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On Embedding Theorem in Variable Lebesgue Spaces with Mixed Norm

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Abstract. In this paper, we study theorem on continuously embedding between variable exponent Lebesgue spaces with mixed norm. In particular, we found a criterion characterizing the embedding between variable exponent Lebesgue spaces with mixed norm.

Key Words and Phrases: variable exponent Lebesgue spaces, variable exponent Lebesgue spaces with mixed norm, embedding theorem.

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1. Introduction

It is well known that the variable Lebesgue space in the literature for the first time was studied by Orlicz [11] in 1931. In [11], Hölder's inequality for variable discrete Lebesgue space was proved. Orlicz also considered the variable Lebesgue space on the real line, and proved the Hölder inequality in this setting. However, this paper is essentially the only contribution of Orlicz to the study of the variable Lebesgue spaces (see also [8]). The next step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [9] and [10]. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Musielak and others Polish mathematicians (see [7]). In particular, the variable Lebesgue spaces were objects of interest during the last two decades(see [4, 5]). The further investigation of these spaces being undertaken in [6, 13, 14] and e.t.c. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [4, 12, 15]).

In this paper, we study theorem on continuously embedding between variable exponent Lebesgue spaces with mixed norm. In particular, we found a criterion characterizing the embedding between variable exponent Lebesgue spaces with mixed norm.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. The main results are stated and proved in Section 3.

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2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x=(x_1,...,x_n)$ and let Ω be a Lebesgue measurable subset in $\mathbb{R}^n$. Suppose that $\mathbf{p}(x)=(p_1\,(x_1,\ldots,x_n)\,,p_2\,(0,x_2,\ldots,x_n)\,,\ldots,p_n\,(0,\ldots,0,x_n))$ is a vector function defined on $\mathbb{R}^n$ with Lebesgue measurable components $p_i\left(x^{(i)}\right)$, such that $1\le p_i(x^{(i)})<\infty$ and $x^{(i)}=(0,\ldots,0,x_i,\ldots,x_n)$ $(i=1,\ldots,n)$. Further in this paper all sets and functions are supposed to be Lebesgue measurable and $x^{(1)}=x$, $x^{(n)}=(0,\ldots,0,x_n)$. Throughout this paper $\underline{p}_i=\operatorname*{ess\,inf}_{x^{(i)}\in\mathbb{R}^n}p_i\left(x^{(i)}\right)$, $\overline{p}_i=\operatorname*{ess\,sup}_{x^{(i)}\in\mathbb{R}^n}p_i\left(x^{(i)}\right)$, $\underline{q}_i=\operatorname*{ess\,inf}_{x^{(i)}\in\mathbb{R}^n}q_i\left(x^{(i)}\right)$, $\overline{q}_i=\operatorname*{ess\,sup}_{x^{(i)}\in\mathbb{R}^n}q_i\left(x^{(i)}\right)$ and $p_n\left(x^{(n)}\right)=p_n\left(x_n\right)$. We denote by $\mathbf{p}'(x)=\left(p_1'\left(x\right),p_2'\left(x^{(2)}\right),\ldots,p_n'\left(x^{(n)}\right)\right)$ the conjugate exponent vector-function defined by
$\dfrac{1}{\mathbf{p}(x)}+\dfrac{1}{\mathbf{p}'(x)}=1$, $x\in\mathbb{R}^n$, i.e. $\dfrac{1}{p_i\left(x^{(i)}\right)}+\dfrac{1}{p_i'\left(x^{(i)}\right)}=1$, $i=1,\ldots,n$.

By $L_{p_1(x),x_1}\left(\mathbb{R}^n\right)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_1>0$

$$I_{1,\,p_1}f\left(x_2,\ldots,x_n\right)=\int\limits_{\mathbb{R}}\left(\frac{|f(x)|}{\lambda_1}\right)^{p_1(x)}dx_1<\infty.$$

The expression

$$\|f\|_{L_{p_1(x),\,x_1}(\mathbb{R}^n)}=\|f\|_{p_1(\cdot),\,x_1}=\inf\left\{\lambda>0:\ \int\limits_{\mathbb{R}}\left(\frac{|f(x)|}{\lambda}\right)^{p_1(x)}dx_1\le 1\right\}$$

is the norm in $L_{p_1(x),\,x_1}\left(\mathbb{R}^n\right)$ with respect to the variable x_1. It is obvious that the result is a function of variables $x_2,\ldots,x_n$, i.e. $\|f\|_{p_1(\cdot),\,x_1}=\|f\|_{p_1(\cdot),\,x_1}\left(x_2,\ldots,x_n\right)$.

