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A Mixed Problem for a Class of Nonlinear Tymoshenko Systems

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Abstract. In this paper a mixed problem for semilinear systems of equations describing the oscillations of a thin-walled bar is considered. Reducing the problem under consideration to a differential equation, a theorem on local solvability is proved.

Key Words and Phrases: system of equations of a bar vibration, mixed problem, local solvability.

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Let us consider the bars described by a system of two differential equations in the domain $Q = [0,T] \times [0,l]$

$$\left.\begin{aligned} EIy_{xxxx} + \rho Ay_{tt} - \rho Ae\theta_{tt} = f_1(t,x,y,\theta) \\ EC_w\theta_{xxxx} - GC\theta_{xx} - \rho Aey_{tt} + \rho\left(I + Ae^2\right)\theta_{tt} = f_2\left(t,x,y,\theta\right) \end{aligned}\right\} \tag{1}$$

with boundary conditions

$$\left.\begin{aligned} y\left(0,t\right) = 0,\ y\left(l,\ t\right) = 0,\ y_{xx}\left(0,\ t\right) = 0\ ,\ y_{xx}\left(l,\ t\right) = 0 \\ \theta\left(0,t\right) = 0\ ,\ \theta\left(l,\ t\right) = 0,\quad \theta_{xx}\left(0,\ t\right) = 0\ ,\quad \theta_{xx}\left(l,\ t\right) = 0 \end{aligned}\right\} \tag{2}$$

with initial conditions

$$\left.\begin{aligned} y\left(x,0\right) = y_0\left(x\right), \qquad y_t\left(x,0\right) = y_1\left(x\right) \\ \theta\left(x,0\right) = \theta_0\left(x\right), \qquad \theta_t\left(x,0\right) = \theta_1\left(x\right) \end{aligned}\right\} \tag{3}$$

where $0 < x < l$, $0 < t < T$, $l > 0$, $T > 0$ are given numbers, $y\left(x,t\right)$ is a transverse displacement, $\theta\left(x,t\right)$ is an angle of cross-section of the bar, E is the Young's modulus, I is a polar moment of inertia of the cross section with respect to its center of gravity, ρ is a density of the material of the bar, A is a cross-sectional area, e is a distance from center of gravity to center of torsion, C_w is a sectorial moment of inertia of the cross section, G is a shear modulus, C is a geometric rigidity of free torsion, EC_w is a stiffness of bending torsion, GC is a stiffness of free torsion. Here, f_1 and f_2 are functions depending on t, x, y and θ (see e.g. [1, 2]).

The system of equations (1), (2) can be written as follows

$$Rw_{tt} + Sw + Nw = F(t,\ x,\ y,\ \theta), \tag{4}$$

$$w\,(0) = w_0, \qquad w_t\,(0) = w_1 \tag{5}$$

where

$$R = \begin{pmatrix} \rho A & -\rho Ae \\ -\rho Ae & \rho\left(I + Ae^2\right) \end{pmatrix}, S = \begin{pmatrix} EI\partial^4 & 0 \\ 0 & EC_w\partial^4 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 \\ 0 & -GC\partial^2 \end{pmatrix},$$

$$w = \begin{pmatrix} y \\ \theta \end{pmatrix}, w_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix}, w_1 = \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

Let us consider the functional space $\mathscr{H} = L_2\,(0,1) \times L_2(0,1)$ with a scalar product:

$$\left\langle w^1,\, w^2 \right\rangle = \left\langle w^1,\, w^2 \right\rangle_{\mathscr{H}} = \frac{I}{C_w}\left\langle y^1,\, y^2 \right\rangle_{L_2(0,1)} + \left\langle \theta^1,\, \theta^2 \right\rangle_{L_2(0,1)},$$

where

$$w^i = \left(y^i, \theta^i\right) \in \mathscr{H}, \quad i = 1, 2.$$

Let us define $\hat{H}_0^2$ and $\hat{H}_0^4$ in the following way:

$$\hat{H}_0^2 = \left\{u \ \ : \ \ u \in H^2,\ u(0) = u(l) = 0\right\},$$

$$\hat{H}_0^4 = \left\{u : \ \ u \in H^4,\ u(0) = u(1) = u_{xx}(0) = u_{xx}(l) = 0\right\}.$$

Denote by $\mathscr{H}_1$ the space $\widehat{H_0^2} \times \widehat{H_0^2}$, and by $\mathscr{H}_2$ the space $\widehat{H_0^4} \times \widehat{H_0^4}$.

