^{-1}(R)\right]^2 u_i^2 dx \leq$$

$$\leq 16r^2 \int_{\Theta_r} \sum_{i,j=1}^{n} \left[\omega_i^{-1}(R)\,\omega_j^{-1}(R)\right]^2 u_{ij}^2 dx.$$

Now it suffices to apply Lemma 1 in order to complete the proof of estimate (20).
Let $\Theta_r' = \Theta_{r/2}(x') = \Theta_{R:\frac{r}{2}}(x')$.

Lemma 5. *Let be satisfied all conditions of Lemma 3, then it holds an estimate for any function* $u(x) \in C^\infty\left(\bar{\Theta}_r\right)$ *and* $\varepsilon > 0$ *:*

$$\|U\|_{W^2_{2,\lambda}(\Theta_r')} \leq C_8 \left\|L'u\right\|_{L_2(\Theta_r)} + \varepsilon \|u\|_{W^2_{2,\lambda}(\Theta_r)} + \frac{C_{11}(\gamma, n)}{\varepsilon r^2 R^2} \|u\|_{L_2(\Theta_r)}. \tag{21}$$

Proof. Fix arbitrary $\varepsilon' > 0$. Let $\zeta(x) \in C_0^\infty(\Theta_{R:r}(x'))$, $0 \leq \zeta(x) \leq 1$, $\zeta(x) = 1$ in $\Theta_{R:\frac{r}{2}}(x')$, and $\zeta(x) = 0$ on the complement of $\Theta_{R:\frac{3r}{4}}(x')$, moreover

$$|\zeta_i| \leq \frac{C_{12}(n)}{r\omega_i^{-1}(R)}, \quad |\zeta_{ij}| \leq \frac{C_{12}(n)}{r^2\omega_i^{-1}(R)\,\omega_j^{-1}(R)}, \qquad i,j = 1,...,n. \tag{22}$$

It is clear that, $u(x) \cdot \zeta(x) \in C_0^\infty(\Theta_r)$. Applying Lemma 4 for this function, we get

$$\|U\|_{W^2_{2,\lambda}(\Theta_r')} \leq C_8 \left\|L'(u(x) \cdot \zeta(x))\right\|_{L_2(\Theta_r)} \tag{23}$$

On the other hand

$$L'(u \cdot \zeta) = \zeta \cdot L'u + 2\sum_{i,j=1}^{n} a_{ij}(x)\, u_i \zeta_j + u \cdot L'\zeta.$$

Therefore, and using (22), it follows

$$\left|L'(u \cdot \zeta)\right| \leq \left|L'u\right| + 2\left|\sum_{i,j=1}^{n} a_{ij}(x)\, u_i \zeta_j\right| + |u| \cdot \left|L'\zeta\right| \leq$$

$$\leq \left|L'u\right| + 2\left(\sum_{i,j=1}^{n} a_{ij}(x)\, u_i u_j\right)^{1/2} \cdot \left(\sum_{i,j=1}^{n} a_{ij}(x)\, \zeta_i \zeta_j\right)^{1/2} +$$

$$+|u|\cdot\left|\sum_{i,j=1}^{n}a_{ij}(x)\,\zeta_{ij}\right|\le\left|L'u\right|+2\gamma^{-1}\left|\sum_{i=1}^{n}\lambda_i(x)\,u_i^2\right|\times\left(\sum_{i=1}^{n}\lambda_i(x)\,\zeta_i^2\right)^{1/2}+$$

$$+|u|\cdot\sum_{i,j=1}^{n}a_{ij}(x)\cdot\frac{C_{12}}{r^2\omega_i^{-1}(R)\,\omega_j^{-1}(R)}\le\left|L'u\right|+2\gamma^{-1}\left(\sum_{i=1}^{n}\lambda_i(x)\,u_i^2\right)^{1/2}\cdot$$

$$\cdot\left[\sum_{i,=1}^{n}C_2\left(\frac{\omega_i^{-1}(R)}{R}\right)^2\cdot\frac{C_{12}^2}{r^2\left(\omega_i^{-1}(R)\right)^2}\right]^{1/2}+|u|\cdot\frac{C_{12}}{\gamma r^2}\cdot\sum_{i=1}^{n}\lambda_i(x)\cdot\frac{1}{\left(\omega_i^{-1}(R)\right)^2}\le\left|L'u\right|+$$

$$+\frac{2C_{12}\sqrt{nC_2}}{\gamma rR}\left(\sum_{i=1}^{n}\lambda_i(x)\,u_i^2\right)^{1/2}+|u|\,\frac{nC_2C_{12}}{\gamma r^2R^2}. \tag{24}$$

Taking into the account this inequality it follows from (23) that

$$\|u\|_{W^2_{2,\lambda}(\Theta'_r)}\le C_8\left\|L'u\right\|_{L_2(\Theta_r)}+\frac{C_{13}(\gamma,n)}{r^2R^2}\|u\|_{L_2(\Theta_r)}+$$

$$+\frac{C_{14}(\gamma,n)}{rR}\left\|\sqrt{\sum_{i=1}^{n}\lambda_i(x)\,u_i^2}\right\|_{L_2(\Theta_r)}. \tag{25}$$

On other hand

$$J^2=\left\|\sqrt{\sum_{i=1}^{n}\lambda_i(x)\,u_i^2}\right\|^2_{L_2(\Theta_r)}=\int_{\Theta_r}\sum_{i=1}^{n}\lambda_i(x)\,u_i^2dx\le$$

$$\le\frac{C_2}{R_2}\int_{\Theta_r}\sum_{i=1}^{n}\left(\omega_i^{-1}(R)\right)u_i^2dx=\frac{C_2}{R_2}\sum_{i=1}^{n}\left\|\omega_i^{-1}(R)\,u_i\right\|^2_{L_2(\Theta_r)}.$$

Therefore

$$J\le\frac{\sqrt{C_2}}{R_2}\left\|\sum_{i=1}^{n}\left(\omega_i^{-1}(R)\right)u_i\right\|_{L_2(\Theta_r)}=\frac{\sqrt{C_2}}{R}\sum_{i=1}^{n}\|\tilde u_i\|_{L_2(\tilde\Theta_r)}.$$

