Caspian Journal of Applied Mathematics, Ecology and Economics
V. 5, No 2, 2017, December
ISSN 1560-4055

Global Bifurcation from Zero and Infinity in Nonlinear Beam Equation with Indefinite Weight

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Abstract. We consider a nonlinear eigenvalue problem for the beam equation with an indefinite weight function. We investigate the bifurcation from zero and infinity for this problem and prove the existence of unbounded continua bifurcating from the principal eigenvalues of the corresponding linear problem contained in the classes of positive and negative functions.

Key Words and Phrases: nonlinear eigenvalue problem, bifurcation point, principal eigenvalues, global continua, indefinite weight.
2010 Mathematics Subject Classifications: 34C10, 34C23, 47J10, 47J15

1. Introduction

We consider the following fourth order boundary value problem

$$(\ell u)(t) \equiv (p(t)u''(t))''(t) - (q(t)u'(t))' = \lambda g(t)f(u(t)),\ t \in (0,1), \tag{1}$$

$$\begin{array}{l} u'(0)\cos\alpha - (pu'')(0)\sin\alpha = 0, \\ u(0)\cos\beta + Tu(0)\sin\beta = 0, \\ u'(1)\cos\gamma + (pu'')(1)\sin\gamma = 0, \\ u(1)\cos\delta - Tu(1)\sin\delta = 0, \end{array} \tag{2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (pu'')' - qu'$, $p \in C^2[0,1]$ with $p(t) > 0$, $t \in [0,1]$, $q \in C^1[0,1]$ with $q(t) \geq 0$, $t \in [0,1]$, $g \in C[0,l]$ is a sign-changing weight function (i.e. meas$\{t \in (0,1) : \sigma u(t) > 0\} > 0$ for each $\sigma \in \{+,-\}$) and $\alpha, \beta, \gamma, \delta \in [0, \frac{\pi}{2}]$ with except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$. The nonlinear term $f \in C(\mathbb{R};\mathbb{R})$ and satisfies the conditions: $tf(t) > 0$ for $t \in \mathbb{R}\backslash\{0\}$ and there exist $f_0, f_\infty \in (0,+\infty)$ such that

$$f_0 = \lim_{|t|\to 0} \frac{f(t)}{t},\ \ f_\infty = \lim_{|t|\to\infty} \frac{f(t)}{t}\ . \tag{3}$$

It is well known that fourth-order problems arise in many applications (see [8, 24] and the references therein); problem (1)-(2) in particular, is often used to describe the deformation of an elastic beam, which is subject to axial forces (see [8]). Problems with sign-changing weight arise from population modeling. In this model, weight function g changes sign corresponding to the fact that the intrinsic population growth rate is positive at same points and is negative at others, for details, see [10, 15].

The purpose of this work is to study the global bifurcation of solutions of problem (1)-(2) in the classes of positive and negative functions, emanating from the zero and infinity.

It should be noted that the nonlinear problem (1)-(2) is closely related to the following linear eigenvalue problem

$$\begin{array}{l} (p(t)u''(t))''(t) - (q(t)u'(t))' = \lambda g(t)u(t), \ t \in (0,1), \\ u \in B.C., \end{array} \tag{4}$$

where by $B.C.$ we denote the set of boundary conditions (2). The nonlinear problem (1)-(2) and linear problem (4) in the case $p \equiv 1$, $q \equiv 0$ and $\alpha = \gamma = \frac{\pi}{2}$, $\beta = \delta = 0$ was previously considered in [23] the results of which contain gaps.

The problems (4) and (1)-(2) in the case when the first condition in (3) is satisfied are studied in [18], where, in particular, it was shown that there exist two positive and negative principal eigenvalues, λ_1 and λ_{-1}, respectively, of the linear problem (4) and the corresponding eigenfunctions have no zeros in $(0,1)$; moreover, also proved that for each $k \in \{1, -1\}$ and each $\nu \in \{+, -\}$ there exists a continuum (connected closed set) $\mathfrak{L}_k^\nu$ of solutions of problem (1)-(2) bifurcating from the point $(\lambda_k, 0)$, which is unbounded in $\mathbb{R} \times C^3[0,1]$, and $\nu \operatorname{sgn} y(x) = 1$, $x \in (0,1)$ for each $(\lambda, y) \in \mathfrak{L}_k^\nu$. Note that, similar problems have been considered before in, for example, [10] and [30], but the results of these works are not true (see [4]).