Let the operator L be defined in the space $\mathscr{H}$:

$$D\,(L) = \mathscr{H}.$$

$$Lw = R^{-1}Sw = \begin{bmatrix} \frac{E(I+Ae^2)}{\rho A}\frac{\partial^4}{\partial x^4} & \frac{eEC_w}{\rho I}\frac{\partial^4}{\partial x^4} \\ \frac{eE}{\rho}\frac{\partial^4}{\partial x^4} & \frac{EC_w}{\rho I}\frac{\partial^4}{\partial x^4} \end{bmatrix} w, \quad where \quad w = \begin{pmatrix} y \\ \theta \end{pmatrix} \in D(L).$$

We also define the linear operator L_1 as follows:

$$D\,(L_1) = \mathscr{H}_1.$$

$$L_1 w = R^{-1}Cw = \begin{bmatrix} 0 & -\frac{eGC}{\rho I}\frac{\partial^2}{\partial x^2} \\ 0 & -\frac{GC}{\rho I}\frac{\partial^2}{\partial x^2} \end{bmatrix} w, \quad where \ \ w = \begin{pmatrix} y \\ \theta \end{pmatrix} \in D(L_1) \in \mathscr{H}_1.$$

We define the nonlinear operator $G(.)$ in the following way

$$G\,(t,\ w) = \begin{pmatrix} g_1(t, x, w) \\ g_2(t, x, w) \end{pmatrix},$$

where

$$g_1(t,x,w)=\frac{I+Ae^2}{\rho AI}f_1(t,x,y,\theta)+\frac{e}{\rho I}f_2(t,x,y,\theta),$$

$$g_2(t,x,w)=\frac{e}{\rho I}f_1(t,x,y,\theta)+\frac{1}{\rho I}f_2(t,x,y,\theta).$$

Then the problem (4), (5) can be written in the form

$$w_{tt}+Lw+L_1w=G(t,\ w), \tag{6}$$

$$w(0)=w_0,\ w'(0)=w_1. \tag{7}$$

Lemma 1. *L is a positive self-adjoint operator in $\mathscr{H}$.*

Proof. Let $w^i=(y^i,\theta^i)\in D(L)$.

$$Lw^1=\left(\frac{E\left(I+Ae^2\right)}{\rho A}y^1_{xxxx}+\frac{eEC_w}{\rho I}\theta^1_{xxxx},\ \frac{eE}{\rho}y^1_{xxxx}+\frac{EC_w}{\rho I}\theta^1_{xxxx}\right).$$

Hence we obtain that

$$\left\langle Lw^1,w^2\right\rangle=\frac{I}{C_w}\left\langle\frac{E\left(I+Ae^2\right)}{\rho A}y^1_{xxxx}+\frac{eEC_w}{\rho I}\theta^1_{xxxx},y^2\right\rangle_{L_2(0,1)}+$$

$$+\left\langle\frac{eE}{\rho}y^1_{xxxx}+\frac{EC_w}{\rho I}\theta^1_{xxxx},\theta^2\right\rangle_{L_2(0,1)}=$$

$$=\frac{E\left(I+Ae^2\right)}{\rho C_wA}\left\langle y^1_{xx},y^2_{xx}\right\rangle_{L_2(0,1)}+\frac{eE}{\rho}\left\langle\theta^1_{xx},y^2_{xx}\right\rangle_{L_2(0,1)}+$$

$$+\frac{eE}{\rho}\left\langle y^1_{xx},\theta^2_{xx}\right\rangle_{L_2(0,1)}+\frac{EC_w}{\rho I}\left\langle\theta^1_{xx},\theta^2_{xx}\right\rangle_{L_2(0,1)}. \tag{8}$$

