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Global Bifurcation for Half-linearizable Sturm-Liouville Problems with Spectral Parameter in the Boundary Condition

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Abstract. We consider half-linearizable Sturm-Liouville problems with spectral parameter in the boundary condition. We study the structure of the set of bifurcation points and the behaviour of global sets of solutions of this problem bifurcating from the points of the line of trivial solutions.

Key Words and Phrases: half-linearizable Sturm-Liouville problems, half-eigenvalue, half-eigenfunction, bifurcation point, global sets of solutions.

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1. Introduction

In the present paper, we continue the study [2] of the boundary value problem

$$\ell(y) \equiv -\left(p(x)y'\right)' + q(x)y = \lambda r(x)y + h(x, y, y', \lambda),\ x \in (0, \pi), \tag{1}$$

$$b_0 y(0) = d_0 y'(0), \tag{2}$$

$$(a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi), \tag{3}$$

where λ is a real parameter, the functions $p \in C^1[0, \pi],\ q, r \in C^0[0, \pi]$, and $b_0,\ d_0,\ a_1,\ b_1,\ c_1,\ d_1$ are real numbers such that $|b_0| + |d_0| > 0$ and

$$a_1 d_1 - b_1 c_1 > 0. \tag{4}$$

We also assume that p and r are strictly positive on $[0, \pi]$. The nonlinear term h has a representation $h = \alpha y^+ + \beta y^- + g$, where α, β are the continuous functions on $[0, \pi]$, $y^+ = \max\{y, 0\}$, $y^- = \max\{-y, 0\}$, and g is a continuous function on $[0, \pi] \times \mathbb{R}^3$, satisfying the condition:

$$g(x, u, s, \lambda) = o(|u| + |s|), \tag{5}$$

near $(u, s) = (0, 0)$, uniformly in $x \in [0, \pi]$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

The purpose of this paper is to study the structure of the set of bifurcation points on real axis and more accurately describe the structure and behaviour of bifurcation branches of solutions of problem (1)-(3).

In nonlinear analysis an important role is played bifurcation theory of nonlinear eigenvalue problems. The study of nonlinear eigenvalue problems has an applied interest since problems of this type arise in the theory of vibrations, thermal convection theory, hydrodynamics, the theory of critical modes of operation of nuclear and chemical reactors, the theory of critical loads and the theory of elasticity (see, for example, [7, 9, 10]).

Bifurcation problems for nonlinear Sturm-Liouville problems when the spectral parameter is not involved in the boundary conditions was considered by many authors (see [2, 3, 6, 12, 13, 15]). In these papers prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ corresponding to the usual nodal properties and emanating from bifurcation points or bifurcation intervals (in $\mathbb{R} \times \{0\}$ which we identify with $\mathbb{R}$) surrounding the eigenvalues of the corresponding linear problem. It should be noted that in a recent paper [1] of the first author obtained similar results for nonlinear eigenvalue problems for ordinary differential equations of fourth order.

In [3] was also studied problem (1)-(3) in the case of $a_1 = c_1 = 0$ where shown that for this problem possessing different linearizations as $y \to 0^+$ and $y \to 0^-$, the half-eigenvalues of the half-linear problem (1)-(3) with $a_1 = c_1 = 0$ and $g \equiv 0$ correspond to bifurcation points in a global sense.

Problem (1)-(3) in a more general case (i.e. when the nonlinear term h is of the form $h = f + g$, f being continuous and satisfying the condition $|f(x, u, s, \lambda)| < M|u|$ in a neighborhood of $u = s = 0$, uniformly in $x \in [0, \pi]$ and in $\lambda \in \Lambda$) was considered in [2]. In this paper prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ emanating from bifurcation intervals surrounding the eigenvalues of the linear problem obtained from (1)-(3) by setting $h \equiv 0$.

In [5] Browne uses the Prüfer angle techniques for half-eigenvalue problem (1)-(3) with $g \equiv 0$ obtain the existence of two sequences of half-eigenvalues which are different according to the sign of the corresponding half-eigenfunctions in a neighborhood of 0. He studies also oscillatory properties of the corresponding half-eigenfunctions, but in the case $c_1 \neq 0$ could not accurately determine the serial numbers of the half-eigenfunctions which have the same number of zeros in the interval $(0, 1)$. And it is prevents to detailed study of global bifurcation of solutions of problem (1)-(3) in the case $c_1 \neq 0$.