In Section 2, a family of sets to exploit oscillatory properties of eigenfunctions of problem (4) and their derivatives is introduced. The existence of global continua of solutions of the problem (1)-(2) bifurcating simultaneously from the zero and infinity, and contained in these sets is proved in Section 3. Here we give the application of global bifurcation technique to the study of positive or negative solutions for the some nonlinear boundary value problems.

2. Preliminary

In [23] the authors note that there are few papers discussing the existence and multiplicity of positive solutions to (4), the main reason of which is that the spectrum of the linear eigenvalue problem is not clear. They showed that the problem (4) has exactly two principal eigenvalues, one positive and one negative, and the corresponding eigenfunctions do not change its sign on $(0,1)$. But it should be noted that in the proof of this fact, the authors did not give a correct reference to the work [16]. However until recently there no results on the multiplicities of the first m ($m > 2$) (for the definition of m, see [19, 21]) eigenvalues and on the oscillatory properties for the corresponding eigenfunctions of the following problem

$$\begin{aligned} &(p(t)u''(t))''(t) - (q(t)u'(t))' + h(t)u(t) = \mu u(t), \ t \in (0,1), \\ &u \in B.C., \end{aligned} \tag{5}$$

where $h \in C([0,1];\mathbb{R})$. In [19, 21] it was shown that, in the case of $h(t)$ not identically vanishing on any subinterval of $[0,1]$, the eigenvalues of problem (5) are real, and simple, except, possibly, the first m eigenvalues, and the corresponding eigenfunctions with numbers larger than m have the Sturm oscillation properties, i.e. the eigenfunction has only simple nodal zeros and the number of zeros of the eigenfunction is equal to the serial number of the corresponding eigenvalue increased by 1. But, in [23], the authors in proving Theorem 2.1 recall the work [16] and claim that the eigenfunction, corresponding to the first eigenvalue of the problem (5), has no zeros in the interval $(0,1)$. Unfortunately in [16] oscillatory properties of eigenfunctions of the problem (4) were not studied. Recently, in [3] (see also [5, 6]) it was established that all eigenvalues of the problem (5) are simple and the corresponding eigenfunctions have the Sturm oscillation properties.

For the linear eigenvalue problem (4) we have the following result.

Theorem 1. [18, Theorem 2.1] . *The spectral problem* (4) *has two sequences of real eigenvalues*

$$0 < \lambda_1^+ \leq \lambda_2^+ \leq ... \leq \lambda_k^+ \mapsto +\infty,$$

and

$$0 > \lambda_1^- \geq \lambda_2^- \geq ... \geq \lambda_k^- \mapsto -\infty$$

and no other eigenvalues. Moreover, λ_1^+ *and* λ_1^- *are simple principal eigenvalues, i.e. the corresponding eigenfunctions* $u_1^+(t)$ *and* $u_1^-(t)$ *have no zeros in the interval* $(0,1)$.

Similar problems have been considered in [9, 13, 14, 17, 22].

Let E be the Banach space of all continuously three times differentiable functions on $[0,1]$ which satisfy the conditions $B.C.$ and is equipped with its usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max\limits_{t\in[0,1]} |u(t)|$.

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(t) \neq 0, Tu(t) \neq 0, \ t \in [0,1], \ i = 0, \ 1, \ 2\,\}$$

and

$S_2 = \{u \in E :$ there exists $i_0 \in \{0,\ 1,\ 2\}$ and $t_0 \in (0,1)$ such that $u^{(i_0)}(t_0) = 0$, or $Tu(t_0) = 0$ and if $u(t_0)u''(t_0) = 0$, then $u'(t)Tu(t) < 0$ in a neighborhood of t_0, and if $u'(t_0)Tu(t_0) = 0$, then $u(t)u''(t) < 0$ in a neighborhood of $t_0\}$.

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ (see [1-3, 5, 6, 20]) of the Prüfer-type transformation

$$\begin{cases} u(x) = \rho(x) \sin\psi(x) \cos\theta(x), \\ u'(x) = \rho(x) \cos\psi(x) \sin\varphi(x), \\ (pu'')(x) = \rho(x) \cos\psi(x) \cos\varphi(x), \\ Tu(x) = \rho(x) \sin\psi(x) \sin\theta(x), \end{cases} \tag{6}$$

does not vanish on $(0, 1)$.

