Caspian Journal of Applied Mathematics, Ecology and Economics
V. 5, No 2, 2017, December
ISSN 1560-4055

(L_p, L_q)-boundedness of the Fractional Integral Operator with Rough Kernels on Heisenberg Groups

G.A. Dadashova

Abstract. Let Ω is an homogeneous of degree zero function on Heisenberg group $\mathbb{H}_n$, integrable to a power $s > 1$ on the unit sphere generated by the corresponding Heisenberg metric. We study $L_p(\mathbb{H}_n)$-boundedness of the maximal operator M_Ω with rough kernels Ω in Heisenberg groups and the $\big(L_p(\mathbb{H}_n), L_q(\mathbb{H}_n)\big)$-boundedness of the fractional maximal and integral operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$, $0 < \alpha < Q$ with rough kernels.

Key Words and Phrases: fractional maximal function, fractional integral, Heisenberg group.

2010 Mathematics Subject Classifications: Primary: 42B25, 42B35, 43A15.

---

1. Introduction

The Heisenberg group [3, 4, 7, 9] appears in quantum physics and many fields of mathematics, including harmonic analysis, the theory of several complex variables and geometry. In this paper, we establish the norm inequalities for the maximal operator on the Heisenberg group in Lebesgue spaces. We begin with some basic notation. The Heisenberg group $\mathbb{H}_n$ a non-commutative nilpotent Lie group with the product spaces $\mathbb{R}^{2n+1}$ that have the multiplication

$$xy = \Big(x' + y', x_{2n+1} + y_{2n+1} + 2\sum_{k=1}^{n} x_k y_{n+k} - x_{n+k} y_k\Big),$$

where $x = (x', x_{2n+1})$, and $y = (y', y_{2n+1})$. By the definition, the identity element on $\mathbb{H}_n$ is $0 \in \mathbb{R}^{2n+1}$, while the inverse element of $x = (x', t)$ is $x^{-1} = (-x', -t)$.

The corresponding Lie algebra is generated by the left-invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n,$$
$$X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, \dots, n,$$

$$X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

The only non-trivial commutator relations are

$$\big[X_j, X_{n+j}\big] = -4X_{2n+1}, \quad j = 1, \dots, n.$$

The non-isotropic dilation on $\mathbb{H}_n$ is defined as $\delta_t(x', x_{2n+1}) = (tx', t^2 x_{2n+1})$ for $t > 0$. The Haar measure dx on this group coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is easy to check that

$$d\big(\delta_t x\big) = r^Q dx.$$

In the above, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}_n$.

The norm of $x = (x', x_{2n+1}) \in \mathbb{H}_n$ is given by

$$|x|_h = (|x'|^4 + x_{2n+1}^2)^{1/4},$$

where $|x'|^2 = \sum\limits_{k=1}^{2n} x_k^2$. The norm satisfies the triangle inequality and leads to the left-invariant distance $d(x, y) = |xy^{-1}|_h$. With this norm we define the Heisenberg ball,

$$B(x, r) = \{y \in \mathbb{H}_n : |xy^{-1}| < r\},$$

where x is the center and r is the radius. The volume of $B(x, r)$ is $C_n r^{2n+2}$, where C_n is the volume of the unit ball $B_1 \equiv B(e, 1)$, i.e.,

$$C_n = \frac{2\pi^{n+\frac{1}{2}}\Gamma\big(\frac{1}{2}\big)}{(n+1)\Gamma(n)\Gamma\big(\frac{n+1}{2}\big)}.$$

Let $S_H = \{x \in \mathbb{H}_n : |x|_h = 1\}$ be the unit sphere in $\mathbb{H}_n$ equipped with the normalized Haar surface measure $d\sigma$ and Ω be δ_t-homogeneous of degree zero, i.e. $\Omega(\delta_t x) \equiv \Omega(x)$, $x \in \mathbb{H}_n$, $t > 0$. The fractional maximal function $M_{\Omega,\alpha} f$ and the fractional integral $I_{\Omega,\alpha} f$ by with rough kernels, $0 < \alpha < Q$ of a function $f \in L_1^{\mathrm{loc}}(\mathbb{H}_n)$ are defined by

$$M_{\Omega,\alpha} f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{Q}} \int_{B(x,t)} |\Omega(y^{-1}x)| \; |f(y)| dy,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y^{-1}x) \; f(y)}{\big|y^{-1}x\big|_h^{Q-\alpha}} dy.$$

If $\Omega \equiv 1$, then $M_\alpha \equiv M_{1,\alpha}$ and $I_\alpha \equiv I_{1,\alpha}$ are the fractional maximal operator and the fractional integral operator, respectively. If $\alpha = 0$, then $M_\Omega \equiv M_{\Omega,0}$ is the maximal operator with rough kernel. It is well known that the fractional maximal operator on Heisenberg groups play an important role in harmonic analysis (see [4, 8]).