By applying the results of works [2, 5, 8, 11] (in the case of $c_1 \neq 0$ for additional restrictions on the functions $\alpha(x)$ and $\beta(x)$) we update the oscillatory properties of corresponding half-linear problem (1)-(3) with $g \equiv 0$, and show that the set of bifurcation points of problem (1)-(3) consists of all half-eigenvalues of problem (1)-(3) with $g \equiv 0$. Using the approximation technique from [3] and combining it with the global bifurcation results in [2, 8, 11] we prove the existence of global sets of solutions emanating from bifurcation points which are similar to those obtained in [3].

2. Preliminary

Along with problem (1)-(3), we consider the following boundary value problem

$$\begin{cases} \ell(y) = \lambda r(x)y + \alpha(x)y^{+} + \beta(x)y^{-}, \ x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi), \end{cases} \tag{6}$$

The problem (6) is non-linear, but is positively homogeneous (in the sense that if y is a solution of this problem, then αy is also a solution for all $\alpha > 0$) and linear in the cones $y > 0$ and $y < 0$. Hence nonlinear eigenvalue problems of this type called "half-linear" by Berestycki [3].

The following definitions are given by Berestycki [3] (see also [14]). We say that λ is a half-eigenvalue of problem (6) if there exists a nontrivial solution (λ, y_λ) of this problem; the function y_λ be called a half-eigenfunction. In this situation, the set $\{(\lambda, ty_\lambda) : t > 0\}$ is a half-line of nontrivial solutions of problem (6). The number λ is said to be simple if all solutions (λ, v) of (6) with v and y_λ, having the same sign in a deleted neighborhood of 0, are on this half-line. There may exist another half-line of solutions $\{(\lambda, v_\lambda) : t > 0\}$, but then we say that λ is simple if v_λ and y_λ have different signs in a deleted neighborhood of 0, and all solutions (λ, v) of problem (6) lie on these two half-lines.

From now on ν will denote an element of $\{+, -\}$ that is, either $\nu = +$ or $\nu = -$.

Half-linear problem (6) in the case $|a_1| + |c_1| > 0$ was investigated in [5], where the author shows that for each ν there exists infinitely increasing sequence $\{\lambda_k^\nu\}_{k=1}^\infty$, of real and simple half-eigenvalues of this problem. The corresponding half-eigenfunctions $y_k^\nu(x)$, $k = 1\,2, \ldots,$ have the following properties:

(i) $\nu y_k^\nu > 0$ in a deleted neighborhood of 0;

(i) if $c_1 = 0$, then function $y_k^\nu(x)$, $k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in $(0, \pi)$;

(iii) if $c_1 \neq 0$, then $y_k^\nu(x)$ has exactly $k-1$ simple nodal zeros for $k \leq N_0^\nu$, and exactly $k-2$ simple nodal zeros for $k > N_0^\nu$ in the interval $(0, \pi)$, where a positive integer N_0^ν is determined from the inequality $\mu_{N(0)-1}^\nu < -\frac{d_1}{c_1} \leq \mu_{N(0)}^\nu$; μ_k^ν, $k \in \mathbb{N}$, is the kth half-eigenvalue of equation (1) with the boundary conditions (2) and $y(\pi) = 0$ (by a nodal zero we mean the function changes sign at the zero and at a simple nodal zero, the derivative of the function is nonzero).

For $c_1 \neq 0$ let N_0 be an integer such that $\tau_{N_0-1} < -\frac{d_1}{c_1} \leq \tau_{N_0}$, where τ_k, $k \in \mathbb{N}$, is the kth eigenvalue of the Sturm-Liouville equation $\ell(y) = \lambda r(x)y$, $x \in (0, \pi)$, with the boundary conditions (2) and $y(\pi) = 0$; here we take $\tau_0 = -\infty$.