For each $u \in S$ we define $\rho(u,t)$, $\theta(u,t)$, $\varphi(u,t)$, $w(u,t)$ to be the continuous functions on $[0,1]$ satisfying

$$\rho(u,t) = u^2(t) + u'^2(t) + (p(t)u''(t))^2 + (Tu(t))^2,$$

$$\theta(u,t) = \operatorname{arctg}\frac{Tu(t)}{u(t)},\ \theta(u,0) = \beta - \pi/2\,,$$

$$\varphi(u,t) = \operatorname{arctg}\frac{u'(t)}{(pu'')(t)},\ \varphi(u,0) = \alpha\,,$$

$$w(u,t) = \operatorname{ctg}\psi(u,t) = \frac{u'(t)\cos\theta(u,t)}{u(t)\sin\varphi(u,t)},\ w(u,0) = \frac{u'(0)\sin\beta}{u(0)\sin\alpha}\,,$$

and $\psi(u,t) \in (0, \pi/2)$, $t \in (0,1)$, in the cases of $u(0)u'(0) > 0$; $u(0) = 0$; $u'(0) = 0$ and $u(0)u''(0) > 0$, $\psi(u,t) \in (\pi/2, \pi)$, $t \in (0,1)$, in the cases $u(0)u'(0) < 0$; $u'(0) = 0$ and $u(0)u''(0) < 0$; $u'(0) = u''(0) = 0$, $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is apparent that $\rho,\ \theta,\ \varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 3.1. By (7) for each $u \in S$ the function $w(u,t)$ can be determined by one of the following relations

$$\text{a) } w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{(pu'')(x)\cos\theta(u,x)}{u(x)\cos\varphi(u,x)},\ w(u,0) = \frac{(pu'')(0)\sin\beta}{u(0)\cos\alpha}\,,$$

$$\text{b) } w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{(pu'')(x)\sin\theta(u,x)}{Tu(x)\cos\varphi(u,x)},\ w(u,0) = -\frac{(pu'')(0)\cos\beta}{Tu(0)\cos\alpha}\,,$$

$$\text{c) } w(u,x) = \operatorname{ctg}\psi(u,x) = \frac{u'(x)\sin\theta(u,x)}{Tu(x)\sin\varphi(u,x)},\ w(u,0) = -\frac{u'(0)\cos\beta}{Tu(0)\sin\alpha}\,.$$

For each $\nu \in \{+, -\}$ let S_1^ν denotes the subset of such $u \in S$ that:

1) $\theta(u,1) = \pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(u,1) = 0$;

2) $\varphi(u,1) = 2\pi - \gamma$ or $\varphi(u,1) = \pi - \gamma$ in the case $\psi(u,0) \in [0, \pi/2)$; $\varphi(u,1) = \pi - \gamma$ in the case $\psi(y,0) \in [\pi/2, \pi)$, where $\gamma = 0$ in the case $\psi(y,l) = \pi/2$;

3) for fixed u, as t increases from 0 to 1, the function $\theta(u,t)$ ($\varphi(u,t)$) strictly increasingly takes values of $m\pi/2$, $m \in \{-1, 0, 1\}$ ($s\pi$, $s \in \{0, 1, 2\}$); as t decreases from 1 to 0, the function $\theta(u,t)$ ($\varphi(u,t)$), strictly decreasing takes values of $m\pi/2$, $m \in \{-1, 0, 1\}$ ($s\pi$, $s \in \{0, 1, 2\}$);

4) the function $\nu u(t)$ is positive in a neighborhood of $t = 0$.

By [2; Theorem 4.4], [6; Theorem 1.1], [7; Lemma 2.2, Theorems 5.1, 5.2, 6.1, 6.3-6.5] and Theorem 2.1 we have $u_1^+, u_1^- \in S_1$, i.e the sets S_1^+ and S_1^- are nonempty. It immediately follows from the definition of these sets that they are disjoint and open in E. Moreover, by [2; Lemma 2.2] if $u(t) \in \partial S_1^\nu \cap C^4[0,1]$, $\nu \in \{+, -\}$, then $u(t)$ has at least one zero of multiplicity 4 in $(0,1)$.

Let $u_{1,+}^+$ $(u_{1,+}^-)$ denote the unique eigenfunction of (4) corresponding to the eigenvalue λ_k^+ (λ_k^-) such that $u_{1,+}^+ \in S_1^+$ $(u_{1,+}^- \in S_1^+)$ and $||u_{1,+}^+||_3 = 1$ $(||u_{1,+}^-||_3 = 1)$.