Further, by $L_{\left(p_1(x),\,p_2\left(x^{(2)}\right)\right),\,x_1,\,x_2}\left(\mathbb{R}^n\right)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_2>0$

$$I_{2,\,p_2}f\left(x_3,\ldots,x_n\right)=\int\limits_{\mathbb{R}}\left(\frac{\|f\|_{p_1(\cdot),x_1}\left(x_2,\ldots,x_n\right)}{\lambda_2}\right)^{p_2\left(x^{(2)}\right)}dx_2<\infty.$$

The expression

$$\|f\|_{L_{\left(p_1(x),\,p_2\left(x^{(2)}\right)\right),\,x_1,\,x_2}(\mathbb{R}^n)}=\left\|\|f\|_{p_1(\cdot),x_1}\right\|_{p_2(\cdot),\,x_2}$$

$$=\inf\left\{\mu>0:\ \int\limits_{\mathbb{R}}\left(\frac{\|f\|_{p_1(\cdot),x_1}\left(x_2,\ldots,x_n\right)}{\mu}\right)^{p_2\left(x^{(2)}\right)}dx_2\le 1\right\}$$

is the norm in $L_{\left(p_1(x),\,p_2\left(x^{(2)}\right)\right),\,x_1,\,x_2}\left(\mathbb{R}^n\right)$. It is obvious that the result is a function of variables $x_3,\ldots,x_n$.

Definition 1. *By $L_{\mathbf{p}(x)}(\mathbb{R}^n) = L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),\ldots,p_n(x_n)\right)}(\mathbb{R}^n)$ we denote the space of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda_n > 0$*

$$I_{n,p_n} f = \int\limits_{\mathbb{R}} \left(\frac{\left\| \cdots \left\| \|f\|_{p_1(\cdot),x_1} \right\|_{p_2(\cdot),\, x_2} \cdots \right\|_{p_{n-1}(\cdot),\, x_{n-1}} (x_n)}{\lambda_n} \right)^{p_n(x_n)} dx_n < \infty.$$

The expression

$$\|f\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)} = \left\| \cdots \left\| \|f\|_{p_1(\cdot),x_1} \right\|_{p_2(\cdot),\, x_2} \cdots \right\|_{p_n(\cdot),\, x_n}$$

$$= \inf \left\{ \nu > 0 : \int\limits_{\mathbb{R}} \left(\frac{\left\| \cdots \left\| \|f\|_{p_1(\cdot),x_1} \right\|_{p_2(\cdot),\, x_2} \cdots \right\|_{p_{n-1}(\cdot),\, x_{n-1}} (x_n)}{\nu} \right)^{p_n(x_n)} dx_n \le 1 \right\}$$

defines a norm in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$.

Remark 1. *Let $\mathbf{p}(x) = (p_1, \ldots, p_n) = \mathbf{p} \ge \mathbf{1}$, i.e. $1 \le p_i\left(x^{(i)}\right) = p_i = const$, $i = 1, \ldots, n$. It is well known that usually Lebesgue spaces with mixed norm was introduced and studied in [3]. The variable Lebesgue spaces with mixed norm was introduced and studied in [1] and [2].*

Suppose that $\Omega \subset R^n$ is a measurable set and $f : \Omega \mapsto \mathbb{R}$. The norm in the space $L_{\mathbf{p}(x)}(\Omega)$ is defines as

$$\|f\|_{L_{\mathbf{p}(x)}(\Omega)} = \|f\chi\Omega\|_{L_{\mathbf{p}(x)}(\mathbb{R}^n)},$$

where $\chi_\Omega(x)$ is a characteristic function of a set Ω.

Remark 2. *Let $\mathbf{p}(x) = (p_1, \ldots, p_n) = \mathbf{p} \ge \mathbf{1}$, i.e. $1 \le p_i\left(x^{(i)}\right) = p_i = const$, $i = 1, \ldots, n$. Then $L_{\mathbf{p}(x)}(\mathbb{R}^n)$ coincides with the usual mixed norm Lebesgue spaces.*

Remark 3. *Let $p_1(x_1, \ldots, x_n) = p_2\left(x^{(2)}\right) = \ldots = p_n\left(x^{(n)}\right) = p(x_n)$, i.e. $\mathbf{p}(x) = (p(x_n), \ldots, p(x_n))$. Then $L_{\mathbf{p}(x)}(\mathbb{R}^n) = L_{p(x_n)}(\mathbb{R}^n)$.*

Now we introduce a analog of generalized Hölder inequality in variable Lebesgue space with mixed norm.