The boundedness of classical operators of the real analysis, such as the maximal operator and singular integral operators etc, from one Lebesgue space to another one is well studied by now, and there are well known various applications of such results in partial differential equations. In this paper, we study the L_p-boundedness of the maximal operator with rough kernels in Heisenberg groups, including also the case of weak boundedness. Also we obtain $\big(L_p(\mathbb{H}_n), L_q(\mathbb{H}_n)\big)$-boundedness of the fractional maximal and integral operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$, $0<\alpha<Q$ with rough kernels.

Throughout the paper, for a measurable set E, $|E|$ denotes the normalized Haar measure of E, i.e., $|B_1| = \int_{B_1} dx = 1$. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. For a number p, p' denotes the conjugate exponent of p. d B are equivalent.

2. Boundedness of the fractional integral operators in the spaces $L_p(\mathbb{H}_n)$

In this section we prove the $L_p(\mathbb{H}_n)$-boundedness of the operator M_Ω and the $\big(L_p(\mathbb{H}_n), L_q(\mathbb{H}_n)\big)$-boundedness of the operators $I_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$.

Theorem 1. *Let $\Omega \in L_s(S_H)$, $1 < s \le \infty$, be δ_t-homogeneous of degree zero. Then the operator M_Ω is bounded in the space $L_p(\mathbb{H}_n)$, $p > s'$.*

Proof.

In the case $s=\infty$ the statement of Theorem 1 is known and may be found in [2] and [8]. So we assume that $1<s<\infty$.

Note that

$$\begin{aligned} \|\Omega(\cdot^{-1}x)\|_{L_s(B(x,t))} &= \Big(\int_{B(0,t)} |\Omega(y)|^s dy\Big)^{1/s} \\ &= \Big(\int_0^t r^{Q-1}dr \int_{S_H} |\Omega(\omega)|^s d\sigma(\omega)\Big)^{1/s} \\ &= c_0\ \|\Omega\|_{L_s(S_H)}\ |B(x,t)|^{1/s}, \end{aligned} \tag{1}$$

where $c_0 = \big(Qv_H\big)^{-1/s}$ and $v_H = |B(0,1)|$.

The case $p=\infty$ is easy. Indeed, making use of (1), we get

$$\begin{aligned} \|M_\Omega f\|_{L_\infty(\mathbb{H}_n)} &\le \|f\|_{L_\infty(\mathbb{H}_n)} \sup_{t>0} |B(x,t)|^{-1+\frac{1}{s'}} \|\Omega(\cdot^{-1}x)\|_{L_s(B(x,t))} \\ &\le c_0\ \|\Omega\|_{L_s(S_H)}\ \|f\|_{L_\infty(\mathbb{H}_n)}. \end{aligned}$$

So we assume that $s'<p<\infty$. Applying Hölder's inequality, we get

$$M_\Omega f(x) \le \sup_{t>0} |B(x,t)|^{-1} \|\Omega(\cdot^{-1}x)\|_{L_s(B(x,t))}\ \|f\|_{L_{s'}(B(x,t))}. \tag{2}$$

Then from (2) and (1) we have

$$M_\Omega f(x) \le c_0\ \|\Omega\|_{L_s(S_H)} \sup_{t>0} |B(x,t)|^{-1+1/s}\ \|f\|_{L_{s'}(B(x,t))}$$

$$= c_0 \, \|\Omega\|_{L_s(S_H)} \Big(\sup_{t>0} |B(x,t)|^{-1} \| |f|^{s'} \|_{L_1(B(x,t))} \Big)^{1/s'}$$

$$= c_0 \, \|\Omega\|_{L_s(S_H)} \Big(M(|f|^{s'})(x) \Big)^{1/s'}. \tag{3}$$

Therefore, from (3) for $1 \le s' < p < \infty$ we get

$$\|M_\Omega f\|_{L_p(\mathbb{H}_n)} \le c_0 \, \|\Omega\|_{L_s(S_H)} \big\| \big(M(|f|^{s'})(\cdot) \big)^{1/s'} \big\|_{L_p(\mathbb{H}_n)}$$
$$= c_0 \, \|\Omega\|_{L_s(S_H)} \big\| M\big(|f|^{s'}\big) \big\|^{1/s'}_{L_{p/s'}(\mathbb{H}_n)} \lesssim \| |f|^{s'} \|^{1/s'}_{L_{p/s'}(\mathbb{H}_n)} = \|f\|_{L_p(\mathbb{H}_n)}.$$

We prove the boundedness of the fractional maximal and integral operators $M_{\Omega,\alpha}$, $I_{\Omega,\alpha}$ with rough kernel from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/Q$, and from the space $L_1(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/Q$.

