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On Wiman-Valiron Type Estimations for Evolution Equations

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Abstract. In the paper we establish Wiman-Valiron-type estimates for evolution equations in Hilbert spaces containing a pseudo-differential operator of the Hormander class $L^m_{p,\delta}$. Using asymptotic formulas for the function $N(\lambda)$ for the given operator in the equation, we prove Wiman-Valiron-type theorems characterizing the behavior of the solution depending on the properties of Fourier coefficients of the solution.
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1. Introduction

Let $f(z) = \sum\limits_0^\infty a_n z^n$ be an integer

$$M(r) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \max_n |a_n| r^u, \;\; \mu(r) \to \infty, \;\; M(r) \to \infty, \;\; r \to \infty.$$

The estimation $\mu(r) \leq M(r)$ is always true. But it is very important to get the estimation $M(r)$ from above by $\mu(r)$. In the papers of Wiman [1] and Valiron, the estimation of the following form

$$M(r) \leq \mu(r)(\log \mu(r))^{\frac{1}{2}+\varepsilon}$$

that is fulfilled out of some set $E \subset (0, \infty)$ of finite logarithmic measure, was established. In 1966, Rosenbloom [3] established more general result: for some class of functions $\varphi(y) > 0$, $y > 0$ the estimation of type

$$M(r) \leq \mu(r)\sqrt{\varphi(\log M(r))} \tag{1}$$

is fulfilled out of some set of weighted measure. In 1966, Kovari [4] established similar results for power series with finite radius of convergence. In the author's (see [5]) theory of Wiman-Valiron-Rosenbloom type estimations was constructed for evolution equations in Hilbert space. In the present paper we establish estimations of type (1) for evolution (parabolic) equations containing pseudo-differential operator of the Hormander class.

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2. Problem statement

Let us consider the equation

$$u'(t) + A(t)\,u(t), \tag{2}$$

where $A(t) \in L^m_{p,\delta}$ is a positive self-adjoint pseudo-differential (2) operator with a discrete spectrum. Let on $D(t)$ the strong derivative $A'(t)$ be determined and for $U \in D(A)$ the condition of the form

$$\left(A'(t)\,u, u\right) \leq k(t)\,(A(t)\,u, u), \quad 0 < k(t) \in L_1(0, \infty) \tag{$*$}$$

be fulfilled.

Denote by $N(\lambda)$ the amount of all eigen values $\lambda_k(t)$ of the operator $A(t)$ not exceeding λ (with regard to multiplicity). The following lemma was proved in the paper (see [5], p. 84) of the author.

Lemma 1. *The following differential inequality*

$$e^{2g(t)} \leq \mu(t)\,P\left(g', g''\right), \tag{3}$$

where $0 < t < T$, $g(t) = \frac{1}{2}\log(u(t), u(t))$ *the* $u(t)$ *is solution of equation (2),*

$$\mu(t) = \max_k |(u(t), \varphi_k(t))|,$$

$$P(a; b) = N\left(a + C\sqrt{b + k(t)\,a}\right) - N\left(a - C\sqrt{b + k(t)\,a}\right). \tag{4}$$

(Here and in the sequel denotes C absolute an constant, but not always identical). We briefly note the basic idea (fragments) of the method of proof based on probability. This method was constructed by us and is a very significant and strong modification of Rosembloom's problem constructed by him only for entire functions.

Associate to the function $u(t)$ some random variable ξ whose range of values is the set of eigen-values $\lambda_k(t)$ of the operator $A(t)$, and distribution of probabilities (dependent on parameter t) we define by the

$$P_k = P(\xi = \lambda_k(t)) = C_k(t)^2\,\|u(t)\|^2,$$

where $C_k(t) \equiv (u(t), \varphi_k(t))$ are the Fourier coefficients of the function $u(t)$ with respect to orthonormed system $\{\varphi_k(t)\}$ of eigen functions of the operator $A(t)$.

