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Approximation of Hypersingular Integral Operators on Hölder Spaces

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Abstract. In the present paper, the hypersingular integral operator is approximated by a sequence of operators of the special form and is obtained the estimate of the convergence rate in Hölder spaces.

Key Words and Phrases: hypersingular integral, Hölder space, approximating operators, convergence rate.

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1. Introduction

An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. (see [4, 5, 10, 13, 15, 20, 22, 23, 26, 27]). Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works [3-9,11,12,14,16-19,21-25, 27-31] are devoted to their development. In the present paper hypersingular integral operator In the present paper hypersingular integral operator

$$\left(S^{(0)}\varphi\right)(t) = \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau)}{|\tau - t|} d\tau$$

is approximated by a sequences of operators of the form

$$\left(S_n^{(0)}\varphi\right)(t) = \sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\, \varphi\left(\tau_k^{(t)}\right), \qquad t \in \gamma_0$$

in the unit circle $\gamma_0 = \{t \in C : |t| = 1\}$, where $\tau_k^{(t)} = e^{k\theta i} \cdot t$, $k = \overline{0,\, 2n}$, $\theta = \frac{\pi}{n}$, $n \in N$, $\alpha_k^{(n)}(t)$ – are continuous functions in γ_0 , $k = \overline{0, 2n-1}$, $n \in N$.

It should be noted that, the determination of the inverse operator $\left[S_n^{(0)}\right]^{-1}$ is equivalent to the study of the equation

$$\sum_{k=0}^{2n-1} \alpha_k^{(n)}(t)\,\varphi\left(\tau_k^{(t)}\right) = f(t)\ ,\quad t\in\gamma_0$$

at the points $\tau_0^{(t)}, \tau_1^{(t)}, ..., \tau_{2n-1}^{(t)}$, because solving the resulting system of linear algebraic equations with respect to $\left(\varphi\left(\tau_0^{(t)}\right),\ \varphi\left(\tau_1^{(t)}\right), ..., \varphi\left(\tau_{2n-1}^{(t)}\right)\right)$, we obtain the function $\varphi(t) = \varphi\left(\tau_0^{(t)}\right)$.

Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and its applications to the singular integral equations are given in the papers [1] and [2], analogous approximations for hypersingular integral operators with Cauchy kernel are given in [3].

2. Hypersingular integral operator

Consider the following integral

$$\int_a^b \frac{g(x)}{|x-x_0|}dx\ ,\quad x_0\in(a,b) \tag{1}$$

where the function $g(x)$ is defined in the interval $[a,b]$. If we define this integral similar to the Cauchy integral, even if $g\equiv 1$, we get the divergent integral:

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{1}{|x-x_0|}dx+\int_{x_0+\varepsilon}^b\frac{1}{|x-x_0|}dx\right) = \lim_{\varepsilon\to 0+}\left(-2\ln\varepsilon+\ln(x_0-a)(b-x_0)\right)=\infty.$$

Therefore, using the idea of Hadamard finite part integral [15], we will define the integral (1) as follows:

Definition 1. *If a finite limit*

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{g(x)\,dx}{|x-x_0|}+\int_{x_0+\varepsilon}^b\frac{g(x)\,dx}{|x-x_0|}+2g(x_0)\ln\varepsilon\right)$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{g(x)}{|x-x_0|}$*,* $x_0\in(a,b)$ *on* $[a,b]$ *and is denoted by* $\int_a^b\frac{g(x)}{|x-x_0|}dx$ *.*

Now consider the integral

$$\int_{\gamma_0}\frac{\varphi(\tau)\,d\tau}{|\tau-t|},\quad t\in\gamma_0\,, \tag{2}$$

where the function $\varphi(t)$ *is defined in the unit circle*$\gamma_0=\{t\in C:\ |t|=1\}$*.*

From the definition 1.1 for the hypersingular integral on interval, we define the integral (2) *as follows.*

Definition 2. *If a finite limit*

$$\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}+2it\varphi(t)\ln\varepsilon\right)$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{\varphi(\tau)}{|\tau-t|}$, $t\in\gamma_0$ *on the circle*γ_0 *and is denoted by* $\int_{\gamma_0}\frac{\varphi(\tau)d\tau}{|\tau-t|}$, *where* $\gamma_\varepsilon=\{\tau\in\gamma_0:\,|\tau-t|>\varepsilon\}$.