Theorem 2. *Suppose that $0 < \alpha < Q$ and the function $\Omega \in L_{\frac{Q}{Q-\alpha}}(S_H)$ is δ_t-homogeneous of degree zero. Let $1 \le p < \frac{Q}{\alpha}$ and $1/p - 1/q = \alpha/Q$. Then the fractional integration operator $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ for $p > 1$ and from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ for $p = 1$.*

Proof. We denote

$$K(x) := \frac{\Omega(x)}{|x|_h^{Q-\alpha}}$$

for brevity, and may assume that $K(x) \ge 0$. We have

$$\Big| \{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > \lambda\} \Big| \le \Big| \{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > C_{Q,\alpha}^{-1} \lambda\} \Big| \le I_1 + I_2,$$

where

$$I_1 := \Big| \Big\{ x \in \mathbb{H}_n : |K^1_\mu * f(x)| > \frac{\lambda}{2} \Big\} \Big|, \quad I_2 := \Big| \Big\{ x \in \mathbb{H}_n : |K^2_\mu * f(x)| > \frac{\lambda}{2} \Big\} \Big|,$$

$$K^1_\mu(x) = (K(x) - \mu)\chi_{E(\mu)}(x) \ \text{ and } \ K^2_\mu(x) = K(x) - K^1_\mu(x),$$

$\mu > 0$ and $E(\mu) = \{x \in \mathbb{H}_n : |K(x)| > \mu\}$. Note that

$$|E(\mu)| \le B \mu^{\frac{Q}{Q-\alpha}}. \tag{4}$$

where $B = \frac{1}{\alpha} \|\Omega\|^{\frac{Q}{Q-\alpha}}_{L_{\frac{Q}{Q-\alpha}}(S_H)}$ as seen from the following estimation:

$$|E(\mu)| \le \frac{1}{\mu} \int_{E(\mu)} \frac{|\Omega(x)|}{|x|_h^{Q-\alpha}} dx$$
$$= \frac{1}{\mu} \int_{S_H} \Omega(x') d\sigma(x') \int_0^{\left(\frac{|\Omega(x')|}{\mu}\right)^{\frac{1}{Q-\alpha}}} r^{\alpha-1} dr = B \mu^{\frac{Q}{Q-\alpha}}.$$

By means of (4) we can prove the estimate

$$\|K_\mu^2\|_{L_{p'}(\mathbb{H}_n)} \le \Big(\frac{Q-\alpha}{Q}Bq\Big)^{\frac{1}{p'}} \mu^{\frac{Q}{(Q-\alpha)\,q}}, \quad 1 \le p < \frac{Q}{\alpha}.$$

For $p = 1$ it easily follows from (4), and for $p > 1$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |K_\mu^2(x)|^{p'} dx &= p' \int_0^\mu t^{p'-1} |E(t)| dt \\ &\le p' B \int_0^\mu t^{p'-1-\frac{Q}{Q-\alpha}} dt \\ &= \frac{Q-\alpha}{Q} Bq \mu^{\frac{Q}{Q-\alpha}\frac{p'}{q}}.\end{aligned}$$

Then by the Young inequality we obtain

$$\|K_\mu^2 * f\|_{L_\infty(\mathbb{H}_n)} \le \|K_\mu^2\|_{L_{p'}} \|f\|_{L_p(\mathbb{H}_n)} \le \Big(\frac{Q-\alpha}{Q}Bq\Big)^{\frac{1}{p'}} \mu^{\frac{Q}{(Q-\alpha)\,q}} \ \|f\|_{L_p(\mathbb{H}_n)}.$$

Now for a $\lambda > 0$, we choose μ such that

$$\Big(\frac{Q-\alpha}{Q}Bq\Big)^{\frac{1}{p'}} \mu^{\frac{Q}{(Q-\alpha)\,q}} \ \|f\|_{L_p(\mathbb{H}_n)} = \frac{\lambda}{2},$$

then

$$\Big|\Big\{x \in \mathbb{H}_n : |K_\mu^2 * f(x)| > \frac{\lambda}{2}\Big\}\Big| = 0.$$

Thus

$$\begin{aligned}\Big|\{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > \lambda\}\Big| &\le \Big|\Big\{x \in \mathbb{H}_n : |K_\mu^1 * f(x)| > \frac{\lambda}{2}\Big\}\Big| \\ &\le \Big(\frac{2}{\lambda} \ \|K_\mu^1 * f\|_{L_p(\mathbb{H}_n)}\Big)^p. \end{aligned} \tag{5}$$