Having calculated the mathematical expectation $M\xi$, and variance $D\xi$, we find:

$$M\xi = -g'(t), \quad D\xi \leq g''(t) \quad\quad k(t)\,g'(t).$$

Applying the Chebyshev known inequality from probability theory

$$P(|\xi - M\xi| > \varepsilon) \leq D\xi|\,\varepsilon^2,$$

we get (take $\varepsilon = C\sqrt{D\xi}$)

$$\left| -P\left(\left|\xi + g'\right| \le C\sqrt{D\xi}\right) \le C\sqrt{D\xi} \le 1 \right| C^2.$$

Hence we have

$$1 - \frac{1}{C^2} \le P\left(\left|\xi + g'\right| \le \varepsilon\right) = \sum_{k\in I} p_k = \frac{T}{\|u\|^2}\sum_{k\in I} C_k^2,$$

where $I = \{k : |\lambda_k + g'| \le \varepsilon\}$. Consequently, we have:

$$\|u\,(t)\|^2 \le C\mu\,(t)^2 \sum_{k\in I} 1 \tag{5}$$

It is clear that by (4),

$$\sum_{k\in I} 1 = N\left(\left|g'\right| + C\sqrt{g'' + k\,(t)\,g'}\right) - N\left(\left|g'\right| - C\sqrt{g'' + k\,(t)\,g'}\right) \equiv$$

$$\equiv P\left(\left|g'\right|, g''\right).$$

Then, (5) yields the estimation of the form

$$\|u\,(t)\|^2 \le C\mu\,(t)^2\, P\left(\left|g'\right|, g''\right). \tag{6}$$

If we find a function $\Psi\,(y)$, $y > 0$ such that in some sense the inequality of following form

$$P\left(\left|g'\right|, g''\,(t)\right) \le \Psi\,(g\,(t))\,, \tag{7}$$

is valid, then, from (6) we get that it holds the estimation of type

$$\|u\,(t)\| \le C\mu\,(t)\,\sqrt{\psi\,(\log\|u\,(t)\|)},$$

that is Riman-Valiron type estimation for evolution equation (2). The conditions under which inequalities of type (8) are fulfilled, were studied in the papers (see [5]).

In the following theorem there is an assumption on asymptotic behavior of the function $N\,(\lambda)$, and Wiman-Valiron-Rosenbloom type estimations for solving equation (2) are established on its bases.

Theorem 1. *Let the function $N\,(\lambda)$ for the operator $A \in L^m_{p,\delta}\,(\Omega)$ of the Hormander class satisfy the following conditions:*

$$N\,(\lambda) \le C\lambda^{s+1}\ln\lambda, \quad s+1>0, \quad C>0 \tag{8}$$

and for $\lambda > \delta > 0$, $\lambda \to \infty$ the inequality of type $(0 \le \nu \le 1)$:

$$\Delta N\,(\lambda,\delta) \equiv N\,(\lambda+\delta) - N\,(\lambda-\delta) \le C\delta\lambda^s\,(1+\lambda^\nu)\,(1+lg\lambda). \tag{9}$$

Let the function $\varphi(y) > 0$, $y > 0$ do not decrease and be such that for some $\alpha > 0$, following the condition be fulfilled

$$\int^{\infty} \left(\int^{y} \varphi(t)\, dt \right)^{-\alpha} dy < +\infty. \tag{10}$$

Then, out of possibly some certain set $EC(0,\infty)$ of finite measure, the following Wiman-Valirob type estimation is valid:

$$\|u(t)\| \le C\mu(t) \sqrt[4]{\varphi(\log \|u(t)\|)}. \tag{11}$$

Proof. We immediately note that conditions (9) and (10) satisfy the function, for example, of type:

$$N(\lambda) = \lambda^{p} \ln \lambda, \quad N(\lambda) = \lambda^{\frac{m}{n}} + O\left(\lambda^{\frac{n-1}{m}}\right), \quad N(\lambda) = \lambda^{p} l(\lambda),$$

where $l(\lambda)$ is a slowly growing function, i.e. $\lim\limits_{\lambda\to\infty} \lambda \frac{l'(\lambda)}{l(\lambda)} = 0$. For example, the functions $l(\lambda) = \ln \lambda$, $l(y) = \ln \ln \lambda$, $l(\lambda) = (\ln \lambda)^{2}$, $\alpha > 0$ and others are this type functions.