From definitions 1.1 and 1.2, it follows that if $t=e^{ix_0}$, $x_0\in(-\pi,\pi)$, then

$$\int_{\gamma_0}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}=\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}+2it\varphi(t)\ln\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{x_0+\delta(\varepsilon)}^{x_0+2\pi-\delta(\varepsilon)}\frac{\varphi\left(e^{ix}\right)ie^{ix}dx}{\left|e^{ix}-e^{ix_0}\right|}+2ie^{ix_0}\varphi\left(e^{ix_0}\right)\ln\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{[-\pi,\pi]/(x_0-\delta(\varepsilon),\,x_0+\delta(\varepsilon))}\frac{\varphi\left(e^{ix}\right)ie^{ix}}{|x-x_0|}\left|\frac{x-x_0}{e^{ix}-e^{ix_0}}\right|\cdot dx+2ie^{ix_0}\varphi\left(e^{ix_0}\right)\ln\varepsilon\right)=$$

$$=\int_{-\pi}^{\pi}\frac{\varphi\left(e^{ix}\right)ie^{ix}dx}{\left|e^{ix}-e^{ix_0}\right|}+2ie^{ix_0}\varphi\left(e^{ix_0}\right)\cdot\lim_{\varepsilon\to 0+}\left(\ln\varepsilon-\ln\delta(\varepsilon)\right)=\int_{-\pi}^{\pi}\frac{\varphi\left(e^{ix}\right)ie^{ix}dx}{\left|e^{ix}-e^{ix_0}\right|}\,,\qquad(3)$$

where $\delta(\varepsilon)=2\arcsin\frac{\varepsilon}{2}$.

Equation (3) shows that, by means of change of variables $t=e^{ix}$ the hypersingular integral on a circle is reduced to hypersingular integral on an interval.
We will calculate the hypersingular integral $\int_{\gamma_0}\frac{d\tau}{|\tau-t|}$, $t=e^{ix_0}\in\gamma_0$. We have

$$\int_{\gamma_0}\frac{d\tau}{|\tau-t|}=\lim_{\varepsilon\to 0+}\left(\int_{[x_0-\pi,x_0+\pi]/(x_0-\varepsilon,\,x_0+\varepsilon)}\frac{ie^{ix}}{\left|e^{ix}-e^{ix_0}\right|}dx+2it\ln\varepsilon\right)=$$

$$=\lim_{\varepsilon\to 0+}\left(\int_{[x_0-\pi,x_0+\pi]/(x_0-\varepsilon,\,x_0+\varepsilon)}\frac{ie^{ix}}{2\left|\sin\frac{x-x_0}{2}\right|}dx+2it\ln\varepsilon\right)=$$

$$=2it\cdot\lim_{\varepsilon\to 0+}\left(\int_{x_0+\varepsilon}^{x_0+\pi}\frac{\cos(x-x_0)}{2\sin\frac{x-x_0}{2}}+\ln\varepsilon\right)=2it\cdot(\ln 4-2)\,,\qquad(4)$$

where $\delta(\varepsilon)=2\arcsin\frac{\varepsilon}{2}\sim\varepsilon$ as $\varepsilon\to 0+$. From equation (4) it follows that,

$$\int_{\gamma_0}\frac{\varphi(\tau)\,d\tau}{|\tau-t|}=\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau+(\ln 4-2)\,2it\varphi(t)\,.\qquad(5)$$

Let $H_\alpha(\gamma_0)$, $0<\alpha\le 1$ be the space of Hölder continuous functions with exponent α in γ_0 , i.e. the space of the functions which satisfies the following condition

$$\exists C>0\quad\forall t_1,\ t_2\in\gamma_0:\quad|\varphi(t_1)-\varphi(t_2)|\le C\cdot|t_1-t_2|^\alpha$$

with the norm

$$\|\varphi\|_{\alpha} = \|\varphi\|_{\infty} + H(\varphi;\alpha)$$

where

$$\|\varphi\|_{\infty} = \max_{t\in\gamma_0} |\varphi(t)|, H(\varphi;\alpha) = \sup\left\{ \frac{|\varphi(t_1) - \varphi(t_2)|}{|t_1 - t_2|} : \ t_1, t_2 \in \gamma_0,\ t_1 \neq t_2 \right\}.$$