Lemma 1. (see [1, 2]) *If* $(\lambda, u) \in \mathbb{R} \times E$ *is a solution of* (1)-(2) *and* $u \in \partial S_1^\nu$, $\nu \in \{+, -\}$, *then* $u \equiv 0$.

3. Global bifurcation from zero and infinity for the problem (1)-(2)

It should be noted that in order to prove the existence of at least one solution of the problem (1)-(2) in the class of positive functions, in [23], the authors used global bifurcation results (see [23, p. 6598]) which also contains gaps. This result is similar to that for the nonlinear Sturm-Liouville problems which has been obtained by Rabinowitz [26]. In the nonlinear Sturm-Liouville problem considered in [26] nodal properties are preserved on the continuous branch of nontrivial solutions emanating from bifurcation points and this prevents the first alternative in part (ii) of [29; Lemma 2.6] occurring. But for the nonlinear fourth order eigenvalue problem nodal properties need not be preserved, so we must considered this alternative. Therefore, in the study of nonlinear fourth-order eigenvalue problem there is a need to study the following questions: to construct the classes of functions that preserve the oscillation properties of eigenfunctions of the linear problem (4) and their derivatives, such that if the solution of the nonlinear problem is contained on the boundary of this set, then this must be identically zero (if means that continuous branch of solutions can not go from the boundary of this set). This question was solved in a recent paper [3] (see also [2]), in which global bifurcation from zero of solutions of the nonlinear eigenvalue problems for ordinary differential equations of fourth order was studied.

Let $\mathfrak{L}$ denotes the closure of the set of nontrivial solutions of (1)-(2).

Theorem 2. *For each* $k \in \{-1, 1\}$ *and each* $\nu \in \{-, +\}$ *there exists a continuum* $\mathfrak{L}_k^\nu$ *of solutions of problem* (1)-(2) *in* $(\mathbb{R} \times S_1) \cup \left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)\right\} \cup \left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)\right\}$ *which meets* $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)$ *and* $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$ *in* $R^{\mathrm{sgn}k} \times E$*, where* $R^{\mathrm{sgn}k} = \{\chi \in \mathbb{R} : 0 < \chi \mathrm{sgn} k \leq +\infty\}$.

Proof. By virtue of (3) there exists the functions $\tau \in C(\mathbb{R}, \mathbb{R})$ and $\varepsilon \in C(\mathbb{R}, \mathbb{R})$ such that

$$f(u) = f_0 u + \tau(u), \;\; f(u) = f_\infty u + \varepsilon(u), \tag{7}$$

and

$$\lim_{|u|\to 0} \frac{\tau(u)}{u} = 0, \quad \lim_{|u|\to +\infty} \frac{\varepsilon(u)}{u} = 0. \tag{8}$$

It follows from (7) that the problem (1)-(2) can be rewritten in the following form

$$\begin{array}{l} (\ell u)(t) = \lambda f_0 g(t) u(t) + \lambda g(t) \tau(u(t)),\ t \in (0,1), \\ u \in B.C. \end{array} \tag{9}$$

or

$$\begin{array}{l} (\ell u)(t) = \lambda f_\infty g(t) u(t) + \lambda g(t) \varepsilon(u(t)),\ t \in (0,1), \\ u \in B.C.. \end{array} \tag{10}$$

Since $\lambda = 0$ is not eigenvalue of the linear problem (5) for $h \equiv 0$ it follows that the problems (9) and (10) are equivalent to the following integral equations

$$u(t) = \lambda f_0 \int_0^1 K(t,s) g(s) u(s) ds + \lambda \int_0^1 K(t,s)\, g(s)\, \tau(u(s))\, ds, \tag{11}$$

$$u(t) = \lambda f_\infty \int_0^1 K(t,s) g(s) u(s) ds + \lambda \int_0^1 K(t,s)\, g(s)\, \varepsilon(u(s))\, ds, \tag{12}$$

respectively, where $K(t,s)$ is a Green's function of differential expression $\ell(u)$ with respect to the $B.C.$.