Similarly we obtain that

$$Lw^2=\left(\frac{E\left(I+Ae^2\right)}{\rho A}y^2_{xxxx}+\frac{eEC_w}{\rho I}\theta^2_{xxxx},\ \frac{eE}{\rho}y^2_{xxxx}+\frac{EC_w}{\rho I}\theta^2_{xxxx}\right).$$

$$\left\langle w^1,Lw^2\right\rangle=\frac{I}{C_w}\left\langle u^1,\frac{E\left(I+Ae^2\right)}{\rho A}y^2_{xxxx}+\frac{eEC_w}{\rho I}\theta^2_{xxxx}\right\rangle_{L_2(0,1)}$$

$$+\left\langle v^1,\frac{eE}{\rho}y^2_{xxxx}+\frac{EC_w}{\rho I}\theta^2_{xxxx}\right\rangle_{L_2(0,1)}=$$

$$=\frac{E\left(I+Ae^2\right)}{\rho C_wA}\left\langle y^1_{xx},y^2_{xx}\right\rangle_{L_2(0,1)}+\frac{eE}{\rho}\left\langle y^1_{xx},\theta^2_{xx}\right\rangle_{L_2(0,1)}+$$

$$+\frac{eE}{\rho}\left\langle\theta_{xx}^{1},\,y_{xx}^{2}\right\rangle_{L_2(0,1)}+\frac{EC_w}{\rho I}\left\langle\theta_{xx}^{1},\,\theta_{xx}^{2}\right\rangle_{L_2(0,1)}. \tag{9}$$

Comparing (8) and (9), we obtain that

$$\left\langle Lw^1,\,w^2\right\rangle=\left\langle w^1,\,Lw^2\right\rangle.$$

On the other hand, the operator L is invertible.

Indeed, let $h=(h_1,\,h_2)\in\mathscr{H}$. Consider the equation

$$Lw=h,\qquad w=(y,\theta)\in D(L). \tag{10}$$

Equation (10) has the following form

$$\begin{cases}\frac{E(I+Ae^2)}{\rho A}y_{xxxx}+\frac{eEC_w}{\rho I}\theta_{xxxx}=h_1,\\ \frac{eE}{\rho}y_{xxxx}+\frac{EC_w}{\rho I}\theta_{xxxx}=h_2.\end{cases} \tag{11}$$

Hence we obtain that

$$\begin{cases}\frac{EI}{\rho A}u_{xxxx}=h_1-eh_2,\\ y\left(0\right)=y\left(l\right)=y_{xx}\left(0\right)=y_{xx}\left(l\right)=0.\end{cases} \tag{12}$$

The problem (11) has a unique solution $y\in\widehat{H_0^4}$. Similarly we obtain that the problem (11) has a unique solution

$$w=\left(y,\theta\right),\ \text{where}\ y,\ \theta\in\widehat{H_0^4},\quad i.e.\ \ w\in\mathscr{H}.$$

From the definition of L and from the scalar product in $\mathscr{H}$, we get that

$$\left\langle Lw,\,w\right\rangle=\frac{EI\left(I+Ae^2\right)}{\rho C_w A}\left\|y_{xx}\right\|_{L_2(0,1)}^2+\frac{2eE}{\rho}\left\langle y_{xx},\,\theta_{xx}\right\rangle_{L_2(0,1)}+\frac{EC_w}{\rho I}\left\|\theta_{xx}\right\|_{L_2(0,1)}^2. \tag{13}$$

Using the Holder's and Young's inequality, we obtain that

$$\left|2e\left\langle y_{xx},\theta_{xx}\right\rangle\right|=2\left|\left\langle e\sqrt{\frac{I}{C_w}}y_{xx},\,\sqrt{\frac{C_w}{I}}\theta_{xx}\right\rangle\right|\le e^2\frac{I}{C_w}\left\|y_{xx}\right\|_{L_2}^2+\frac{C_w}{I}\left\|\theta_{xx}\right\|_{L_2}^2. \tag{14}$$

From (13) and (14) we obtain that

$$\left\langle Lw,\,w\right\rangle\ge 0.$$

Thus, L is a positive self-adjoint operator.

Lemma 2. *Linear operator* L_1 *is subjected to the operator* $L^{\frac{1}{2}}$.