From equation (5) it follows that, if $\varphi \in H_{\alpha}(\gamma_0)$ then hypersingular integral $\int_{\gamma_0} \frac{\varphi(\tau)d\tau}{|\tau - t|}$ exists for all $t \in \gamma_0$

Consider the hypersingular integral operator:

$$\left(S^{(0)}\varphi\right)(t) = \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau)}{|\tau - t|} d\tau.$$

Theorem 1. *Hypersingular integral operator $S^{(0)}$ is bounded from the space $H_{\alpha}(\gamma_0)$ into the space $H_{\alpha-\varepsilon}(\gamma_0)$ for all $0 < \alpha \le 1$ and $0 < \varepsilon < \alpha$.*

Proof. From equation (5) it follows that, it is sufficient to prove the stated theorem for the following operator:

$$(T\varphi)(t) = \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau) - \varphi(t)}{|\tau - t|} d\tau.$$

Let $\varphi \in H_{\alpha}(\gamma_0)$. Then

$$\|T\varphi\|_{\infty} = \max_{t\in\gamma_0}\left| \frac{1}{\pi i}\int_{\gamma_0} \frac{\varphi(\tau) - \varphi(t)}{|\tau - t|} d\tau \right| \le \frac{1}{\pi}\left| \int_{\gamma_0} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|} |d\tau| \right| \le$$

$$\le \frac{1}{\pi}\left| \int_{\gamma_0} \frac{H(\varphi;\alpha)}{|\tau - t|^{1-\alpha}} |d\tau| \right| \le C_1 \cdot H(\varphi;\alpha) \le C_1 \cdot \|\varphi\|_{\alpha}, \tag{6}$$

where C_1 – constant which only depends on α.

Estimate the difference $(T\varphi)(t_1) - (T\varphi)(t_2)$ for any two points $t_1,\ t_2 \in \gamma_0,\ t_1 \neq t_2$. If $|t_1 - t_2| \ge \frac{1}{2}$, then from inequality (6) it follows that,

$$|(T\varphi)(t_1) - (T\varphi)(t_2)| \le 2C_1 \cdot \|\varphi_{\alpha}\| \le 4C_1 \cdot \|\varphi_{\alpha}\| \cdot |t_1 - t_2|. \tag{7}$$

Consider the case $|t_1 - t_2| < \frac{1}{2}$. We plot the circle centered at the t_1 with radius $\delta = 2 \cdot |t_1 - t_2|$. This circle and γ_0 intersect at two points, which we will denote by a and b. Denote by l the part of γ_0 which is inside of this circle .

Represent the difference $(T\varphi)(t_1) - (T\varphi)(t_2)$ as follows:

$$(T\varphi)(t_1) - (T\varphi)(t_2) = \frac{1}{\pi i}\int_{l} \frac{\varphi(\tau) - \varphi(t_1)}{|\tau - t_1|} d\tau - \frac{1}{\pi i}\int_{l} \frac{\varphi(\tau) - \varphi(t_2)}{|\tau - t_2|} d\tau +$$

$$+\frac{1}{\pi i}\int_{\gamma_0\setminus l} \left\{ \frac{\varphi(\tau) - \varphi(t_1)}{|\tau - t_1|} - \frac{\varphi(\tau) - \varphi(t_2)}{|\tau - t_2|} \right\} d\tau =$$

$$= \frac{1}{\pi i}\int_l \frac{\varphi(\tau)-\varphi(t_1)}{|\tau-t_1|}d\tau - \frac{1}{\pi i}\int_l \frac{\varphi(\tau)-\varphi(t_2)}{|\tau-t_2|}d\tau + \frac{1}{\pi i}\int_{\gamma_0\backslash l} \frac{\varphi(t_2)-\varphi(t_1)}{|\tau-t_1|}d\tau +$$

$$+\frac{1}{\pi i}\int_{\gamma_0\backslash l}[\varphi(\tau)-\varphi(t_2)]\left[\frac{1}{|\tau-t_1|}-\frac{1}{|\tau-t_2|}\right]d\tau = J_1+J_2+J_3+J_4. \qquad (8)$$