According to the interpolation inequality from [13], for any $\varepsilon'>0$ there exists a constant $C_{15}(n)$ such that

$$\sum_{i=1}^{n}\|\tilde u_i\|_{L_2(\Theta_r)}\le\varepsilon'\sum_{i,j=1}^{n}\left\|\tilde u_{ij}\right\|_{L_2(\tilde\Theta_r)}+\frac{C_{15}}{\varepsilon'}\|\tilde u\|_{L_2(\tilde\Theta_r)}.$$

Coming back to the variables x and using Lemma 1, it follows

$$\frac{C_{14}}{r^R}\cdot J\le\frac{C_{14}}{r^R}\cdot\frac{\sqrt{C_2}}{R}\cdot\varepsilon'\cdot\sum_{i,j=1}^{n}\left\|\tilde u_{ij}\right\|_{L_2(\tilde\Theta_r)}+$$

$$+\frac{C_{14}}{r^R}\cdot\frac{\sqrt{C_2}}{R}\cdot\frac{C_{15}}{\varepsilon'}\cdot\|\tilde{u}\|_{L_2(\tilde{\Theta}_r)}=\frac{\varepsilon'\cdot C_{14}\sqrt{C_2}}{rR^2}\sum_{i,j=1}^{n}\left\|\omega_i^{-1}(R)\,\omega_j^{-1}(R)\,u_{ij}\right\|_{L_2(\Theta_r)}+$$

$$+\frac{C_{14}C_{15}\sqrt{C_2}}{\varepsilon' rR^2}\cdot\|u\|_{L_2(\Theta_r)}\leq\frac{\varepsilon'\cdot C_{14}\sqrt{C_2}\cdot R^2}{rR^2\cdot C_1}\|u\|_{W^2_{2,\lambda}(\Theta_r)}+\frac{C_{14}C_{15}\sqrt{C_2}}{\varepsilon' rR^2}\cdot\|u\|_{L_2(\Theta_r)}.\quad(26)$$

Finally, choose $\varepsilon'=\frac{\varepsilon C_1\cdot r}{C_{14}\sqrt{C_2}}$ in order to get the needed estimate (21) using (25) and (26).

3. Main estimation on coercivity

Lemma 6. *Let* $A_R=\Theta_{R:1+\frac{r}{2}+\frac{r^2}{16}}(0)\backslash\overline{\Theta}_{R:1+\frac{r}{2}-\frac{r^2}{16}}(0)$. *Then for the countable ellipsoids system*

$$\Theta'_r(x^\nu)=\Theta_{R:\frac{r}{2}}(x^\nu),\quad x^\nu\in\partial\Theta_{R:1+\frac{r}{2}}(0),\quad \nu=1,2,...$$

there exists a covering for the set A_R.

Proof. Let $x\in A_R$. Without losing the generality, we may assume that $x_1\neq 0$. Choose α_1 so that a point $x^\nu=(\alpha_1,x_2,...,x_n)$ belongs to $\partial\Theta_{R:1+\frac{r}{2}}(0)$:

$$\left(\sum_{i=2}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}+\frac{\alpha_1^2}{\left(\omega_1^{-1}(R)\right)^2}\right)^{1/2}=1+\frac{r}{2},$$

then

$$1+\frac{r}{2}-\frac{r^2}{16}<\left(\sum_{i=1}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}<1+\frac{r}{2}+\frac{r^2}{16},$$

where from it follows that there exists such an α_1. Assume that $sign x_1=sign\alpha_1$. Let for the convenience, $|x_1|\geq|\alpha_1|$, then

$$\frac{x_1^2-\alpha_1^2}{\left(\omega_1^{-1}(R)\right)^2}=\sum_{i=1}^{n}\frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}-\left(\sum_{i=2}^{n}\frac{x_i^2}{\left(\omega_1^{-1}(R)\right)^2}+\frac{\alpha_1^2}{\left(\omega_i^{-1}(R)\right)^2}\right)\leq$$

$$\leq\left(1+\frac{r}{2}+\frac{r^2}{16}\right)^2-\left(1+\frac{r}{2}\right)^2=\left(1+\frac{r}{2}\right)^2+2\left(1+\frac{r}{2}\right)\cdot\frac{r^2}{16}+\frac{r^4}{256}-\left(1+\frac{r}{2}\right)^2=$$

On other hand $x_1^2-\alpha_1^2\geq(x_1-\alpha_1)^2$. Therefore,

$$\left(\sum_{i=1}^{n}\frac{(x_i-x_i^\nu)}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}=\left(\frac{(x_1-\alpha_1)^2}{\left(\omega_1^{-1}(R)\right)^2}\right)^{1/2}\leq\left(\frac{\left(x_1^2-\alpha_1^2\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}<\frac{r}{2},$$

which completes the proof of Lemma.

Lemma 7. *Let $\bar{A}_{R_0} \subset D$ and for $m = 1, 2, ...$ it is $R_m = R_0 \cdot a^m$, where the number a is so that*

$$\frac{\alpha}{\beta} \le a < 1.$$

Therefore

$$\Theta_{R_0 : 1 + \frac{r}{2} + \frac{r^2}{16}} (0) \setminus \{0\} \subset \bigcup_{m=0}^{\infty} A_{R_m}$$

Proof. For $m = 1, 2, ...$ it suffices to set

$$\Theta_{R_{m+1} : 1 + \frac{r}{2} + \frac{r^2}{16}} (0) \supset \bar{\Theta}_{R_m : 1 + \frac{r}{2} - \frac{r^2}{16}} (0) . \tag{27}$$

The inclusion (27) is equivalently to that of

$$\left(1 + \frac{r}{2} - \frac{r^2}{16}\right) \omega_i^{-1} (R_m) \le \left(1 + \frac{r}{2} + \frac{r^2}{16}\right) \omega_i^{-1} (R_{m+1}) .$$

for $m = 1, 2, ...,\ \ i = 1, ..., n$. It follows from (4) that

$$\alpha R_m \le \beta R_{m+1}$$

i.e.

$$\frac{\alpha}{\beta} \le \frac{R_{m+1}}{R_m} = a < 1.$$

This completes the lemma.

Remark 1. *It holds an inclusion*

$$\bigcup_{\nu=1}^{\infty} \Theta_r (x^\nu) \subset B_R = \Theta_{R : 1 + \frac{3r}{2}} (0) \supset \bar{\Theta}_{R_m : 1 - \frac{r}{2}} (0) ,$$

where is a cover with ellipsoids $\Theta_r (x^\nu) = \Theta_{R:k} (x^\nu)$ has a finite multiplicity $N_1 (n, r)$ and $x^\nu \in \partial \Theta_{R : 1 + \frac{r}{2}} (0)$.