The following estimations take (4) into account:

$$\begin{aligned}\int_{\mathbb{H}_n} |K_\mu^1(x)| dx &= \int_{E(\mu)} \big(|K(x)| - \mu\big) dx \\ &\le \int_0^\infty |E(t+\mu)| dt \\ &\le B \int_\mu^\infty t^{-\frac{Q}{Q-\alpha}} dt \\ &= \frac{\alpha B}{Q-\alpha} \ \mu^{-\frac{\alpha}{Q-\alpha}}.\end{aligned} \tag{6}$$

For all $f \in L_\infty(\mathbb{H}_n)$ and $x \in \mathbb{H}_n$, from (6) it follows that

$$|K^1_\mu * f(x)| \le \|f\|_{L_\infty(\mathbb{H}_n)} \int_{\mathbb{H}_n} |K^1_\mu(x)|dx \le \frac{\alpha B}{Q-\alpha}\, \mu^{-\frac{\alpha}{Q-\alpha}}\, \|f\|_{L_\infty(\mathbb{H}_n)}. \tag{7}$$

For all $f \in L_1(\mathbb{H}_n)$, from (6) follows

$$\begin{aligned} \|K^1_\mu * f\|_{L_1(\mathbb{H}_n)} &\le \int_{\mathbb{H}_n}\int_{\mathbb{H}_n} |K^1_\mu(x-y)||f(y)|dxdy \\ &\le \frac{\alpha B}{Q-\alpha}\, \mu^{-\frac{\alpha}{Q-\alpha}}\, \|f\|_{L_1(\mathbb{H}_n)}. \end{aligned} \tag{8}$$

Thus from (7) and (8) follows that the operator $T_1 : f \to K^1_\mu * f$ is of (∞,∞) and $(1,1)$-type. Then by the Riesz-Thorin theorem the operator T_1 is also of (p,p)-type, $1 < p < \infty$, and

$$\|T_1 f\|_{L_p(\mathbb{H}_n)} \le \frac{\alpha B}{Q-\alpha}\, \mu^{-\frac{\alpha}{Q-\alpha}}\, \|f\|_{L_p(\mathbb{H}_n)}. \tag{9}$$

From (5) and (9) we get

$$\begin{aligned} \Big|\{x \in \mathbb{H}_n : I_{\Omega,\alpha} f(x) > \lambda\}\Big| &\le \Big(\frac{2}{\lambda}\, \|K^1_\mu * f\|_{L_p(\mathbb{H}_n)}\Big)^p \\ &\le C\Big(\frac{1}{\lambda}\, \|f\|_{L_p(\mathbb{H}_n)}\Big)^q, \end{aligned} \tag{10}$$

where C is independent of λ and f.

To finish the proof, i.e. prove that the operator $I_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ for $1 < p < \frac{Q}{\alpha}$ and $1/p - 1/q = \alpha/Q$, observe that the inequality (10) tells us that $I_{\Omega,\alpha}$ is bounded from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ with $1 - 1/q = \alpha/Q$. We choose any p_0 such that $p < p_0 < \frac{Q}{\alpha}$, and put $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{Q}$. By (10) the operator $I_{\Omega,\alpha}$ is of weak (p_0, q_0)-type. Since it is also of weak $(1, q)$-type by the Marcinkiewicz interpolation theorem, we conclude that $I_{\Omega,\alpha}$ is of $\big(L_p, L_q\big)$-type.

Corollary 1. *Under the assumptions of Theorem 2, the fractional maximal operator $M_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ for $p > 1$ and from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ for $p = 1$.*

Proof. It suffices to refer to the known fact that

$$M_{\Omega,\alpha} f(x) \le C_{Q,\alpha} I_{\Omega,\alpha} f(x), \qquad C_{Q,\alpha} = |B(0,1)|^{\frac{Q-\alpha}{Q}},$$

References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math., **42**, 1975 765-778.

[2] R.R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homognes.* (French) Étude de certaines intégrales singulières. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.

[3] G.B. Folland, *Harmonic Analysis in Phase Space*, vol. 122 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, USA, 1989.

[4] G.B. Folland, E.M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes, 28, Princeton Univ. Press, Princeton, 1982.

[5] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian)

[6] V.S. Guliyev, *Integral operators, function spaces and questions of approximation on Heisenberg groups*, Elm, Baku, 1996. (in Russian)

[7] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, vol. 159 of Progress in Mathematics, Birkhauser, Boston, Mass, USA, 1998.

[8] E.M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[9] E. M. Stein, Harmonic analysis: Real-variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton, 1993.

Gulgayit A. Dadashova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: gdova@mail.ru

Received 14 August
Accepted 27 September