We also consider the following eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x)y, \ x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)y(\pi) = (c_1\lambda + d_1)y'(\pi). \end{cases} \tag{7}$$

It is known [4] that the eigenvalues of problem (7) are real, simple, and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^\infty$. The corresponding eigenfunctions $v_k(x)$, $k = 1, 2, \ldots,$ have the following oscillation properties : (a) if $c_1 = 0$, then $v_k(x)$, $k \in \mathbb{N}$, has exactly $k-1$ simple nodal zeros in $(0, \pi)$; (b) if $c_1 \neq 0$, then $v_k(x)$ has exactly $k-1$ simple nodal zeros for $k \leq N_0$, and exactly $k-2$ simple nodal zeros for $k > N_0$ in the interval $(0, \pi)$.

Should be noted that in [5] for the case $c_1 \neq 0$ is not given the connection between the natural numbers N_0^ν, $\nu \in \{+, -\}$ and N_0. Therefore, in this case, applying the global bifurcation result from [2] (see [2, Theorem 3.4]), it is impossible to study the structure of all the bifurcation branches of the solutions of problem (1)-(3).

3. Global bifurcation of solutions of problem (1)-(3)

Let E be the Banach space of all continuously differentiable functions on $[0, \pi]$ which satisfy the boundary condition (2). E is equipped with its usual norm $||y||_1 = \max\limits_{x\in[0,\pi]} |y(x)| + \max\limits_{x\in[0,\pi]} |y'(x)|$. Let S_k^+ be the set of functions $y \in E$ which have exactly $k-1$ simple nodal zeros in $(0, \pi)$ and which are positive near $x = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. The sets S_k^+ and S_k^- are disjoint and open in E. We say that $(\lambda, 0)$ is a bifurcation point of (1)-(3) with respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, if in every small neighborhood of this point there is a solution to this problem which is contained in $\mathbb{R} \times S_k^\nu$.

Let $J_k = [\lambda_k - \frac{M}{r_0}, \lambda_k + \frac{M}{r_0}]$, $k \in \mathbb{N}$. For $c_1 = 0$ let $I_k = J_k$, $k \in \mathbb{N}$, and for $c_1 \neq 0$ let

$$I_k = \begin{cases} \tilde{J}_k, & \text{if } k \neq N_0, \\ [\lambda_{N_0} - \frac{M}{r_0}, \lambda_{N_0+1} + \frac{M}{r_0}], & \text{if } k = N_0, \end{cases}$$

where $M = \max\limits_{x\in[0,\pi]} \{|\alpha(x)| + |\beta(x)|\}$, $r_0 = \min\limits_{x\in[0,\pi]} r(x)$ and $\tilde{J}_k = \begin{cases} J_k, & \text{if } k < N_0, \\ J_{k+1}, & \text{if } k > N_0. \end{cases}$

It is obvious that

$$|\alpha(x)y^+(x) + \beta(x)y^-(x)| \leq M|y(x)|, \; x \in [0, \pi].$$

Hence for the boundary value problem (1)-(3) the assertions of section 3 of the work [2] is true. Therefore, for this problem have the following results.

Lemma 1. *The set of bifurcation points of problem* (1)-(3) *is nonempty.*

Lemma 2. *If* $(\lambda, 0)$ *is a bifurcation point of* (1)-(3)*, then* λ *is an half-eigenvalue of problem* (6)*.*

Proof. Let $(\lambda_n, y_n) \in \mathbb{R} \times E$, $y_n \neq 0$, be a sequence of solutions of problem (1)-(3) converging to $(\lambda, 0)$. Let $v_n = \frac{y_n}{||y_n||_1}$. Then dividing (1)-(3) by $||y_n||_1$ and setting

$$v_n(x) = \frac{y_n(x)}{||y_n||_1} \quad \text{and} \quad g_n(x) = \frac{g(x, y_n(x), y_n'(x), \lambda_n)}{||y_n||_1}$$

we have

$$\begin{cases} \ell(v_n)(x) = \lambda_n v_n(x) + \alpha(x)v_n^+(x) + \beta(x)v_n^-(x) + g_n(x), x \in (0,\pi), \\ v_n \in BC_\lambda \ , \end{cases} \tag{8}$$