Remark 2. *It holds an inclusion*

$$\bigcup_{m=0}^{\infty} B_{R_m} \backslash \Theta_{R : 1 + \frac{3r}{2}} (0) ,$$

where is a cover with spherical layers B_{R_m} has a finite multiplicity $N_2 (n, r)$.

Let

$$\Theta^1_{R_0} (\bar{x}) = \Theta_{R_0 : 1 + \frac{r}{2} + \frac{r^2}{16}} (\bar{x}) , \Theta^1_{R_0} (0) = \Theta^1_{R_0}, \quad \Theta^2_{R_0} (\bar{x}) = \Theta_{R_0 : 1 + \frac{3r}{2}} (\bar{x}) , \Theta^2_{R_0} (0) = \Theta^2_{R_0}.$$

It is easy to see that

$$\Theta^1_{R_0} \subset \Theta^2_{R_0},\ \Theta^1_{R_0} = \bigcup_{m=0}^{\infty} A_{R_m},\ \ \Theta^2_{R_0} = \bigcup_{m=0}^{\infty} B_{R_m}. \tag{28}$$

Lemma 8. *Let be satisfied the conditions* (3) *and* (5), *then for a function* $u(x) \in C^{\infty}\left(\Theta^2_{R_0}\right)$ *it holds an estimate for any* $\varepsilon > 0$ *:*

$$\|u\|_{W^2_{2,\lambda}\left(\Theta^1_{R_0}\right)} \le C_{16}\left(L', n\right)\left\|L'u\right\|_{L_2\left(\Theta^2_{R_0}\right)} + \varepsilon\|u\|_{W^2_{2,\lambda}\left(\Theta_{R^2_0}\right)} + \frac{C_{17}\left(L',n\right)}{\varepsilon}\sup_{\Theta^2_{R_0}}\|u\|. \quad (29)$$

Proof. Fix arbitrary $\varepsilon > 0$. It follows from Lemma 5 that for any $\varepsilon' > 0$ and $\nu = 1, 2, ...$ it holds the estimate

$$\|u\|^2_{W^2_{2,\lambda}(\Theta'_r(x^{\nu}))} \le C_{18}\left(L', n\right)\left\|L'u\right\|_{L_2(\Theta_r(x^{v}))} + \left(\varepsilon'\right)\|u\|^2_{W^2_{2,\lambda}(\Theta_r(x^{\nu}))} + \frac{C_{19}(\gamma, n)}{\left(\varepsilon'\right)^2 r^4 R^4}\|u\|^2_{L_2(\Theta_r(x^{\nu}))}, \quad (30)$$

on the ellipsoids Θ'_r and Θ_r where is $R = R_m$, $m = 0, 1, 2, \ldots$.

Summing all inequalities (30) over ν and using Lemma 6 with help of Remark 1 to Lemma 7, we infer

$$\|u\|^2_{W^2_{2,\lambda}(A_{R_m})} \le C_{20}\left(L', n\right)\left\|L'u\right\|^2_{L_2(B_{R_m})} + N_1\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}(B_{R_m})} + \frac{C_{21}\left(L', n\right)}{\left(\varepsilon'\right)^2 r^4 R^4_m}\|u\|^2_{L_2(B_{R_m})}$$

On other hand, it is

$$\|u\|^2_{L_2(B_{R_m})} = \int_{B_{R_m}} u^2 dx \le \left(\sup_{\Theta^2_{R_0}}|u|\right)^2 \cdot mes\Theta^2_{R_0}.$$

Thus

$$\|u\|^2_{W^2_{2,\lambda}(A_{R_m})} \le C_{20}\left\|L'u\right\|^2_{L_2(B_{R_m})} + N_1\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}(B_{R_m})} + \frac{C_{22}\left(L', n\right)}{\left(\varepsilon'\right)^2}\left(\sup_{\Theta^2_{R_0}}|u|\right)^2. \quad (31)$$

After summing all inequalities (31) over m beginning from zero to infinity and applying Lemma 7, Remark 2 on it, we come to the inequality

$$\|u\|^2_{W^2_{2,\lambda}\left(\Theta^1_{R_0}\right)} \le C_{23}\left\|L'u\right\|^2_{L_2\left(\Theta^2_{R_0}\right)} + N_1N_2\cdot\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}\left(\Theta^2_{R_0}\right)} + \frac{C_{24}\left(L', n\right)}{\left(\varepsilon'\right)^2}\left(\sup_{\Theta^2_{R_0}}|u|\right)^2$$

Finally, choosing $\varepsilon' = \frac{\varepsilon}{\sqrt{N_1N_2}}$ we complete the proof on needed estimation (29).

Remark 3. *Since the operator* L' *degenerates on a point 0, the estimate* (29) *takes place in the ellipsoids* $\Theta^1_{R_0}(\bar{x})$ *and* $\Theta^2_{R_0}(\bar{x})$ *provided that* $\bar{\Theta}^2_{R_0}(\bar{x}) \subset D$, $\Theta^2_{R_0}(x) \bigcap \Theta_{R_0:1}(0) = \emptyset$. *Also, the mentioned estimate takes place for any* $R \in (0, R_0]$.

Let $D(\rho) = \left\{x : x \in D, \Theta^2_{\rho}(x) \subset D\right\}$, for $\rho > 0$, and $\Theta^2_{\rho}(x) = \Theta_{\rho:1+\frac{3r}{2}}(x)$, $\Theta^1_{\rho}(x) = \Theta_{\rho:1+\frac{r}{2}+\frac{r^2}{16}}(x)$.