From the condition $\varphi \in H_\alpha(\gamma_0)$, $\delta = 2\cdot|t_1-t_2|$ we have the following estimate

$$|J_1| \le \frac{1}{\pi}\int_l \frac{|\varphi(\tau)-\varphi(t_1)|}{|\tau-t_1|}|d\tau| \le \frac{H(\varphi;\alpha)}{\pi}\int_l \frac{|d\tau|}{|\tau-t_1|^{1-\alpha}} \le$$

$$\le \frac{2H(\varphi;\alpha)}{\pi}\int_0^\delta \frac{dr}{(r/2)^{1-\alpha}} = \frac{4H(\varphi;\alpha)}{\pi\alpha}\cdot|t_1-t_2|^\alpha \le \frac{4\|\varphi\|_\alpha}{2^\varepsilon\pi\alpha}\cdot|t_1-t_2|^{\alpha-\varepsilon}.$$

Absolutely analogously

$$|J_2| \le \frac{1}{\pi}\int_l \frac{|\varphi(\tau)-\varphi(t_2)|}{|\tau-t_2|}|d\tau| \le \frac{H(\varphi;\alpha)}{\pi}\int_l \frac{|d\tau|}{|\tau-t_2|^{1-\alpha}} \le$$

$$\le \frac{2H(\varphi;\alpha)}{\pi}\int_0^{3\delta/2} \frac{dr}{(r/2)^{1-\alpha}} = \frac{6H(\varphi;\alpha)}{\pi\alpha}\cdot|t_1-t_2|^\alpha \le \frac{6\|\varphi\|_\alpha}{2^\varepsilon\pi\alpha}\cdot|t_1-t_2|^{\alpha-\varepsilon}.$$

We estimate the integral J_3as follows:

$$|J_3| \le \frac{|\varphi(t_2)-\varphi(t_1)|}{\pi}\left|\int_{\gamma_0\backslash l}\frac{d\tau}{|\tau-t_1|}\right| \le \frac{H(\varphi;\alpha)\cdot|t_1-t_2|^\alpha}{\pi}\left|\int_{\gamma_0\backslash l}\frac{d\tau}{|\tau-t_1|}\right| \le$$

$$\le 4H(\varphi;\alpha)\cdot|t_1-t_2|^\alpha\cdot\ln\frac{\pi}{|t_1-t_2|} \le C_2\cdot\|\varphi\|_\alpha\cdot|t_1-t_2|^{\alpha-\varepsilon},$$

where C_2 – constant which only depends on α and ε.

Now turn to the estimate of the integral J_4.

$$|J_4| = \frac{1}{\pi}\left|\int_{\gamma_0\backslash l}\frac{[\varphi(\tau)-\varphi(t_2)]\cdot[|\tau-t_2|-|\tau-t_1|]}{|\tau-t_1|\cdot|\tau-t_2|}d\tau\right| \le$$

$$\le \frac{H(\varphi;\alpha)}{\pi}\left|\int_{\gamma_0\backslash l}\frac{|\tau-t_2|-|\tau-t_1|}{|\tau-t_1|\cdot|\tau-t_2|^{1-\alpha}}d\tau\right| \le \frac{H(\varphi;\alpha)\cdot|t_1-t_2|}{\pi}\int_{\gamma_0\backslash l}\frac{|d\tau|}{|\tau-t_1|\cdot|\tau-t_2|^{1-\alpha}} =$$

$$= \frac{H(\varphi;\alpha)\cdot|t_1-t_2|}{\pi}\int_{\gamma_0\backslash l}|\tau-t_1|^{\alpha-2}\left|\frac{\tau-t_1}{\tau-t_2}\right|^{\alpha-1}|d\tau|.$$

Since for any $\tau \in \gamma_0\backslash l$, the following inequality is holds

$$|\tau-t_1| \le \frac{1}{3}|\tau-t_2|,$$

then we have

$$|J_4| \leq \frac{H(\varphi;\alpha)\cdot|t_1-t_2|}{3^{1-\alpha}\pi}\int_{\gamma_0\setminus l}|\tau-t_1|^{\alpha-2}|d\tau| \leq C_3\cdot\|\varphi\|_\alpha\cdot|t_1-t_2|^{\alpha-\varepsilon},$$

where C_3 - constant which only depends on α and ε.