where denote by BC_λ the set of boundary conditions (2)-(3). Since $\{v_n\}_{n=1}^\infty$ is bounded in $C^1[0,\pi]$, α, β are bounded in $C^0[0,\pi]$, and $g_n \to 0$ in $C^0[0,\pi]$ (by (5)), it follows from (8) that $\{v_n\}_{n=1}^\infty$ is bounded in $C^2[0,\pi]$. Therefore, by the Arzela-Ascoli theorem, we may assume that $v_n \to v$ in $C^1[0,\pi]$, $||v||_1 = 1$, and thus also $v_n \to v$ in $C^2[0,\pi]$ by equation (8). Consequently, by passing to the limit as $n \to \infty$ in (8) we obtain

$$\begin{cases} \ell(v)(x) = \lambda v(x) + \alpha(x)v^+(x) + \beta(x)v^-(x), x \in (0,\pi), \\ v \in BC_\lambda \ . \end{cases}$$

The proof of this lemma is complete.

As an immediate consequence of Lemmas 3.1, 3.2 and [2, Corollary 3.1], we obtain the following result.

Lemma 3. *The set of bifurcation points of problem* (1)-(3) *with respect to* $\mathbb{R}\times S_k^\nu$ *nonempty.*

Lemma 4. *If* $(\lambda, 0)$ *is a bifurcation point of* (1)-(3) *with respect to* $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, *then* $\lambda \in I_k$*; moreover,* $\lambda = \lambda_k^\nu$ *if* $k < N_0$, $\lambda = \lambda_{k+1}^\nu$ *if* $k > N_0$, *and either* $\lambda = \lambda_{N_0}^\nu$ *or* $\lambda = \lambda_{N_0+1}^\nu$ *if* $k = N_0$.

We define the positive numbers γ_k, $k \in \mathbb{N} \cup \{0\}$, as follows:

$$\gamma_k = \lambda_{k+1} - \lambda_k, \ k \in \mathbb{N}, \ \gamma_0 = \min\{\gamma_k \, : \, k \in \mathbb{N}\}.$$

It is known (see [4]) that $\lim\limits_{k\to\infty} \gamma_k = +\infty$.

Throughout what follows, for $c_1 \neq 0$ we shall assume that the following condition fulfilled:

$$M < \frac{1}{2} r_0 \gamma_0. \tag{9}$$

Then for any $k, m \in \mathbb{N}$ $(k \neq m)$, we have

$$J_k \cap J_m = \emptyset. \tag{10}$$

Hence it follows by Lemmas 3.3, 3.4 and [2, Theorem 3.5] that

Lemma 5. *For each* $k \in \mathbb{N}$ *the following relation hold:*

$$\lambda_k^\nu \in J_k.$$

Corollary 1. *If* $k' > k \geq 1$, *then*

$$\lambda_{k'}^{\nu'} > \lambda_k^\nu \ \ for \ \ each \ \ \nu', \nu \in \{+\,,\,-\}.$$

Now we introduce the approximate problem

$$\begin{cases} \ell(y) = \lambda r(x)y + \alpha(x)||y||_1^{\varepsilon}y^+ + \beta(x)||y||_1^{\varepsilon}y^- + \\ \qquad\qquad + g(x, y, y', \lambda),\ x \in (0, \pi), \\ (\lambda, y) \in BC_\lambda, \end{cases} \tag{11}$$

where $\varepsilon \in (0, 1]$. This type of problem has been considered in [1-3, 6, 11, 13].

For each $y \in E$ we define the function $\tilde{g}(y) \in C[0, \pi]$ as follows:

$$\tilde{g}(y)(x) = \alpha(x)y^+(x) + \beta(x)y^-(x),\ x \in [0, \pi].$$

Since $\alpha(x), \beta(x) \in C[0, \pi]$, the map $\tilde{g} : E \to C[0, \pi]$ is continuous and satisfies the condition

$$||\tilde{g}(y)||_\infty \leq M||y||_1. \tag{12}$$

Problem (11) can be rewritten in the following equivalent form:

$$\begin{cases} \ell(y) = \lambda r(x)y + \tilde{g}(||y||_1^{\varepsilon}y) + g(x, y, y', \lambda),\ x \in (0, \pi), \\ (\lambda, y) \in BC_\lambda. \end{cases} \tag{13}$$