Lemma 9. *For a function $u(x) \in C^{\infty}\left(\Theta^2_{R_0}\right)$, a number $\varepsilon > 0$, and sufficiently small $\rho > 0$ it holds an estimate*

$$\|u\|_{W^2_{2,\lambda}(D(\rho))} \leq C_{25}\left(L', n, \rho, D\right)\left\|L'u\right\|_{L_2(D)} + \varepsilon\|u\|_{W^2_{2,\lambda}(D)} + $$

$$+\frac{C_{26}\left(L', n, \rho, D\right)}{\varepsilon}\sup_D|u| \tag{32}$$

Proof. Fix a number $\varepsilon > 0$ and sufficiently small $\rho > 0$. Cover the set $\overline{D(\rho)}$ with finite $N_3(n, \rho, D)$ number ellipsoids $\left\{\Theta^1_\rho(x^v)\right\}$. According to the Lemma 8, for a $\varepsilon' > 0$ it holds

$$\|u\|^2_{W^2_{2,\lambda}\left(\Theta^1_\rho(x^v)\right)} \leq C_{27}\left(L', n\right)\left\|L'u\right\|^2_{L_2\left(\Theta^2_\rho(x^v)\right)} + \left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}\left(\Theta^2_{R_0}(x^v)\right)} + $$

$$+\frac{C_{26}}{\left(\varepsilon'\right)^2}\left(\sup_D|u|\right)^2, \quad \nu = 1, ..., N_3. \tag{33}$$

Summing all inequalities (33) over the ν from 1 to N_3, we get

$$\|u\|^2_{W^2_{2,\lambda}(D(\rho))} \leq C_{27}\cdot N_3\left\|L'u\right\|^2_{L_2(D)} + N_3\left(\varepsilon'\right)^2\|u\|^2_{W^2_{2,\lambda}(D)} + $$

$$+\frac{C_{28}\left(L', n, \rho, D\right)}{\left(\varepsilon'\right)^2}\cdot N_3\left(\sup_D|u|\right)^2.$$

Now it suffices to set $\varepsilon' = \frac{\varepsilon}{\sqrt{N_3}}$ in order to get the estimate (32).

For a $\rho > 0$ set $D_\rho = \{x : x \in D,\ dist\,(x, \partial D) > \rho\}$.

Lemma 10. *Let the conditions* (3) *and* (5) *be satisfied. Then for a function $u(x) \in W^2_{2,\lambda}(D)$ it holds an estimate for any $\varepsilon > 0$ and $\rho > 0$:*

$$\|u\|^2_{W^2_{2,\lambda}(D_\rho)} \leq C_{29}(n, \rho, D, \gamma)\left(\int_D\left(L'u\right)^2dx + \varepsilon\|u\|^2_{W^2_{2,\lambda}(D)} + \frac{1}{\varepsilon}\left(\sup_D|u|\right)^2\right). \tag{34}$$

Proof. Fix a number $\varepsilon > 0$ and arbitrary small $\rho > 0$. Cover $\bar{D}_\rho$ with finite $N_4(n, \rho, D, \gamma)$ number ellipsoids $\Theta^1_\rho(x^v)$ applying Lemma 8 in the everyone.

Lemma 11. *Let the conditions* (3) *and* (5) *be satisfied. Then for a function $u(x) \in \overset{0}{W}{}^2_{2,\lambda}(D)$ it holds the estimate for a $\rho > 0$:*

$$\|u\|^2_{W^2_{2,\lambda}(D\backslash D_\rho)} \leq C_{30}(n, \gamma, \rho, D)\left(\int_D\left|L'u\right|^2dx + \left(\sup_D|u|\right)^2\right). \tag{35}$$

Proof. Since $\partial D \subset C^2$, according to [2], for sufficiently small ρ such that $(D \backslash D_\rho) \bigcap \Theta_{\rho:1}(0) = \emptyset$, it holds that

$$\|u\|^2_{W_2^2(D\backslash D_\rho)} \leq C_{31}(n,\gamma,\rho,D)\left(\int_D \left|L'u\right|^2 dx + \int_D |u|^2 dx\right)$$

Now, in order to complete the proof of the following Theorem, it suffices to apply

$$\|u\|_{W_2^2(D\backslash D_\rho)} \leq C_{32}(n,\rho)\cdot\|u\|_{W_2^2(D\backslash D_\rho)}$$

and the inequality

$$\int_D |u|^2 dx \leq mesD \cdot \left(\sup_D |u|\right)^2.$$

Theorem 1. *Let the coefficients of operator L' satisfy the conditions* (3) *and* (5), *then for a function $u(x) \in \overset{0}{W}{}^2_{2,\lambda}(D)$ the estimate*

$$\|u\|^2_{W^2_{2,\lambda}(D)} \leq C_{33}(n,\gamma,D)\left(\left|\left|L'u\right|\right|^2_{L_2(D)} + \left(\sup_D |u|\right)^2\right) \tag{36}$$

takes place.

Proof. Fix the sufficiently small number $\rho > 0$ and sum the inequalities (34) and (35). We get

$$\|u\|^2_{W^2_{2,\lambda}(D)} \leq (C_{29}+C_{30})\int_D \left|L'u\right|^2 dx + C_{29}\cdot\varepsilon\cdot\|u\|^2_{W^2_{2,\lambda}(D)} +$$
$$+\left(\frac{C_{29}}{\varepsilon}+C_{30}\right)\left(\sup_D |u|\right)^2.$$

Now, it suffices to set $\varepsilon = \frac{1}{2C_{29}}$ and $C_{33} = \max\left\{2(C_{29}+C_{30});2\left(2C_{29}^2+C_{30}\right)\right\}$ in order to complete the proof.

For proving the estimate (36) for operator L, we need the following imbedding assertion from [2].

Theorem 2. *For a function $u(x) \in C^\infty(D)$, with $u|_{\partial D} = 0$ it holds the estimate*

$$\sum_{i=1}^n \|u_i\|_{L_p(D)} \leq C_{34}(p,q,n)\sum_{i,j=1}^n \|u_{ij}\|_{L_q(D)},$$

provided that

$$p \geq q \geq 1,\ \ 1-\left(\frac{1}{q}-\frac{1}{p}\right)\cdot(n+2) \geq 0, \tag{37}$$

and

$$\|u\|_{L_{p_1}(D)} \leq C_{35}\left(p_1, q_1, n\right) \sum_{i,j=1}^{n} \|u_{ij}\|_{L_{q_1}(D)},$$

provided that

$$p_1 \geq q_1 \geq 1, \;\; 2-\left(\frac{1}{q_1}-\frac{1}{p1}\right)(n+2) \geq 0. \tag{38}$$