Introduce a change of variables:

$$\xi(t) = \int_{0}^{t} \Phi(p)\, d\rho, \tag{12}$$

where

$$\Phi(\rho) = \int_{\rho}^{t} k(\tau)\, d\tau.$$

For any function $h(t)$ denote $\widetilde{h}(\xi) = h(t(\xi))$, where $t(\xi)$ is determined from the relation (13). We get:

$$\sqrt{g''(t) + k(t)\, g'(t)} = \sqrt{\widetilde{g}''(t)}\Phi(t), \quad \widetilde{g}'' > 0.$$

From condition (10) we get an inequality of the form $\left(\lambda = \widetilde{g}', \quad \delta = \sqrt{\widetilde{g}''}\right)$:

$$\Delta N(\lambda, \delta) \le C\sqrt{\widetilde{g}''}\widetilde{g}'^{s}\left(1 + \widetilde{g}'^{\nu}\right). \tag{13}$$

Thus, the problem is reduced to the fact that it is necessary to find such a function $\varphi(y)$ that the inequality (in the sequel, instead of $\widetilde{g}$ we simply write g):

$$\sqrt{g''}g'\left(1 + g'\nu\right) \le \sqrt{\varphi(g)} \tag{14}$$

is fulfilled.

Thus, we get the system of differential inequalities

$$\left\{ \begin{array}{l} \sqrt{g''}g'^{s+\nu} \leq \alpha\sqrt{\varphi(g)} \\ \sqrt{g''}g'^{s} \leq \beta\sqrt{\varphi(g)} \end{array} \right. \qquad (\alpha+\beta \leq 1)\,. \tag{15}$$

Let $E = \left\{\sqrt{g''}g'^{s+\nu} > \alpha\sqrt{\varphi(g)}\right\}$. Consider the first inequality of the system:

$$g''g'^{2(s+\nu)+1} \leq \alpha^2\varphi(g)\,g'$$

$$\left(g'^{p}\right)' \leq C\varphi(g)\,g', \qquad p = 2(s+\nu)+2,$$

$$g' \leq \left(C\int^{g}\varphi(t)\,dt\right)^{1/p} \equiv \psi_1(g)\,.$$

Then

$$E = \left\{g'(t) > \varphi_1(g)\right\}.$$

From condition (11) we have

$$mesE = \int_E dt < \int_E \frac{g'(t)\,dt}{\varphi_1(g)} \leq \int_{g(E)} \frac{dg}{\varphi_1(g)} \leq \int_0^{\infty}\frac{dg}{\varphi_1(g)} = \int^{\infty}\left(\int^{\infty}\varphi(t)\,dt\right)^{-\frac{1}{p}} dg < \infty.$$

Thus, subject to the condition (11), out of the set E, $mesE < \infty$ the first inequality of the system (16) is fulfilled. In the similar way, we obtain that the second inequality of this system is also fulfilled out of some set of finite measure. Consequently, the system (1) is true out of E, $mesE < \infty$. Then the inequality (15) is fulfilled out of E. Consequently, $\Delta N(g', g'') \leq \varphi(g)$.

Then, by Lemma 1, estimation (12) is valid.

Theorem 2. *Let for $\lambda > \delta > 0$, $\lambda \to \infty$ the condition of the form*

$$\Delta N(\lambda,\delta) \leq C\lambda^{n/m}\left(\delta + \lambda^{-\frac{1}{m}}\right)(1+\ln\lambda) \tag{16}$$

be fulfilled. Then estimation of type (12) is valid. The proof is similar.

Remark 1. *The condition of type (12) appears when for differential operator $A \in L^m_{p,\delta}(R^n)$ of order m in R^n, the function $N(\lambda)$ grows as $\lambda \to \infty$ faster than power λ, for example as $\lambda^p \ln\lambda$.*

In Shubin's monograph [p. 130], there is an example of the operator for which the function $N(\lambda)$ grows faster than power λ

$$N(\lambda) = C\lambda^{k_0}(\ln\lambda)^l, \quad \lambda \to \infty,$$

where $k_0 > 0$ while l is a natural number that is equal to the order of the pole at the point $z = -k_0$ of zeta function

$$\zeta(z) = \int_0^{\infty} t^z dN(t)\,.$$

For a self-adjoint positive elliptic operator of order m, Hormander [see [7], p. 134] obtained for $N(\lambda)$ [see also [8]-[10]] the exact formula of the form

$$N(\lambda) = C\lambda^{n/m} + O\left(\lambda^{\frac{n-1}{m}}\right).$$

In Shubin's papers this result was proved by the original method owing to which this formula was developed.