By (12), for each fixed $\varepsilon \in (0, 1]$

$$||\tilde{g}(||y||_1^{\varepsilon}y)||_\infty = o(||y||_1) \text{ as } ||y||_1 \to 0.$$

Hence the assertion of [2, Theorem 2.2] holds for (13) (that is, for problem (11)): for each $k \in \mathbb{N}$ and each ν there exists an unbounded continuum $C_{k,\varepsilon}^{\nu}$ of solutions of problem (13) such that

$$(\lambda_k, 0) \in C_{k,\varepsilon}^{\nu} \subset (\mathbb{R} \times T_k^{\nu}) \cup \{(\lambda_k, 0)\}. \tag{14}$$

We define the positive numbers $\tilde{\gamma}_0$ and δ_0 as follows:

$$\tilde{\gamma}_0 = \lambda_{N_0+1} - \lambda_{N_0},\ \delta_0 = \frac{\gamma_0}{4} - \frac{M}{2r_0}.$$

Lemma 6. *There exits $\sigma_0 \in (0, 1)$ such that for any $\varepsilon \in (0, 1)$ the problem (11) has no solution (λ, w) satisfying the conditions $\delta_0 < \operatorname{dist}\{\lambda, J_k\} < 2\delta_0$, $k \in \{N_0, N_0+1\}$, $w \in S_{N_0}^{\nu}$ and $||w||_1 < \sigma_0$.*

Proof. To prove this statement, assume the contrary. Then for any $\sigma \in (0, 1)$ there exit $\varepsilon_\sigma \in (0, 1)$ such that problem (13) with $\varepsilon = \varepsilon_\sigma$ has a nontrivial solution $(\lambda_\sigma, v_\sigma)$ satisfying the conditions

$$\delta_0 < \operatorname{dist}\{\lambda_\sigma, J_k\} < 2\delta_0,\ k \in \{N_0, N_0 + 1\},\ v_\sigma \in S_{N_0}^{\nu} \text{ and } ||v_\sigma||_1 < \sigma.$$

Let $\{\sigma_n\}_{n=1}^{\infty} \subset (0, 1)$ be a sequence such that $\lim\limits_{n\to\infty} \sigma_n = 0$. Then for each $n \in \mathbb{N}$ problem (13) with $\varepsilon = \varepsilon_n$ ($\varepsilon_n = \varepsilon_{\sigma_n}$) has a solution $(\lambda_n, v_n) = (\lambda_{\sigma_n}, v_{\sigma_n})$ such that

$$2\delta_0 < \operatorname{dist}\{\lambda_n, J_k\} < 2\delta_0,\ k \in \{N_0, N_0 + 1\},\ v_n \in S_{N_0}^{\nu} \text{ and } ||v_n||_1 < \sigma_n.$$

Let $w_n(x) = \frac{v_n(x)}{||v_n||_1}$. Then by (13) we have

$$\begin{cases} \ell(w_n) = \lambda r(x) w_n + \tilde{g}(||v_n||_1^{\varepsilon_n} w_n) + \frac{g(x, v_n, v_n', \lambda_n)}{||v_n||_1}, \ x \in (0, \pi), \\ (\lambda_n, v_n) \in BC_\lambda. \end{cases} \tag{15}$$

Hence it follows from (15) that the sequence $\{(\lambda_n, w_n)\}_{n=1}^{\infty}$ is bounded in $\mathbb{R} \times C^2[0, \pi]$. Then there exists a subsequence $\{(\lambda_{n_s}, w_{n_s})\}_{s=1}^{\infty}$ converging to $(\tilde{\lambda}, \tilde{w})$ in $\mathbb{R} \times E$. Moreover, we may assume that $||v_{n_s}||_1^{\varepsilon_{n_s}} \to \tilde{\tau}$ as $s \to \infty$ for some $\tilde{\tau} \in [0, 1]$. By (15) some subsequence $\{(\lambda_{n_s}, w_{n_s})\}_{s=1}^{\infty}$ also converges to $(\tilde{\lambda}, \tilde{w})$ in $\mathbb{R} \times C^2[0, \pi]$. In addition, $\delta_0 < \mathrm{dist}\{\tilde{\lambda}, J_k\} < 2\delta_0$, $k \in \{N_0, N_0 + 1\}$, $||\tilde{w}|| = 1$, $\tilde{w} \in \overline{S_{N_0}^{\nu}}$ and