Comparing obtained estimates for $J_1,\ J_2,\ J_3$ andJ_4, from equation (8) and inequality (7) it follows the validity of the theorem. This completes the proof of the theorem.

3. Approximation of hypersingular integral operator

Consider the sequences of operators

$$\left(S_n^{(0)}\varphi\right)(t)=\frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|}\Delta\tau_{2k+1}^{(t)}\ +(\ln 4-2)\,2it\varphi(t)\,,\ t\in\gamma_0,\quad n=1,2,..,$$

where $\tau_k^{(t)}=e^{k\theta i}\cdot t,\ \Delta\tau_k^{(t)}=\left(\tau_{k+1}^{(t)}-\tau_{k-1}^{(t)}\right)\frac{\theta}{\sin\theta}=2ie^{k\theta i}\cdot t\cdot\theta\,,\ k=\overline{0,2n}\,,\ \theta=\frac{\pi}{n}.$

It is clear that, operators $S_n^{(0)}$, $n=1,2,..$ is bounded from the space $H_\alpha(\gamma_0)$ into the space $H_\alpha(\gamma_0)$ for all $0<\alpha\leq 1$.

Theorem 2. *For any $\varphi\in H_\alpha(\gamma_0)$, $0<\alpha\leq 1$, the following estimate holds*

$$\left\|S_n^{(0)}\varphi-S^{(0)}\varphi\right\|_\infty\leq\frac{C_4\ln(n+1)}{n^\alpha}\cdot H(\varphi;\alpha)\,,n=1,2,..,\tag{9}$$

where C_4 – constant which only depends on α.

Proof. From equation (5) it follows that, for all $t\in\gamma_0$

$$\left|\left(S_n^{(0)}\varphi\right)(t)-\left(S^{(0)}\varphi\right)(t)\right|=\left|\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau-\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|}\Delta\tau_{2k+1}^{(t)}\right|\leq$$

$$\leq\frac{1}{\pi}\sum_{k=0}^{n-1}\left|\int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}}\frac{\varphi(\tau)-\varphi(t)}{|\tau-t|}d\tau-\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|}\Delta\tau_{2k+1}^{(t)}\right|=\frac{1}{\pi}\sum_{k=0}^{n-1}I_k\,.\tag{10}$$

Estimate the difference I_k, $k=\overline{0,\,n-1}$. For the difference I_0we have

$$I_0\leq\int_{\tau_0^{(t)}\tau_2^{(t)}}\frac{|\varphi(\tau)-\varphi(t)|}{|\tau-t|}|d\tau|+\frac{\left|\varphi\left(\tau_1^{(t)}\right)-\varphi(t)\right|}{\left|\tau_1^{(t)}-t\right|}\left|\Delta\tau_1^{(t)}\right|\leq$$

$$\leq H(\varphi;\alpha)\cdot\left[\int_{\tau_0^{(t)}\tau_2^{(t)}}\frac{|d\tau|}{|\tau-t|^{1-\alpha}}+\frac{\left|\Delta\tau_1^{(t)}\right|}{\left|\tau_1^{(t)}-t\right|^{1-\alpha}}\right]\leq\frac{C_5}{n^\alpha}\cdot H(\varphi;\alpha)\,,$$

where C_5 – constant which depends on α. Analogously it follows that,

$$I_{n-1} \leq \frac{C_5}{n^\alpha} \cdot H(\varphi;\alpha).$$

For I_k, $k=\overline{1, n-2}$ we have

$$I_k \leq \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{|\tau-t|} d\tau - \frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left|\tau_{2k+1}^{(t)}-t\right|} \Delta\tau_{2k+1}^{(t)} \right| +$$

$$+ \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\varphi(\tau)-\varphi\left(\tau_{2k+1}^{(t)}\right)}{|\tau-t|} d\tau \right| = I_k^{(1)} + I_k^{(2)}.$$