Proof. From the definition of L_1 it follows that

$$\|Lw\|^2_{\mathscr{H}} = \frac{(e+1)G^2C^2}{\rho^2 I^2}\int_0^{\partial}\left|\frac{\partial^2\theta}{\partial x^2}\right|^2 dx \le c\left\| L^{\frac{1}{2}}w\right\|^2_{\mathscr{H}},$$

i.e. L_1 is subjected to the operator $L^{\frac{1}{2}}$.

Applying the general theory of nonlinear hyperbolic differential equations, we obtain.

Theorem 1. *Let L be a positive self-adjoint operator and L_1 is subjected to the operator $L^{\frac{1}{2}}$. Suppose that $G(t,\ w)$ acts from $[0,T]\times\mathscr{H}_1$ to $\mathscr{H}$ and satisfies the local Lipschitz condition, i.e. if for any $t_1,t_2\in[0,\ T]$ and $w^1,w^2\in\mathscr{H}_1$*

$$\left\| G(t_1,\ w^1) - G(\ t_2,\ w^2)\right\|_{\mathscr{H}} \le c\left(\left\|w^1\right\|_{\mathscr{H}_1},\ \left\|w^2\right\|_{\mathscr{H}_1}\right)\times\left[|t_1-t_2| + \ \left\|w^1-w^2\right\|_{\mathscr{H}_1}\right].$$

Then for any $w_0\in\mathscr{H}_1$, $w_1\in\mathscr{H}$ there exists T', such that the problem (6), (7) *has a unique solution*

$$w\in C(\left[0,T'\right],\mathscr{H}_1)\cap C^1(\left[0,T'\right],\mathscr{H}).$$

If T_{max} is the length of the maximum interval of existence of solutions, then one of the following alternatives is fulfilled

i) $\lim_{t\to T_{max}-0}\left[\left\|w'(t)\right\|_{\mathscr{H}} + \ \left\|w(t)\right\|_{\mathscr{H}_1}\right] \ = +\infty$

or

ii) $T_{max}=T$.

Note that if $w_0\in\mathscr{H}_0$ and $w_1\in\mathscr{H}_1$, then

$$w\in C(\left[0,T'\right],\mathscr{H}_0)\cap C^1(\left[0,T'\right],\mathscr{H}_1)\cap C^2(\left[0,T'\right],\mathscr{H}).$$

Lemma 3. *Let*

$$f_i(t,x,y,\theta)\in C^1([0,T]\times[0,l]\times R^2).$$

Then $G(t,\ w) = \begin{pmatrix} g_1(t,x,w)\\ g_2(t,x,w)\end{pmatrix}$ *acts from $\mathscr{H}_1$ to $\mathscr{H}$ and satisfies the local Lipschitz condition.*

Proof. Let $t_i\in[0,T]$, $w^i=(y^i,\theta^i)\in\mathscr{H}$. Then

$$\left\| G(t_1,\ w^1) - G(\ t_2,\ w^2)\right\|^2_{\mathscr{H}} \le$$

$$\le c\left\| f_1(t_1,x,y^1,\theta^1) - f_2(t_2,x,y^2,\theta^2)\right\|^2_{L_2(0,l)} + \ c\left\| f_2(t_2,x,y^2,\theta^2)\right\|^2_{L_2(0,l)},$$

where $c = max\left\{\frac{I+Ae+Ae^2}{\rho AI},\ \frac{e+1}{\rho I}\right\}$, on the other hand

$$\left\| f_1(t_1,x,y^1,\theta^1) - f_2(t_2,x,y^2,\theta^2)\right\|^2_{L_2(0,l)} =$$

$$= \int_0^l\left|\int_0^1 f'_{1_t}\left(t_1+\tau(t_2-t_1),y^1+\tau\left(y^2-y^1\right),\theta^1+\tau\left(\theta^2-\theta^1\right)\right)d\tau\right|^2 dx\,|t_1-t_2| +$$

$$+\int_0^l \left|\int_0^1 f'_{1_u}\left(t_1+\tau\left(t_2-t_1\right), y^1+\tau\left(y^2-y^1\right), \theta^1+\tau\left(\theta^2-\theta^1\right)\right) d\tau\right|^2 \left|y^1-y^2\right|^2 dx+$$