Theorem 3. *Let the coefficients of operator L satisfy conditions* (3)-(7)*, then for a function $u(x) \in C^{\infty}\left(\bar{D}\right)$, with $u|_{\partial D}=0$ takes place the estimate*

$$\|u\|_{W^2_{2,\lambda}(D)} \leq C_{36}\left(L, n, D\right)\left(\|Lu\|_{L_2(D)} + \sup_{D}|u|\right). \tag{39}$$

Proof. First, prove that

$$\sum_{i=1}^{n} \|b_i u_i\|_{L_2(D)} \leq C_{37}\left(L, n, D\right) \varphi_{B:m}\left(\sigma\right) \sum_{i,j=1}^{n} \|u\|_{W_{2,\lambda}(D)}, \tag{40}$$

where $b_i\left(x\right) \in L_m\left(D\right), \;\; m=n+2,\, i=1,...,n,$ and $\varphi_{B:M}\left(\sigma\right) = \max\limits_{1\leq i\leq n} \varphi_{b_i:m}, \;\; \sigma = mesD.$

Evidently, (37) takes place for $q=\frac{p(n+2)}{p+n+2}$. Using Holder's inequality, we have

$$W_1 = \sum_{i=1}^{n} \|b_i u_i\|_{L_2(D)} = \sum_{i=1}^{n}\left(\int_D b_i^2 u_i^2 dx\right)^{1/2} \leq$$

$$\leq \sum_{i=1}^{n}\left(\int_D |b_i|^m dx\right)^{1/m} \cdot \left(\int_D |u_i|^{\frac{2m}{m-2}} dx\right)^{\frac{m-2}{2m}} \leq$$

$$\leq \varphi_{B:m}\left(\sigma\right) \cdot \sum_{i=1}^{n} ||u_i||_{L_{\frac{2m}{m-2}}(D)} \leq C_{34}\varphi_{B:m}\left(\sigma\right) \cdot \sum_{i,j=1}^{n} ||u_{ij}||_{L_q(D)},$$

where $p=\frac{2m}{m-2}$ and $q=\frac{2m(n+2)}{m(n+4)-2(n+2)}$ (see [2]). Since $m=n+2$, it is $p=\frac{2(n+2)}{n}$ and $q=2$. Therefore

$$W_1 \leq C_{34}\varphi_{B:m}\left(\sigma\right) \cdot \sum_{i,j=1}^{n} \|u_{ij}\|_{L_2(D)} \leq C_{34}\varphi_{B:m}\left(\sigma\right)\|u_{ij}\|_{W^2_2(D)} \leq C_{37}\varphi_{B:m}\left(\sigma\right)\|u\|_{W^2_{2,\lambda}(D)}. \tag{41}$$

Show that, for a $c\left(x\right) \in L_{\mu}\left(D\right)$ and $\mu=\frac{n+2}{2}$, it holds

$$\|Cu\|_{L_2(D)} \leq C_{38}\left(L, n, D\right) \varphi_{C:\mu}\left(\sigma\right)\|u\|_{W^2_{2,\lambda}(D)}. \tag{42}$$

Evidently, the estimate (38) holds for $q_1=\frac{p_1(n+2)}{2p_1+n+2}$.

We have

$$W_2=\|Cu\|_{L_2(D)}=\left(\int_D C^2u^2dx\right)^{1/2}\leq\left(\int_D C^{\mu}dx\right)^{1/\mu}\cdot\left(\int_D |u|^{\frac{2\mu}{\mu-2}}dx\right)^{\frac{\mu-2}{2\mu}}=$$
$$=\|C\|_{L_\mu(D)}\cdot\|u\|_{L_{\frac{2\mu}{\mu-2}}(D)}\leq C_{35}\varphi_{C:\mu}(\sigma)\cdot\|u_{ij}\|_{L_{q_1}(D)},$$

According to Theorem 2 with $p_1=\frac{2\mu}{\mu-2}$ and $q_1=\frac{2\mu(n+2)}{\mu(n+6)-2(n+2)}$. Since, $\mu=\frac{n+2}{2}$, it is $p_1=\frac{2(n+2)}{n-2}$ and $q_1=2$. Therefore,

$$W_2\leq C_{35}\varphi_{C:\mu}(\sigma)\|u_{ij}\|_{L_2(D)}\leq C_{35}\varphi_{C:\mu}(\sigma)\|u\|_{W_2^2(D)}\leq C_{38}\varphi_{C:\mu}(\sigma)\|u\|_{W^2_{2,\lambda}(D)} \tag{43}$$

From Theorem 1 it follows that

$$\|u\|_{W^2_{2,\lambda}(D)}\leq C_{33}\left(\|Lu\|_{L_2(D)}+\sum_{i=1}^n\|b_iu_i\|_{L_2(D)}+\|Cu\|_{L_2(D)}+\sup_D|u|\right)\leq$$
$$\leq C_{33}\left(\|Lu\|_{L_2(D)}+(C_{37}\varphi_{B:m}(\sigma)+C_{38}\varphi_{C:\mu}(\sigma))\|u\|_{W^2_{2,\lambda}(D)}+\sup_D|u|\right).$$

Now, it suffices to set

$$C_{37}\varphi_{B:m}(\sigma)+C_{38}\varphi_{C:\mu}(\sigma)\leq\frac{1}{2C_{33}},$$

in order to get the estimate (39).

Theorem 4. *Let the conditions* (3)-(7) *be satisfied for the coefficients of operator* L. *Then for a function* $u(x)\in \overset{o}{W}{}^2_{2,\lambda}(D)$, *it holds the estimate too*

$$\|u\|_{W^2_{2,\lambda}(D)}\leq C_{39}(n,L,D)\|Lu\|_{L_q(D)}. \tag{44}$$

Proof. By assumptions, $c(x)\leq 0$ and therefore the Aleksandrov's inequality [14] takes place

$$\sup_D|u|\leq C_{40}(n,D)\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\cdot F_n\left(\left\|\frac{b}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\right), \tag{45}$$

where $F_n(z)=l^{\frac{1}{n\omega_n}\left(\frac{z}{n}\right)^n+\varphi_n(z)}$, moreover φ_n are bounded and $\varphi_n(0)=0$ (in particular, $\varphi_1=0$), and ω_n- is volume of unit $n-$dimensional ball

$$\|b\|_{L_n(D)}=\left\|\sqrt{\sum_{i=1}^n b_i^2}\right\|_{L_n(D)}.$$

Evidently

$$\det(a_{ij}(x))\geq C_{41}(n,D)\prod_{i=2}^n\lambda_i(x)\geq C_{41}\prod_{i=1}^n\left[\frac{\omega_i^{-1}\left(\sum_{i=1}^n\omega_i(|x_i|)\right)}{\sum_{\varepsilon=1}^n\omega_i(|x_i|)}\right]^2.$$