Define $\mathcal{L} : E \to E$ by

$$(\mathcal{L}u)(t) = \int_0^1 K(t,s)\, g(s)\, u(s)\, ds$$

$\mathcal{F} : \mathbb{R} \times E \to E$ by

$$(\mathcal{F}(u))(t) = \int_0^1 K(t,s)\, g(s)\, \tau(u(s))\, ds.$$

and $\mathcal{G} : \mathbb{R} \times E \to E$ by

$$(\mathcal{G}(u))(t) = \int_0^1 K(t,s)\, g(s)\, \varepsilon(u(s))\, ds.$$

It is easily seen that the operator $\mathcal{L}$ is compact in E and the operators $\mathcal{F} : \mathbb{R} \times E \to E$ and $\mathcal{G} : \mathbb{R} \times E \to E$ are completely continuous. Thus problems (11) (or (9)) and (12) (or (10)) can be rewritten in the following equivalent forms

$$u = \lambda f_0 \mathcal{L} u + \lambda \mathcal{F}(u) \tag{13}$$

and

$$u = \lambda f_\infty \mathcal{L} u + \lambda \mathcal{G}(u). \tag{14}$$

By (3) we have

$$\mathcal{F}(u) = o(||u||_3) \text{ as } ||u||_3 \to 0, \tag{15}$$

and

$$\mathcal{G}(u) = o(||u||_3) \text{ as } ||u||_3 \to +\infty\,. \tag{16}$$

By virtue of (15) and (16) the linearization of (13) at $u = 0$ and of (14) at $u = \infty$ are spectral problems

$$u = \lambda f_0 \mathcal{L} u \tag{17}$$

and

$$u = \lambda f_\infty \mathcal{L} u, \tag{18}$$

respectively. Obviously, the problem (17) and (18) are equivalent to the spectral problems

$$\begin{aligned} &\ell u(t) = \lambda f_0 g(t) u(t),\ t \in (0,1), \\ &u \in B.C. \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\ell u(t) = \lambda f_\infty g(t) u(t),\ t \in (0,1), \\ &u \in B.C.\,, \end{aligned} \tag{20}$$

respectively.

The principal eigenvalues $\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}$, $k \in \{-1,\ 1\}$, of problem (19) are the characteristic values of problem (17) and are simple. Hence all the conditions of Theorem 1.3 from [26] are satisfied and there exists a continua $\mathfrak{L}_{\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}} \equiv \mathfrak{L}_k$, $k \in \{-1,\ 1\}$, of the set of solutions of problem (13), as in Theorem 1.3 in [26]. By virtue of [3, Theorem 1.1] (see also [12, Theorem 2]) continua $\mathfrak{L}_k$, $k \in \{-1,\ 1\}$, decomposes into two subcontinua $\mathfrak{L}_k^-$ and $\mathfrak{L}_k^+$ with meets $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)$, are contained in $(\mathbb{R} \times S_1^-) \cup \left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)\right\}$ and $(\mathbb{R} \times S_1^+) \cup \left\{\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)\right\}$, respectively, and both are unbounded in $\mathbb{R}^{\mathrm{sgn}k} \times E$.

On the other hand, since the principal eigenvalues $\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}$, $k \in \{-1,\ 1\}$, of problem (20) are the characteristic values of problem (18) and are simple, by the discussion above and [25; Theorem 2.4] (see also [27, 28]) for each $k \in \{-1, 1\}$ there exists an unbounded component $\mathcal{D}_{\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}} \equiv \mathcal{D}_k \subset \mathbb{R}^{\mathrm{sgn}k} \times E$ of $\mathfrak{L}$ which contains $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$. In addition, if $\Lambda \subset \mathbb{R}^{\mathrm{sgn}k}$ is an interval such that $\Lambda \cap \sigma(L,g) = \frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}$ ($\sigma(L,g)$ is a set of eigenvalues of problem (4)) and $\mathcal{M}$ is a neighborhood of $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{r f_\infty}, \infty\right)$ whose projection on $\mathbb{R}^{\mathrm{sgn}k}$ lies in Λ and whose projection on E is bounded away from 0, then either

(i) $\mathcal{D}_k \backslash \mathcal{M}$ is bounded in $\mathbb{R}^{\mathrm{sgn}k} \times E$, in which case $\mathcal{D}_k \backslash \mathcal{M}$ meets $\mathbb{R}^{\mathrm{sgn}k} \times \{0\}$, or

(ii) $\mathcal{D}_k \backslash \mathcal{M}$ is unbounded; if additionally $\mathcal{D}_k \backslash \mathcal{M}$ has a bounded projection on $\mathbb{R}^{\mathrm{sgn}k}$, then $\mathcal{D}_k \backslash \mathcal{M}$ contains $\left(\frac{\lambda_m^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$, where $m \in \mathbb{N}$ and $m > 1$.