Lemma 1. *Let $\mathbf{p}(x) = (p_1(x), \ldots, p_n(x_n))$, $\mathbf{q}(x) = (q_1(x), \ldots, q_n(x_n))$ and $\mathbf{r}(x) = (r_1(x), \ldots, r_n(x_n))$. Suppose that $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$ and $\dfrac{1}{r_i\left(x^{(i)}\right)} = \dfrac{1}{p_i\left(x^{(i)}\right)} - \dfrac{1}{q_i\left(x^{(i)}\right)}$, $i = 1, \ldots, n$.*

Then the inequality

$$\|fg\|_{L_{\mathbf{p}(\cdot)}(\Omega)} \le \prod_{i=1}^{n} \left(A_i + B_i + \left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)} \right)^{1/\underline{p}_i} \|f\|_{L_{\mathbf{q}(\cdot)}(\Omega)} \, \|g\|_{L_{\mathbf{r}(\cdot)}(\Omega)}$$

holds for any $f \in L_{\mathbf{q}(x)}(\Omega)$, $g \in L_{\mathbf{r}(x)}(\Omega)$, *where* $\Omega_{1,i} = \left\{x \in \Omega : p_i\left(x^{(i)}\right) < q_i\left(x^{(i)}\right)\right\}$, $\Omega_{2,i} = \left\{x \in \Omega : p_i\left(x^{(i)}\right) = q_i\left(x^{(i)}\right)\right\}$, $A_i = \sup\limits_{x\in\Omega_{1,i}} \dfrac{p_i\left(x^{(i)}\right)}{q_i\left(x^{(i)}\right)}$ *and*
$B_i = \sup\limits_{x\in\Omega_{1,i}} \dfrac{q_i\left(x^{(i)}\right) - p_i\left(x^{(i)}\right)}{q_i\left(x^{(i)}\right)}$.

Remark 4. *Note that in the case* $p_1(x) = p_2\left(x^{(2)}\right) = \ldots = p_n(x_n) = 1$, *Theorem Lemma 2.1 was proved in [1]. The proof of Lemma 2.1 is similar, but with some modifications (see [2]).*

3. Main results

Now we introduce an embedding theorem between variable Lebesgue spaces with mixed norm.

Theorem 1. *Let* $\mathbf{p}(x) = (p_1(x), \ldots, p_n(x_n))$, $\mathbf{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *and* $\mathbf{r}(x) = (r_1(x), \ldots, r_n(x_n))$. *Suppose that* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$ *and satisfy condition* $\dfrac{1}{r_i\left(x^{(i)}\right)} = \dfrac{1}{p_i\left(x^{(i)}\right)} - \dfrac{1}{q_i\left(x^{(i)}\right)}$, $i = 1, \ldots, n$.

Then the following conditions are equivalent

a) $L_{\mathbf{q}(x)}(\Omega) \hookrightarrow L_{\mathbf{p}(x)}(\Omega)$;

b) $\|1\|_{L_{\mathbf{r}(\cdot)}(\Omega)} = \left\|\ldots\|\chi\Omega\|_{r_1(\cdot),x_1}\ldots\right\|_{r_n(\cdot),x_n} < \infty$.

Proof. The implication $b) \Rightarrow a)$ immediately implies from Lemma 2.1. Indeed, if we take $g = 1$ in Lemma 2.1 we proved this implication. The proof of implication $a) \Rightarrow b)$ is similar to the case $p_1(x_1, \ldots, x_n) = p_2\left(x^{(2)}\right) = \ldots = p_n\left(x^{(n)}\right) = p(x_n)$ (see [5]).

Now we introduce some particular case of Theorem 3.1.

Let $I = \{x \in \mathbb{R}^n : -\infty \le a_i \le x_i \le b_i \le \infty, i = 1, 2, \ldots, n\}$.

Corollary 1. *[2] Let* $x \in I$, *and let* $\mathbf{p}(x) = (p_1(x), \ldots, p_n(x_n))$ *and* $\mathbf{q}(x) = (q_1(x), \ldots, q_n(x_n))$ *be a vector-functions such that* $1 \le \underline{p}_i \le p_i\left(x^{(i)}\right) \le q_i\left(x^{(i)}\right) \le \overline{q}_i < \infty$. *Suppose that satisfy the following conditions*

$$A_i = \sup_{x^{(i+1)}} \int\limits_{a_i}^{b_i} \left(q_i\left(x^{(i)}\right) - p_i\left(x^{(i)}\right)\right) dx_i < \infty, \quad i = 1, \ldots, n-1,$$

$$A_n = \int\limits_{a_n}^{b_n} \left(q_n(x_n) - p_i(x_n)\right)) dx_n < \infty.$$

Then $L_{\mathbf{q}(x)}(I) \hookrightarrow L_{\mathbf{p}(x)}(I)$ *and the inequality*

$$\|f\|_{L_{\mathbf{p}(\cdot)}(I)} \le \prod_{i=1}^{n} \left[B_i(p_i, q_i)\right]^{\gamma_i} \|f\|_{L_{\mathbf{q}(\cdot)}(I)}$$

holds, where $B_i(p_i, q_i) = \dfrac{1}{s_i} + \dfrac{A_i}{\underline{q}_i}$, $s_i = \operatorname{ess\,inf}_{x^{(i)}} \dfrac{q_i(x^{(i)})}{p_i(x^{(i)})}$ *and* $\gamma_i = \begin{cases} \frac{1}{\underline{p}_i}, & \text{for } B_i(p_i, q_i) \geq 1 \\ \frac{1}{\overline{p}_i}, & \text{for } B_i(p_i, q_i) \leq 1. \end{cases}$

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