In the book [[7], p. 134] for a self-adjoint elliptic operator $A \in L^m_{p,\delta}(\Omega)$ on n-dimensional closed manifold $\Omega \subset R^n$ such that its main symbol $a_m(x,\xi)$ is positive, for the function $N(\lambda)$ it was established the following formula with the unimproved residue

$$N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{-\frac{1}{m}}\right)\right), \lambda \to \infty \tag{17}$$

where the function $V(\lambda)$ is determined by the main symbol $a_m(x,\xi)$ of the equality

$$V(\lambda) = \frac{1}{(2\pi)^n} \int\limits_{a_m(x,\xi)<\lambda} dx d\xi, \quad \lambda \to \infty.$$

In this case, as $\lambda \to \infty$ the asymptotics

$$N(\lambda) = V(\lambda) = C\lambda^{n/m},$$

where

$$C = \frac{1}{(2\pi)^n} \int\limits_{a_m(x,\xi)<\lambda} dx d\xi.$$

But if the operator $A \in L^m_{p,\delta}$ is a general pseudo-differential operator of order m with main symbol $a_m(x,\xi) > 0$, then as was shown in [7], the asymptotics of the function $N(\lambda)$, determined by formula (18), may also have a not power growth, for example as $\lambda^{n/m}\ln\lambda$. Just in such cases a condition of type (17) appears on $N(\lambda)$.

In formula (18) assume $V(\lambda) = \lambda^p \ln\lambda$, $\nu = -1/m$ and consider the difference

$$\Delta N(\lambda,\delta) = (\lambda+\delta)^p \ln(\lambda+\delta) - (\lambda-\delta)^p \ln(\lambda-\delta) +$$

$$+C(\lambda+\delta)^{p-\nu}\ln(\lambda+\delta) - (\lambda-\delta)^{p-\nu}\ln(\lambda-\delta) = A + B;$$

$$A = \lambda^p \left\{ \left(1+\frac{\delta}{\lambda}\right)^p \left[\ln\lambda + \ln\left(1+\frac{\delta}{\lambda}\right)\right] - \left(1-\frac{\delta}{\lambda}\right)^p \left[\ln\lambda + \ln\left(1-\frac{\delta}{\lambda}\right)\right]\right\} =$$

$$= \lambda^p \left\{ \left(1+p\frac{\delta}{\lambda}\right)\left(\ln\lambda + \frac{\delta}{\lambda}\right) - \left(1-p\frac{\delta}{\lambda}\right)\left(\ln\lambda - \frac{\delta}{\lambda}\right)\right\} =$$

$$2\delta\lambda^{p-1}(1+\ln\lambda).$$

Similarly, $B = 2\delta\lambda^{p-1-\nu}(1+\ln\lambda)$.

Consequently: for $\Delta N(\lambda, \delta)$ we get an inequality of type

$$\Delta N(\lambda, \delta) \leq \varepsilon\delta\lambda^{\frac{n}{m}-\nu-1}(1+\lambda^{\nu})(1+\ln\lambda). \tag{18}$$

If we assume $\lambda = g'$, $\delta = \sqrt{g''}$, then obtain $g'' \leq Cg'^2$ (out of the set of finite measure). Thus, for the function ΔN we get an inequality of the form

$$\Delta N(g', g'') \leq C\sqrt{g''}g'^s\left(1+g'^{\nu}\right)\left(1+\ln g'\right), \quad 0<\nu<1, \quad s=\frac{n}{m}-1-\nu.$$

Let $A \in L^m_{p,\delta}$ be an elliptic operator with the main symbol $a_m(x,\xi) > 0$. Consider the function

$$V(t) = \frac{1}{(2\pi)^n}\int\limits_{a_m(x,\xi)<\lambda} dxd\xi. \tag{19}$$

The following statement was established in the paper [7, p. 206].