$$+\int_0^l \left|\int_0^1 f'_{1_u}\left(t_1+\tau\left(t_2-t_1\right), y^1+\tau\left(y^2-y^1\right), \theta^1+\tau\left(\theta^2-\theta^1\right)\right) d\tau\right|^2 \left|\theta^1-\theta^2\right|^2 dx \leq$$

$$\leq \sup_{\substack{0\leq t\leq T\\ x\in[0,l]\\ |\xi|\leq r_0\\ |\eta|\leq r_1}}[\left|f_{1_t}\left(t_1,x,\xi,\eta\right)\right|+\left|f_{1_t}\left(t_1,x,\xi,\eta\right)\right|+\left|f_{1_t}\left(t_1,x,\xi,\eta\right)\right|]\times$$

$$\times[l\left|t_1-t_2\right|+\int_0^l \left|y^1\left(x\right)-y^2\left(x\right)\right|^2 dx+\int_0^l \left|\theta^1\left(x\right)-\theta^2\left(x\right)\right|^2 dx].$$

Hence we obtain that

$$\left\| f_1\left(t_1,x,y^1,\theta^1\right)-f_2\left(t_2,x,y^2,\theta^2\right)\right\|^2_{L_2(0,l)} \leq$$

$$\leq c\left(\left\|y^1\right\|_{\mathscr{H}_1}, \left\|y^2\right\|_{\mathscr{H}_1}, \left\|\theta^1\right\|_{\mathscr{H}_1}, \left\|\theta^2\right\|_{\mathscr{H}_1}\right)\times[\left|t_1-t_2\right|+\left\| y^1-y^2\right\|^2_{L_2(0,l)}+\left\| \theta^1-\theta^2\right\|^2_{L_2(0,l)}] \leq$$

$$\leq c\left(\left\|w^1\right\|_{\mathscr{H}_1}, \left\|w^2\right\|_{\mathscr{H}_1}\right)\cdot[\left|t_1-t_2\right|^2+\left\| w^1-w^2\right\|^2_{\mathscr{H}_1}],$$

where

$$r_0=\max_{x\in[0,l]}[\left|y^1\left(x\right)\right|+\left|y^2\left(x\right)\right|$$

$$r_1=\max_{x\in[0,l]}[\left|\theta^1\left(x\right)\right|+\left|\theta^2\left(x\right)\right|$$

Using Lemmas 1-3 from the Theorem 1, we obtain the following result:

Theorem 2. *Let*

$$f_i\left(t,x,y,\theta\right)\in C^1([0,T]\times[0,l]\times R^2).$$

Then for any $y_0,\ \theta_0\in\widehat{H_0^2},\quad y_1,\ \theta_1\in L_2\left(0,1\right)$ *there exists* $T'>0$ *, such that the problem* (1) -(3) *has a unique solution* $\left(y,\theta\right)$, *where*

$$y,\theta\in C^1(\left[0,T'\right], L_2(0,1))\cap C(\left[0,T'\right], \widehat{H_0^2}).$$

Moreover, if T_{max} *is the length of the maximum interval of existence of solutions, then one of the following alternatives is fulfilled*

i) $\lim_{t\to T_{max}-0}\left[\left\| y_t(t,\cdot)\right\|^2_{L_2(0,l)}+\ \left\|\theta_t(t,\cdot)\right\|^2_{L_2(0,l)}+\left\| y(t,\cdot)\right\|^2_{\widehat{H_0^2}}+\left\|\theta(t,\cdot)\right\|^2_{\widehat{H_0^2}(0,l)}\right] = +\infty$

or

ii) $T_{max}=T.$

References

[1] S.S. Kokhmanyuk, E.G. Yanyutin, L.G. Romanenko, *Oscillations of Deformable Sys- temsunder Impulse and Moving Loads*, Naukova Dumka, Kiev (1980), 231 p. [in Russian].

[2] V. Komkov, *Optimal Control Theory for the Damping of Vibrations of Simple Elastic Systems*, Moscow, Mir, 1975, 158 p.

[3] A.B. Aliev, *The Cauchy problem for high-order quasilinear equations of hyperbolic type with the Volter operator*, DAN SSSR, **280(1)**, 1985, 15-18.

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