By assumptions, the function $\frac{\omega_i(t)}{t}$, decreases on t in $(0,\infty)$ for any $i=1,...,n$. Therefore, the function $\frac{\omega_i^{-1}(t)}{t}$ will be increasing on $(0,\infty)$. On base of inequality $\rho(x)=\sum_{i=1}^{n}\omega_i(|x_i|)\geq\omega_i(|x_i|)$ and that the function $\frac{\omega_i^{-1}(t)}{t}$ is increasing, we get

$$\det(a_{ij}(x))\geq C_{41}\prod_{i=1}^{n}\left[\frac{\omega_i^{-1}(\omega_i(|x_i|))}{\omega_i(|x_i|)}\right]^2=C_{41}\prod_{i=1}^{n}\left(\frac{|x_i|}{\omega_i(|x_i|)}\right)^2.$$

We have

$$\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}=\left(\int_D\frac{|f|^n}{\det(a_{ij})}dx\right)^{1/n}\leq\left(\int_D|f|^{nS}dx\right)^{1/nS}\cdot\left(\int_D\frac{dx}{(\det(a_{ij}))^{S'}}\right)^{1/S'n},$$

where $\frac{1}{S}+\frac{1}{S'}=1$.

Let $q=nS$, then $S=\frac{q}{n}$, $S'=\frac{S}{S-1}=\frac{q}{q-n}$ and

$$\left\|\frac{f}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}\leq\frac{1}{\sqrt[n]{C_{41}}}\|f\|_{L_q(D)}\left(\int_D\prod_{i=1}^{n}\left(\frac{\omega_i(|x_i|)}{(|x_i|)}\right)^{\frac{2q}{q-n}}dx\right)^{\frac{q-n}{qn}}. \tag{46}$$

Here the condition

$$\frac{2q}{q-n}>-1,$$

is needed in order to get the finiteness of the integral in the right hand side. That integral is finite, since $q>\frac{n}{3}$.

Now, prove that the multiplier in the right hand side (45) is finite. Indeed

$$\left\|\frac{b}{\sqrt[n]{\det(a_{ij})}}\right\|_{L_n(D)}=\left\|\sqrt{\sum_{i=1}^{n}\left(\frac{b}{\sqrt[n]{\det(a_{ij})}}\right)^2}\right\|_{L_n(D)}=$$

$$=\left[\int_D\left(\sqrt{\sum_{i=1}^{n}\left(\frac{b_i}{\sqrt[n]{\det(a_{ij})}}\right)^2}\right)dx\right]^{1/n}\leq C_{42}(n)\left(\sum_{i=1}^{n}\int_D\frac{|b_i|^n}{\det(a_{ij})}dx\right)^{1/n}.$$

Therefore, we have

$$\int_D\frac{|b_i|^n}{\det(a_{ij})}dx\leq\left(\int_D|b_i|^m dx\right)^{\frac{n}{m}}\cdot\left(\int_D\frac{dx}{(\det(a_{ij}))^{\frac{m}{m-n}}}\right)^{\frac{m-n}{m}}\leq C_{43}(n,\gamma,D)\|b_i\|^n_{L_m^{(D)}}\cdot$$

$$\cdot\left(\int_D\prod_{i=1}^{n}\left(\frac{\omega_i(|x_i|)}{(|x_i|)}\right)^{\frac{2m}{m-n}}dx\right)^{\frac{m-n}{m}}=C_{43}\|b_i\|^n_{L_m^{(D)}}\cdot\left(\int_D\prod_{i=1}^{n}\left(\frac{\omega_i(|x_i|)}{(|x_i|)}\right)^{n+2}dx\right)^{\frac{2}{n+2}}=$$

$$= C_{43} \left\| b_i \right\|^n_{L_m^{(D)}} \cdot \left\| \prod_{i=1}^{n} \left(\frac{\omega_i \left(|x_i| \right)}{\left(|x_i| \right)} \right) \right\|^2 . \tag{47}$$

Then according to (7), the right hand side (47) is finite. Thus

$$\sup_D |u| \le C_{44} \left(n, \gamma, q, D \right) \left\| Lu \right\|_{L_q(D)} . \tag{48}$$

On other hand

$$\left\| Lu \right\|_{L_2(D)} = \left(\int_D |Lu|^2 \, dx \right)^{1/2} \le \left(\int_D |Lu|^q \, dx \right)^{1/q} \cdot \left(\int_D 1^{\frac{q}{q-2}} dx \right)^{\frac{q-2}{2q}} =$$

$$= \left(mesD \right)^{\frac{q-2}{2q}} \cdot \left\| Lu \right\|_{L_q(D)} . \tag{49}$$

From (39), (45) and (49), we infer

$$\|u\|_{W^2_{2,\lambda}(D)} \le C_{36} \left(\left(mesD \right)^{\frac{q-2}{2q}} + C_{44} \right) \left\| Lu \right\|_{L_q(D)} ,$$

Therefore, the estimate (44) has been proved.

4. Strong solvability of the first boundary value problem

Consider the first boundary value problem (1)-(2) in the domain $D \subset \Re^n$. A function $u(x) \in \overset{o}{W}{}^2_{2,\lambda}(D)$, is called the strong solution of this problem if that satisfies (1) almost everywhere in D.