Estimate for the difference $I_k^{(1)}$ as follows:

$$I_k^{(1)} \leq \left|\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)\right| \cdot \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{d\tau}{|\tau-t|} - \frac{\Delta\tau_{2k+1}^{(t)}}{\left|\tau_{2k+1}^{(t)}-t\right|} \right| \leq$$

$$\leq H(\varphi;\alpha) \cdot \left|\tau_{2k+1}^{(t)}-t\right|^\alpha \cdot \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \left[\frac{1}{|\tau-t|} - \frac{1}{\left|\tau_{2k+1}^{(t)}-t\right|} \right] d\tau \right| \leq$$

$$\leq H(\varphi;\alpha) \cdot \left|\tau_{2k+1}^{(t)}-t\right|^{\alpha-1} \cdot \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\left|\left|\tau_{2k+1}^{(t)}-t\right|-|\tau-t|\right|}{|\tau-t|} |d\tau|. \tag{11}$$

Since for all $\tau \in \tau_{2k}^{(t)}\tau_{2k+2}^{(t)}$ the following inequality holds

$$\left|\left|\tau_{2k+1}^{(t)}-t\right|-|\tau-t|\right| \leq \left|\tau-\tau_{2k+1}^{(t)}\right| \leq \left|\tau_{2k}^{(t)}-\tau_{2k+1}^{(t)}\right| = 2\sin\frac{\theta}{2} \leq \theta = \frac{\pi}{n}, |\tau-t| \geq \frac{1}{2}\left|\tau_{2k+1}^{(t)}-t\right|,$$

then from inequality (11) we get the following estimate:

$$I_k^{(1)} \leq \frac{\pi^2}{2n^2} H(\varphi;\alpha) \cdot \left|\tau_{2k+1}^{(t)}-t\right|^{\alpha-2}.$$

Now turn to the estimate of the integral $I_k^{(2)}$.

$$I_k^{(2)} \leq \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\varphi(\tau)-\varphi\left(\tau_{2k+1}^{(t)}\right)}{|\tau-t|} d\tau \right| \leq H(\varphi;\alpha) \cdot \left| \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{\left|\tau-\tau_{2k+1}^{(t)}\right|^\alpha}{|\tau-t|} d\tau \right| \leq$$

$$\leq H(\varphi;\alpha) \cdot \left|\tau_{2k}^{(t)}-\tau_{2k+1}^{(t)}\right|^\alpha \cdot \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{|d\tau|}{|\tau-t|} \leq H(\varphi;\alpha) \cdot \frac{\pi^\alpha}{n^\alpha} \cdot \int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}} \frac{|d\tau|}{|\tau-t|}.$$

Comparing obtained estimates for I_k, $k=\overline{0,\, n-1}$, from inequality (10) it follows the following inequality:

$$\left|\left(S_n^{(0)}\varphi\right)(t)-\left(S^{(0)}\varphi\right)(t)\right|\le$$

$$\le\frac{H(\varphi;\alpha)}{\pi}\left[\frac{2C_5}{n^\alpha}+\sum_{k=1}^{n-2}\left(\frac{\pi^2}{2n^2}\left|\tau_{2k+1}^{(t)}-t\right|^{\alpha-2}+\frac{\pi^\alpha}{n^\alpha}\int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}}\frac{|d\tau|}{|\tau-t|}\right)\right]. \qquad (12)$$

Since

$$\sum_{k=1}^{n-2}\left|\tau_{2k+1}^{(t)}-t\right|^{\alpha-2}=\sum_{k=1}^{n-2}\left|2\sin\frac{(2k+1)\pi}{2n}\right|^{\alpha-2}\le 2\sum_{k=1}^{n}\left|\frac{2(2k+1)}{n}\right|^{\alpha-2}\le\frac{C_6}{n^{\alpha-2}},$$

$$\sum_{k=1}^{n-2}\int_{\tau_{2k}^{(t)}\tau_{2k+2}^{(t)}}\frac{|d\tau|}{|\tau-t|}=\int_{\gamma_0\setminus\left(\tau_{-2}^{(t)}\tau_{2}^{(t)}\right)}\frac{|d\tau|}{|\tau-t|}\le C_7\ln(n+1),$$

then from inequality (12) it follows the estimate (9). This completes the proof of the theorem.

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