Moreover, $\mathcal{D}_k$, $k \in \{-, +\}$, can be decomposed into two subcontinua $\mathcal{D}_k^-$, $\mathcal{D}_k^+$ and there exists a neighborhood $Q \subset \mathcal{M}$ of $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$ such that $(\lambda, u) \in \mathcal{D}_k^- (\mathcal{D}_k^+) \cap Q$ and $(\lambda, u) \neq \left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right)$ implies

$$(\lambda, u) = (\lambda, s u_{1,+}^{\mathrm{sgn}k} + w),$$

where

$$s < 0 \ (s > 0) \text{ and } |\lambda - \lambda_1^k| = o(1), \ w = o(|s|) \text{ at } |s| = \infty.$$

Consequently,

$$\text{if } (\lambda, u) \in \mathcal{D}_k^\nu \backslash Q, \text{ then } (\lambda, u) \in \mathbb{R}^{\mathrm{sgn}k} \times S_1^\nu. \tag{21}$$

Let

$$(\lambda_n, u_n) \in \mathfrak{L}_k^\nu \text{ and } |\lambda_n| + ||u_n||_3 \to \infty \text{ as } n \to \infty.$$

We note that $\lambda_n \mathrm{sgn}k > 0$ for all $n \in \mathbb{N}$, since $\mathfrak{L} \cap (\{0\} \times E \backslash \{0\}) = \emptyset$. As in the proof of Theorem 1.1 from [23] we can prove that there exists a positive constant M such that

$$|\lambda_n| \leq M, \ n \in \mathbb{N},$$

which implies

$$||u_n||_3 \to \infty \text{ as } n \to \infty.$$

It is obvious that

$$u_n = \lambda_n f_\infty \mathcal{L} u_n + \lambda_n \mathcal{G}(u_n). \tag{22}$$

Let $v_n = \frac{u_n}{||u_n||_3}$. Then by (22) v_n satisfies the relations

$$v_n = \lambda_n f_\infty \mathcal{L} v_n + \lambda_n \frac{\mathcal{G}(u_n)}{||u_n||_3} \tag{23}$$

By virtue of completely continuity of operators $\mathcal{L}$ and $\mathcal{G}$, and the boundedness of $\{\lambda_n\}_{n=1}^\infty$ it follows from (23) that there exists a subsequence of the sequence $\{(\lambda_n, v_n)\}_{n=1}^\infty$ (which we will relabel as $\{(\lambda_n, v_n)\}_{n=1}^\infty$) which is convergent to $(\tilde{\lambda}, v)$ in $\mathbb{R}^{\mathrm{sgn}k} \times E$, with $||v||_3 = 1$, $v \in S_1^\nu$ and

$$v = \tilde{\lambda} f_\infty \mathcal{L} v. \tag{24}$$

Then by Theorem 2.1 it follows from (24) that

$$\tilde{\lambda} = \frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}.$$

Hence

$$(\lambda_n, u_n) \to \left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_\infty}, \infty\right) \text{ as } n \to \infty,$$

which by (21) implies that

$$\mathcal{D}_k^{\nu} \backslash Q \subset \mathfrak{L}_k^{\nu}. \tag{25}$$

Moreover, it follows from the proof of [25; Corollary of Theorem 2.4] that $\mathcal{D}_k^{\nu}$ contains a subcontinuum $\mathfrak{D}_k^{\nu}$ lying in $\mathbb{R} \times S_1^{\nu}$ such that either $\mathfrak{D}_k^{\nu} \backslash Q$ is unbounded or intersects the line $\mathcal{R} = \{(\lambda, 0) \in \mathbb{R} \times E\}$ of trivial solutions at $\left(\frac{\lambda_1^{\mathrm{sgn}k}}{f_0}, 0\right)$. Consequently, by (25) we have $\mathfrak{L}_k^{\nu} = \mathfrak{D}_k^{\nu}$. The proof of this theorem is complete.

Corollary 1. *Let r be a real constant such that*

$$r \in \left(\frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_\infty}, \frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_0}\right)$$

or

$$r \in \left(\frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_0}, \frac{\lambda_1^{\mathrm{sgn}k}\mathrm{sgn}k}{f_\infty}\right), \ k = -1 \text{ or } k = 1.$$

where $f_0 \neq f_\infty$. Then the problem

$$\begin{aligned} &(\ell u)(t) = r g(t) f(u(t)), \ t \in (0,1), \\ &u \in B.C. \end{aligned}$$

has at least one negative and one positive solutions.

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Received 27 August 2017
Accepted 29 September 2017