Theorem 5. *Let the coefficients of operator L are defined in D and it is satisfied the conditions* (3)-(7). *Then for $q > \frac{n}{3}$, the first boundary value problem* (1)-(2) *uniquely solvable in space $\overset{o}{W}{}^2_{2,\lambda}(D)$ for any function $f(x) \in L_q(D)$. Moreover, the function $u(x)$ satisfies to the inequality*

$$\|u\|_{W^2_{2,\lambda}(D)} \le C_{39} \|f\|_{L_q(D)} . \tag{50}$$

Proof. Assume first the littler terms coefficients of equation (1) and the right hand side $f(x)$ be infinitely differentiable in $\bar{D}$. Introduce the integer numbers $s \in \aleph$, $D^+(s) = \left\{ x : x \in D, \, \rho(x) < \frac{1}{s} \right\}$; and $i, j = 1, .., n$

$$\lambda_i^{(S)}(x) = \begin{cases} \lambda_i(x), & if \ \ x \in \bar{D} \backslash D^+(s), \\ \left[\frac{\omega_i^{-1}\left(\frac{1}{s}\right)}{\frac{1}{s}} \right]^2, & if \ \ x \in D^+(s); \end{cases}$$

$a_{ij}^{(s)}(x) = a_{ij}(x)$, for $x \in \bar{D} \backslash D^+\left(\frac{s}{2}\right)$, $a_{ij}^{((s)}(x)$ are extending over $D^+\left(\frac{s}{2}\right)$ such that, $a_{ij}^{(s)}(x) \in C\left(\bar{D}\right)$ and for any $x \in D$ and $\zeta \in E_n$ it satisfies

$$\bar{\gamma} \sum_{i=1}^{n} \lambda_i^{(s)}(x) \zeta_i^2 \le \sum_{i,j=1}^{n} a_{ij}^{(s)}(x) \zeta_i \zeta_j \le \bar{\gamma}^{-1} \sum_{i=1}^{n} \lambda_i^{(s)}(x) \zeta_i^2,$$

where $\bar{\gamma} = \gamma/2$, and $\gamma-$ is a constant from (3).

Set

$$L^{(S)} = \sum_{i,j=1}^{n} a_{ij}^{(s)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i (x) \frac{\partial}{\partial x_i} + c(x).$$

It is clear to see that for an integer s the operator $L(s)$ is uniformly elliptic in D. Let $u^{(s)}(x)-$ be a strong solution of the first boundary value problem

$$L^{(S)} u^{(S)} = f(x), \quad x \in D; \ u^{(S)}\Big|_{\partial D} = 0. \tag{51}$$

Since $a_{ij}^{(s)}(x) \in C(\bar{D})$, according to [2] there exists a strong solution for the problem (51) that is unique and belongs to the space $\overset{o}{W}{}_p^2(D)$ for any $p \in (1, \infty)$. Where $\overset{o}{W}{}_p^2(D)$ denotes the closure of all functions $u(x) \in C^\infty(\bar{D})$ with $u|_{\partial D} = 0$ in the norm

$$\|u\|_{W_p^2(D)} = \left(\int_D \left(|u|^p + \sum_{i=1}^{n} |u_i|^p + \sum_{i,j=1}^{n} |u_{ij}|^p \right) dx \right)^{1/p}.$$

Show that $u^{(s)}(x) \in \overset{o}{W}{}_{2,\lambda}^2(D)$. Let $p > 2-$ be a real number. For $i, j = 1, ..., n$, we have

$$\int_D \lambda_i(x) \lambda_j(x) \left(u_{ij}^S\right)^2 dx \le \left(\int_D \left|u_{ij}^{(S)}\right|^p dx \right)^{2/p} \cdot \left(\int_D [\lambda_i(x) \lambda_j(x)]^{\frac{p}{p-2}} dx \right)^{\frac{p}{p-2}}$$

By using Lemma 1, there exists a large number p, such that

$$\int_D [\lambda_i(x) \lambda_j(x)]^{\frac{p}{p-2}} dx < \infty, \ i, j = 1, ..., n. \tag{52}$$

Evidently

$$\lambda_i(x) \cdot \lambda_j(x) = \left[\frac{\omega_i^{-1}(\rho(x))}{\rho(x)} \right]^2 \cdot \left[\frac{\omega_j^{-1}(\rho(x))}{\rho(x)} \right]^2.$$

From this according to Theorem 4, we infer

$$\left\| u^{(S)} \right\|_{W_{2,\lambda}^2(D)} \le C_{39} \|f\|_{L_q(D)}. \tag{53}$$

It follows from the strong boundedness of the sequence $\left\{u^{(s)}(x)\right\}$ in $\overset{o}{W}{}_{2,\lambda}^2(D)$ that this is a weakly compact sequence in this space. Therefore, there exists a function $u'(x) \in \overset{o}{W}{}_{2,\lambda}^2(D)$ and a subsequence of integer numbers $\{s_k\}$ such that

$$\lim_{k \to \infty} \left(L u^{(s_k)}, \psi \right) = \left(L u', \psi \right), \tag{54}$$

for a function $\psi(x) \in C^{\infty}\left(\bar{D}\right)$ as $k \to \infty$. Where $(g, \psi) = \int_D g(x) \cdot \psi(x)\,dx$. On other hand

$$\left(Lu^{(s_k)}, \psi\right) = \left(L^{(s_k)}u^{(s_k)}, \psi\right) + \left(\left(L - L^{s_k}\right)u^{(s_k)}, \psi\right) =$$

$$= (f, \psi) + \left(\left(L - L^{(s_k)}\right)u^{(s_k)}, \psi\right) = (f, \psi) + i_k,$$

which together with (54) means that

$$(f, \psi) + \lim_{k\to\infty} i_k = \left(Lu', \psi\right). \tag{55}$$

Further, we have

$$|i_k| \le \sum_{i,j=1}^{n} \int_{D^+(S_k/2)} \frac{|a_{ij}(x)|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \sqrt{\lambda_i(x)\lambda_j(x)} \left|u_{ij}^{(S_k)}\right| \cdot |\psi|\,dx +$$

$$+ \sum_{i,j=1}^{n} \int_{D^+(S_k/2)} \frac{\left|a_{ij}^{(S_k)}(x)\right|}{\sqrt{\lambda_i(x)\lambda_j(x)}} \sqrt{\lambda_i(x)\lambda_j(x)} \left|u_{ij}^{(S_k)}\right| \cdot |\psi|\,dx = i_k^1 + i_k^2. \tag{56}$$

From conditions (5)-(6) it follows that $|h_{ij}(x)| \le h_0(L)$ for $x \in D$ and $i, j = 1, ..., n$. Therefore, and using (53), we get

$$i_k^1 \le h_0 \cdot \|u\|_{W_{2,\lambda}^2(D^+(S_k/2))} \cdot \|\psi\|_{L_2(D)}, \quad \text{i.e. } \lim_{k\to\infty} i_k^1 = 0. \tag{57}$$

Arguing by the analogy with preceding it follows that for the equality

$$\lim_{k\to\infty} i_k^2 = 0 \tag{58}$$

it suffices that

$$\int_D \frac{dx}{\lambda_i(x)\lambda_j(x)} < \infty, \ i, j = 1, ..., n. \tag{59}$$

Indeed, by using Lemma 1,

$$\int_D \frac{dx}{\lambda_i(x)\lambda_j(x)} \le \int_D \left(\frac{\omega_i(|x_i|)}{|x_i|}\right)^2 dx < \infty,$$

therefore, the inequality satisfied. From (55)-(58) it follows that for a function $\psi(x) \in C^{\infty}\left(\bar{D}\right)$ it holds the equality

$$\left(Lu', \psi\right) = (f, \psi),$$

therefore, $Lu' = f(x)$ almost everywhere in D.