Proposition 1. *Let at some $\varepsilon > 0$, $\delta > 0$, $c > 0$ for $V(t)$ the condition of type*

$$\frac{V\left(t+Ct^{1-\varepsilon}\right)-V(t)}{V(t)} = O\left(t^{-\delta}\right), \quad t \to +\infty$$

be fulfilled. Then for the function $N(\lambda)$ the asymptotic formula

$$N(\lambda) = V(\lambda)\left(t + O\left(\lambda^{-\delta}\right)\right), \quad t \to +\infty \tag{20}$$

is valid.

Using the method of the paper [7, p. 206] we can formulate a proposition more convenient for application.

Proposition 2. *Let $V(t) > 0$ grow for $t > t_0$ and for some $0 \leq \alpha, \nu \leq 1$ the condition of type*

$$V'(t)|V(t) = O\left(Et^{\alpha+\nu}\right), \quad t \to +\infty \tag{21}$$

be fulfilled.

Then for the function $N(\lambda)$ the asymptotic formula

$$N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{-\nu}\right)\right) \tag{22}$$

is valid.

Indeed, we denote $\varphi(t) = V'(t)|\,V(t)$. Integrating, we get

$$\frac{V(t+at^{\alpha})-V(t)}{V(t)} = \exp\int\limits_t^{t+Ct^2}\varphi(\tau)\,d\tau - 1. \tag{23}$$

As $|\varphi(t)| \leq C_1 t^{-(\alpha+\nu)}$, then for $a+\nu \neq 1$ we have $(\gamma = 1-(a+\nu))$

$$\int\limits_{t}^{t+Ct^2} \varphi(\tau)\, d\tau \leq C_1 t^{\gamma}\left[\left(1+Ct^{\alpha-1}\right)^{\gamma}-1\right] = C_1 t^{\gamma}\left[Ct^{\alpha-1}\right] = C_2 t^{-\nu}.$$

The same estimation is obtained for $\lambda+\nu=1$ as well.

Since $e^{\lambda}-1\tilde{}X$ as $X \to 0$ $nC_2 t^{-\nu} \to 0$ then from (24) we get

$$\frac{V(t+Ct^{\alpha})-V(t)}{V(t)} = O\left(t^{-\nu}\right), \quad t \to +\infty.$$

Hence, formula (23) follows from the above mentioned result of the paper [7].

Note that the asymptotic function $N(\lambda)$ determined from formula (23) may have also a not power series. For example, for the function $V(t) = t^p \ln t$ we have

$$\frac{V'(t)}{V(t)} = \frac{p}{t} + \frac{1}{t\ln t} = O\left(t^{-1}\right), \quad t \to +\infty.$$

Consequently, in proposition 2, $a+\nu=1$. Then by virture of this proposition, for $N(\lambda)$ we get a formula of the form

$$N(\lambda) = \lambda^p \ln\lambda\left(1+O\left(\lambda^{-\nu}\right)\right). \tag{24}$$

Let us consider a simpler example. Then $V(t) = t^p l(t)$, where $l(t) > 0$ is a slowly growing function, i.e.

$$\lim_{t\to\infty} t\frac{l'(t)}{l(t)} = 0. \tag{25}$$

For this function we get

$$\frac{V'(t)}{V(t)} = \frac{p}{t} + \frac{l'(t)}{l(t)}.$$

Taking (26) into account, hence we find $\frac{V'(t)}{V(t)} = O\left(t^{-1}\right)$, i.e. in proposition 2 we have $\alpha+\nu=1$. Consequently, for $N(\lambda)$ the following formula is valid

$$N(\lambda) = \lambda^p l(\lambda)\left(1+O\left(\lambda^{-\nu}\right)\right).$$

Remark 2. *Let the symbol $a(x,\xi)$ of the operator $A \in L^m_{p,\delta}$ satisfy the conditions of the form*

1) $a(y) \to +\infty$ as $|y| \to +\infty$, where $y=(x,\xi)$, $x,\xi \in R^n$

2) $a(y)^{1-\alpha} \leq C\,|(y,\nabla a(y))|$ as $|y| \geq N$, $C>0$, $0 \leq \alpha \leq 1$, where ∇ is the gradient of the function $a(y)$.