$$\begin{cases} \ell(\tilde{w}) = \tilde{\lambda} r(x) \tilde{w} + \tilde{\tau}\, \alpha(x) \tilde{w}^+ + \tilde{\tau}\, \beta(x) \tilde{w}^-, \ x \in (0, \pi), \\ (\tilde{\lambda}, \tilde{w}) \in BC_\lambda. \end{cases} \tag{16}$$

Since $\overline{S_{N_0}^{\nu}} = S_{N_0}^{\nu} \cup \partial S_{N_0}^{\nu}$ and $||\tilde{w}|| = 1$ it follows from the proof of Lemma 1 in [3, p. 379] that $\tilde{w} \in S_{N_0}^{\nu}$. Problem (16) is of the same form as (6) so Lemma 3.5 shows that $\tilde{\lambda} \in J_{N_0} \cup J_{N_0+1}$ in contradiction with the inequality $\delta_0 < \mathrm{dist}\{\tilde{\lambda}, J_k\} < 2\delta_0$, $k \in \{N_0, N_0 + 1\}$. The proof is complete.

Lemma 7. *For each ν the points $(\lambda_{N_0}^{\nu}, 0)$ and $(\lambda_{N_0+1}^{\nu}, 0)$ are bifurcation points of* (1)-(3) *with respect to the set $\mathbb{R} \times S_{N_0}^{\nu}$.*

Proof. Let $\sigma \in (0, \sigma_0)$ is an arbitrary fixed number. Since $C_{N_0, \varepsilon}^{\nu}$ is connected, it follows by (14) and Lemma 3.6 that for each $\varepsilon \in (0, 1)$ problem (11) has a solution $(\lambda_\varepsilon, y_\varepsilon)$ such that $\lambda_\varepsilon \in [\lambda_{N_0} - \frac{M}{r_0} - \delta_0, \lambda_{N_0} + \frac{M}{r_0} + \delta_0]$, $y_\varepsilon \in S_{N_0}^{\nu}$ and $||y_\varepsilon||_1 = \sigma$.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence such that $\lim\limits_{n \to \infty} \varepsilon_n = 0$. Then for each $n \in \mathbb{N}$ problem (11) with $\varepsilon = \varepsilon_n$ has a solution $(\lambda_{\varepsilon_n}, y_{\varepsilon_n})$ such that $\lambda_{\varepsilon_n} \in \Lambda_0 \equiv [\lambda_{N_0} - \frac{M}{r_0} - \delta_0, \lambda_{N_0} + \frac{M}{r_0} + \delta_0]$, $y_{\varepsilon_n} \in S_{N_0}^{\nu}$ and $||y_{\varepsilon_n}||_1 = \sigma$. Using the above argument we can show that there exists a subsequence $\{\lambda_{\varepsilon_{n_s}}, y_{\varepsilon_{n_s}}\}_{s=1}^{\infty}$ converging to $(\lambda_\sigma, y_\sigma)$ in $\mathbb{R} \times C^2[0, \pi]$. Note that $||y_{\varepsilon_{n_s}}||_1^{\varepsilon_{n_s}} \to 1$ as $s \to \infty$. In addition, $\lambda_\sigma \in \Lambda_0$, $||y_\sigma||_1 = \sigma$, $y_\sigma \in S_{N_0}^{\nu}$ and

$$\begin{cases} \ell(y_\sigma) = \lambda_\sigma y_\sigma + \alpha(x) y_\sigma^+ + \beta(x) y_\sigma^- + g(x, y_\sigma, y_\sigma', \lambda_\sigma), \ x \in (0, \pi), \\ (\lambda_\sigma, y_\sigma) \in BC_\lambda\,. \end{cases}$$

which implies that $(\lambda_\sigma, y_\sigma)$ solves (1)-(3).