Consider the general situation. Let O_1 be $n-$ dimensional ball of unit radii and center in the coordinate center, a function $\vartheta_1(x) \subset C_0^{\infty}(E_n)$ be such that $\vartheta_1(x) \ge 0$, $\vartheta_1(x) = 0$ everywhere outside O_1 and $\int_{E_n} \vartheta_1(x)\,dx = 1$.

Set $\vartheta_\varepsilon(x) = \frac{1}{\varepsilon^n}\vartheta_1\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$.

For a locally integrable function $\psi(x)$ in E_n denote $\psi^{\varepsilon}(x)=\int_{E_n}\vartheta_{\varepsilon}(x-y)\,\psi(y)\,dy$ the Frederiche's average of $\psi(x)$ with parameter ε.

Let for $i=1,...,n$ the functions $b_i^{[l]}(x)$, $C^{[l]}(x)$ and $f^{[l]}(x)$ are mollifies of the proper functions with parameter $\frac{1}{l}$

$$L^{[l]}=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i^{[l]}(x)\frac{\partial}{\partial x_i}+C^{[l]}(x),$$

and $u^{[l]}(x)$ be a strong solution from $\overset{o}{W}{}^{2}_{2,\lambda}(D)$ of the first boundary value problem

$$L^{[l]}u^{[l]}=f^{[l]}(x),\ \ x\in D;\ \ u^{[l]}\Big|_{\partial D}=0.$$

According to the preceding results, such a solution exists, moreover the Theorem 4 conforms on the estimate

$$\left\|u^{[l]}\right\|_{W^2_{2,\lambda}(D)}\le C_{39}\left\|f^{[l]}\right\|_{L_q(D)}\le C_{45}(L,n,q,D,f).\tag{60}$$

Therefore, there exists a solution $u(x)\in\overset{o}{W}{}^{2}_{2,\lambda}(D)$ and a subsequence of natural numbers $\{l_k\}$ such that

$$\lim_{k\to\infty}\left(Lu^{[l_k]},\psi\right)=(Lu,\psi)\tag{61}$$

as $k\to\infty$ for a function $\psi(x)\in C^{\infty}\left(\bar{D}\right)$.
On other hand

$$\left(Lu^{[l_k]},\psi\right)=\left(L^{[l_k]}u^{[l_k]},\psi\right)+\left(\left(L-L^{[l_k]}\right)u^{[l_k]},\psi\right)=$$

$$=\left(f^{[l_k]},\psi\right)+\left(L-L^{[l_k]}u^{[l_k]},\psi\right)=\left(f^{[l_k]},\psi\right)+j_k.$$

The (61) and the limit expression

$$\lim_{k\to\infty}\left(f^{[l_k]},\psi\right)=(f,\psi)$$

yields

$$(f,\psi)+\lim_{k\to\infty}j_k=(Lu,\psi).\tag{62}$$

Further, we have

$$|j_k|\le\sum_{i=1}^{n}\int_D\left|b_i(x)-b_i^{[l_k]}(x)\right|\cdot\left|u_i^{[l_k]}\right|\cdot|\psi|\,dx+$$

$$+\int_D\left|C(x)-C^{[l_k]}(x)\right|\cdot\left|u^{[l_k]}(x)\right|\cdot|\psi(x)|\,dx=j_k^1+j_k^2.\tag{63}$$

According to (40) and (42), we infer

$$j_k^1 \leq \|\psi\|_{L_2(D)} \cdot \sum_{i=1}^{n} \int_D \left\| \left(b_i - b_i^{[l_k]} \right) u_i^{[l_k]} \right\|_{L_2(D)} \leq$$

$$\leq C_{46}\left(L,n,D\right) \|\psi\|_{L_2(D)} \cdot \max \left\| b_i - b_i^{[l_k]} \right\|_{L_m(D)} \cdot \left\| u^{[l_k]} \right\|_{W_{2,\lambda}^2(D)};$$

$$j_k^2 \leq \|\psi\|_{L_2(D)} \cdot \left\| \left(C - C^{[l_k]} \right) u^{[l_k]} \right\|_{L_2(D)} \leq$$

$$\leq C_{47}\left(L,n,D\right) \|\psi\|_{L_2(D)} \cdot \left\| C - C^{[l_k]} \right\|_{L_\mu(D)} \cdot \left\| u^{[l_k]} \right\|_{W_{2,\lambda}^2(D)},$$

where the constants m and μ have the meaning as in the condition (7).

Using (60), we get

$$\lim_{k\to\infty} j_k^1 = \lim_{k\to\infty} j_k^2 = 0,$$

which together with (62) and (63) give

$$(Lu, \psi) = (f, \psi). \tag{64}$$

Since the equality (64) is true for a function $\psi(x) \in C^\infty\left(\bar{D}\right)$, then $Lu = f(x)$ for almost everywhere D. Therefore, it has been proved the existence of strong solution of the boundary value problem (1)-(2) and the estimate (50) follows from the Corollary of Theorem 4.

Prove now the uniqueness of the boundary value problem (1)-(2). Let $u_1(x)$ and $u_2(x)$ be different solutions of that problem. Set $\vartheta(x) = u_1(x) - u_2(x)$, then the function $\vartheta(x)$ will be a generalized solution of the problem (1)-(2) with $f(x) \equiv 0$. According to (50) $\vartheta(x) \equiv 0$ almost everywhere in D, i.e. $u_1(x) \equiv u_2(x)$ a.e. in D.

This completes the proof of Theorem 5.

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