Then from the results of the paper [7] (theorem 28.3) we have the estimation of the form

$$\frac{V'(t)}{V(t)} = O\left(t^{\alpha-1}\right), \quad t+\infty.$$

Herewith, if $a(y)$ is an elliptic polynomial with respect to y and of power m, then we can take $\alpha = 0$.

From [7, p. 206] for $\alpha < \nu < 1$ we have:

$$\frac{V\left(t + ct^{1-\nu}\right) - V(t)}{V(t)} = O\left(t^{\alpha-\nu}\right), \quad \to +\infty.$$

Then the estimations of the form

$$N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{\alpha-1}\right)\right), \quad t \to \infty \tag{26}$$

hold.

In particular, for $\alpha = 0$ we get

$$V'(t)\big| V(t) = O\left(\frac{1}{t}\right), \quad N(\lambda) = V(\lambda)\left(1 + O\left(\lambda^{-\nu}\right)\right), \quad 0 < \nu < 1.$$

Thus, for pseudo-differential operator $A \in L^m_{p,\delta}(R^n)$ with properties 1 and 2, the estimations (21) and (22) are valid, where the function $N(\lambda)$ possibly grows in not power way and theorems 1,2 are applicable in such situations.

In conclusion, let us consider as an application the results obtained in the paper, for example, the solutions of head conductivity equation in the domain $(0,T)\times\Omega$, $\Omega \subset R^n$ is a bounded domain with smooth boundary with homogeneous Dirichlet boundary condition on the plane $(0,T) \times \partial\Omega$ in the space $L_2(\Omega)$:

$$\frac{\partial u}{\partial t} = \Delta_x u, \quad u|_{(0,T)\times\partial\Omega} = 0, \quad u|_{t=0} = u_0(x). \tag{27}$$

Let $(u_0, \varphi_n) = \sqrt{n}$, $\lambda_n = \frac{n}{2}$. Then we get

$$\|u(t,\cdot)\|^2 = \sum (u_0, \varphi_n)^2 e^{-2t\lambda_k} = \sum n e^{-nt} = -\frac{d}{dt}\sum e^{nt} = -\frac{d}{dt}\frac{1}{1-e^{-t}} = \frac{e^{-t}}{(1-e^{-t})^2}$$

It is easy to see that $\frac{e^{-t}}{(1-e^{-t})^2} \sim \frac{1}{t^2,}$ $t \to 0$, we have $\|u(t)\|^2 \sim \frac{1}{t^2}$, $t \to 0$, indeed,

$$\lim_{t\to 0}\frac{e^{-t}}{(l-e^{-t})^2} = \lim_{t\to 0}\frac{t^2 e^{-t}}{1 - 2e^{-t} + e^{-2t}} = \lim_{t\to 0}\frac{t^2}{e^t - 2 + e^{-t}} = \lim_{t\to 0}\frac{2t}{e^t - e^{-t}} =$$

$$= \lim_{t\to 0}\frac{2}{e^t + e^{-t}} = 1.$$

Consequently, $\frac{e^{-t}}{(1-e^{-t})^2} \sim \frac{1}{t^2}$.

Calculate $\mu(t)$. Since

$$\mu^2(x) = \max_n n e^{-nt} = \max_n \psi(x),$$

where $\psi(x) = xe^{-xt}$. Then $\psi^l = e^{-xt} - t \times e^{-xt} = 0$, $1 - xt = 0$, $x = \frac{1}{t}$, $\psi\left(\frac{1}{t}\right) = \frac{1}{t}e^{-1}$. Consequently $\mu(t)^2 = \frac{1}{et}$, $\mu(t) = \frac{1}{\sqrt{et}}$. Then we get

$$\|u(t,\cdot)\|^2 = \frac{1}{t} = \frac{\sqrt{e}}{\sqrt{t}}\mu(t) = \mu(t)^2$$

i.e.

$$\|u(t,\cdot)\| = \mu(t) = \frac{1}{\sqrt{t}}, \quad t \to 0.$$

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