Thus we have shown that for each ν and each σ, $0 < \sigma < \sigma_0$, problem (1)-(3) has a solution $(\lambda_\sigma, y_\sigma)$ such that $||y_\sigma||_1 = \sigma$, $\lambda_\sigma \in \Lambda_0$ and $y_\sigma \in S_{N_0}^{\nu}$. Hence it follows that interval $\Lambda_0 \times \{0\}$ contains at least one bifurcation point of problem (1)-(3) with respect to $\mathbb{R} \times S_{N_0}^{\nu}$. Therefore, by virtue of Lemmas 3.2, 3.4 and 3.5, $(\lambda_{N_0}, 0)$ is the unique bifurcation point in $J_{N_0} \times \{0\} \subset \Lambda_0 \times \{0\}$ of (1)-(3) with respect to $\mathbb{R} \times S_{N_0}^{\nu}$.

In a similar way, one can show that $(\lambda_{N_0+1}, 0)$ is the unique bifurcation point in $J_{N_0+1} \times \{0\}$ of (1)-(3) with respect to $\mathbb{R} \times S_{N_0}^{\nu}$. The proof is complete.

If $c_1 = 0$ let $T_k^{\nu} = S_k^{\nu}$, $k \in \mathbb{N}$, if $c_1 \neq 0$ let

$$T_k^{\nu} = \begin{cases} S_k^{\nu}, & \text{if } k \leq N_0, \\ S_{k-1}^{\nu}, & \text{if } k > N_0. \end{cases}$$

We denote by $\mathfrak{L}$ the closure in $\mathbb{R}\times E$ of the set of nontrivial solutions of problem (1)-(3) and by $\mathfrak{L}_k^\nu$ the closure in $\mathbb{R}\times E$ of the set of all solutions (λ, y) of (1)-(3) with $y \in T_k^\nu$.

The next result describes the structure and behaviour of global sets of solutions of problem (1)-(3) bifurcating from the line of trivial solutions.

Theorem 1. *For each $k \in \mathbb{N}$ and each ν there exists an unbounded continuum of solutions of problem* (1)-(3), *$\mathfrak{D}_k^\nu$ such that $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R}\times T_k^\nu)\cup\{(\lambda_k^\nu \times 0)\}$.*

Proof. For each $k \in \mathbb{N}$ and each ν let denote by $\tilde{\mathcal{D}}_k^\nu$ the union of all components $\tilde{\mathcal{D}}_{k,\lambda}^\nu$ of $\mathfrak{L}$ emanating from the bifurcation points $(\lambda, 0) \in I_k\times\{0\}$ of problem (1)-(3) with respect to $\mathbb{R}\times S_k^\nu$. Let $\mathcal{D}_k^\nu = \tilde{\mathcal{D}}_k^\nu \cup (I_k \times 0)$. Note that the set $\mathcal{D}_k^\nu$ is connected in $\mathbb{R}\times E$, but $\tilde{\mathcal{D}}_k^\nu$ my not be connected. Then it follows by [2, Theorem 3.4] that for each $k \in \mathbb{N}$ and each ν, the set $\mathcal{D}_k^\nu$ is unbounded in $\mathbb{R}\times E$ and lies in $(\mathbb{R}\times S_k^\nu)\cup(I_k\times 0)$.

Let $c_1 = 0$. Then by Lemma 3.2 we have $\mathfrak{L}_k^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_k^\nu, 0)\}$. We define $\mathfrak{D}_k^\nu = \mathcal{D}_k^\nu \cap \mathfrak{L}_k^\nu$. Then it follows by the above argument that $\mathfrak{D}_k^\nu$ is an unbounded component of $\mathfrak{L}_k^\nu$ and $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k^\nu\times 0)\}$.

Let $c_1 \neq 0$. Then it follows from Lemma 3.2 that

$$\mathfrak{L}_k^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_k^\nu, 0)\}, \quad \text{if } \ k < N_0,$$

$$\mathfrak{L}_k^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_{k+1}^\nu, 0)\}, \text{ if } k > N_0,$$

$$\mathfrak{L}_{N_0}^\nu \cap (\mathbb{R}\times\{0\}) \subset \{(\lambda_{N_0}^\nu, 0), (\lambda_{N_0+1}^\nu, 0)\}\,.$$

We define $\tilde{\mathfrak{D}}_k^\nu = \mathcal{D}_k^\nu \cap \mathfrak{L}_k^\nu$, $k \in \mathbb{N}$. By Lemma 3.7 the set $\tilde{\mathfrak{D}}_{N_0}^\nu$ has the representation $\tilde{\mathfrak{D}}_{N_0}^\nu = \tilde{\mathfrak{D}}_{N_0,1}^\nu \cup \tilde{\mathfrak{D}}_{N_0,2}^\nu$ such that $(\lambda_{N_0}^\nu, 0) \in \tilde{\mathfrak{D}}_{N_0,1}^\nu$ and $(\lambda_{N_0+1}^\nu, 0) \in \tilde{\mathfrak{D}}_{N_0,2}^\nu$.

Now we define the set $\mathfrak{D}_k^\nu$, $k \in \mathbb{N}$, as follows:

$$\mathfrak{D}_k^\nu = \begin{cases} \tilde{\mathfrak{D}}_k^\nu, & \text{if } k < N_0, \\ \tilde{\mathfrak{D}}_{N_0,1}^\nu, & \text{if } k = N_0, \\ \tilde{\mathfrak{D}}_{N_0,2}^\nu, & \text{if } k = N_0+1, \\ \tilde{\mathfrak{D}}_{k-1}^\nu, & \text{if } k > N_0+1. \end{cases}$$

It is then readily verified that $\mathfrak{D}_k^\nu$ for $k \neq N_0, N_0+1$, is an unbounded component of $\mathfrak{L}_k^\nu$ and $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R}\times T_k^\nu)\cup\{(\lambda_k^\nu, 0)\}$. Moreover, it follows by [8, Theorem 2.1], [11, Theorem 1] that the set $\mathfrak{D}_{N_0}^\nu$ contains $(\lambda_{N_0}^\nu, 0)$ and is either unbounded in $\mathbb{R}\times E$ or meets $(\lambda_{N_0+1}^\nu, 0)$ through $\mathbb{R}\times S_{N_0}^\nu$. (It also shows that a similar result holds for $\mathfrak{D}_{N_0+1}^\nu$.) Hence it follows that, if $\mathfrak{D}_{N_0}^\nu$ is bounded in $\mathbb{R}\times E$, then $\mathfrak{D}_{N_0+1}^\nu$ will also be bounded in $\mathbb{R}\times E$, which contradicts the unboundedness of the set $\tilde{\mathfrak{D}}_{N_0}^\nu = \mathcal{D}_{N_0}^\nu \cap \mathfrak{L}_{N_0}^\nu = \mathfrak{D}_{N_0}^\nu \cup \mathfrak{D}_{N_0+1}^\nu$. Therefore, $\mathfrak{D}_{N_0}^\nu$ and $\mathfrak{D}_{N_0+1}^\nu$ are both unbounded in $\mathbb{R}\times E$. The proof is complete.

If in the case $c_1 \neq 0$ not satisfied condition (9), then it follows from the relation $\lim\limits_{k\to\infty} \gamma_k = +\infty$ that there exists $\tilde{k}_0 \in \mathbb{N}$ such that $\gamma_k > \frac{2M}{r_0}$ for $k > \tilde{k}_0$. Now we define the number $k_0 \in \mathbb{N}$ as follows: $k_0 = \max\{N_0^-, N_0^+, \tilde{k}_0\}$. Then from Theorem 3.1 implies the folllowing result which describes the bifurcation structure of problem (1)-(3) for $k > k_0$.

Theorem 2. *If $c_1 \neq 0$, then for each $k > k_0$ and each ν there exists an unbounded continuum of solutions of problem* (1)-(3), $\mathfrak{D}_k^\nu$ *such that* $(\lambda_k^\nu, 0) \in \mathfrak{D}_k^\nu \subset (\mathbb{R} \times S_{k-1}^\nu) \cup \{(\lambda_k^\nu, 0)\}$.

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