Caspian Journal of Applied Mathematics, Ecology and Economics
V. 5, No 2, 2017, December
ISSN 1560-4055

On Basicity of Perturbed Exponential System in Generalized Lebesgue Spaces

M.I. Aleskerov*, X.M. Gadirova

Abstract. In the present work it is considered perturbed system of exponential functions with piecewise continuous phase. Special cases of this system form systems of eigenfunctios of model first order discontinuous ordinary differential operators. Sufficient conditions on the jumps of phase function, which guarantee the basisness of the system in generalized Lebesgue spaces are provided.

Key Words and Phrases: system of exponentials, basisness, variable exponent, generalized Lebesgue space.

2010 Mathematics Subject Classifications: 33B10; 46E30

1. Introduction

When solving the PDEs of mixed type by Fourier method there frequently appear systems of sines and cosines of the following form

$$\{\cos (n+\alpha)\, t\}_{n\in Z_+}\,, \tag{1}$$

$$\{\sin (n+\alpha)\, t\}_{n\in N}\,, \tag{2}$$

where α is a real number (here, thereafter N is the set of all natural numbers, $Z_+ = \{0\}\bigcup N$). Justification of the Fourier method requires to study the basicity properties of such systems in some function spaces. Some examples of such equations and concrete systems of trigonometric-type functions that appear after applying Fourier method can be found, for example, in [1, 2, 3, 4]. The basicity properties of the systems (1) and (2) are well studied in Lebesgue and Sobolev spaces, as well as, in their weighted settings [5, 6, 7, 8, 9, 10, 11, 12, 27, 28, 29, 30, 31].

During the last two decades, non-standard function spaces became an extremely popular subject because of their appearance in modern problems of analysis and qualitative theory of PDEs. Introduction of Lebesgue spaces with variable exponents at the end of last century and variety of extraordinary results obtained therein were the main motivation

*Corresponding author.

and the inception of this new tendency in analysis. We only mention the monograph [13] and the comprehensive bibliography therein, where thoroughly treatment of these issues can be found.

In the present work it is considered the perturbed I system of exponentials with a piecewise continuous phase. Particular cases of these systems are eigenfunctions of model, discontinuous, ordinary differential operators of the first order. Sufficient conditions are obtained for phase jumps, in the course of which this system forms a basis in generalized Lebesgue spaces.

Notice that, similar problems for the double system of exponents with complex-valued coefficients in Lebesgue spaces with variable exponent were earlier studied in [15, 16, 17, 18, 19]. The basicity properties of the systems (1) and (2) in classical Lebesgue spaces were studied in [20, 24].

2. Preliminaries

We use the following standard denotations: $Z-$the set of all integers; $R-$the set of all real numbers; $C-$complex plane; $(\overline{\cdot})-$complex conjugate of $(\,.\,)$; $\delta_{nk}-$Kronecker delta; $\chi_A(\cdot)-$the indicator function of the set A. $\omega \equiv \{z \in C : |z| < 1\}-$ the unit disc; $\partial\omega \equiv \{z \in C : |z| = 1\}-$ the unit circle.

Let $p : [-\pi, \pi] \to [1, +\infty)-$be a Lebesgue measurable function. We denote by $\mathscr{L}_0$ the set of all Lebesgue measurable functions on $[-\pi, \pi]$. Set

$$I_p(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt$$

and

$$\mathscr{L} \equiv \{f \in \mathscr{L}_0 : I_p(f) < +\infty\}.$$

If $p^+ = \sup\limits_{[-\pi,\pi]} vrai\, p(t) < +\infty$, then $\mathscr{L}$ is a linear space with respect to pointwise linear operations. $\mathscr{L}$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

and we denote it by $L_{p(\cdot)}$. Set

$$WL \overset{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow$$

$$\Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout the paper $q(\cdot)$ denotes the conjugate function of $p(\cdot)$, that is, $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let $p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$. The following generalized Hölder's inequality holds

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

To get the main results we will need the following facts concerning the basicity in generalized Lebesgue spaces $L_{p(\cdot)}(0,\pi)$ of the following single-folded exponentional system, which were obtained in [32]:

$$v_n(t) \equiv a(t)e^{int} - b(t)e^{-int}, \;\; n \in N,$$

where $a(t) = |a(t)|\, e^{i\alpha(t)}$, $b(t) = |b(t)|\, e^{i\beta(t)}$are some complex-valued functions on $[0,\pi]$. It will be assumed that the functions $a(\cdot)$ and $b(\cdot)$ are subjected to the following conditions i)-iv):

i) $a^{\pm 1}(\cdot)$; $b^{\pm 1}(\cdot) \in L_\infty(0,\pi)$;

ii) $\alpha(\cdot)$; $\beta(\cdot)$ are piecewise continuous functions on $(0,\pi)$, with $\{t_k\}_{k\in N}$ and $\{\tau_k\}_{k\in N}$ as their jump points, respectively. Assume that the set $\{\tilde{s}_k\} \equiv \{t_k\} \bigcup \{\tau_k\}$ may have just one limit point $\tilde{s}_0 \in (0,\pi)$ and the function $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ has a finite left and right limits at the point $\tilde{s}_0$.

iii) $\sum_{k=1}^{\infty} |h(\tilde{s}_k)| < +\infty$, where $h(\tilde{s}_k) = \tilde{\theta}(\tilde{s}_k - 0) - \tilde{\theta}(\tilde{s}_k + 0)$ is the jump of the function $\tilde{\theta}(\cdot)$ at point $\tilde{s}_k$.

iv) The jumps $\left\{\tilde{h}_i\right\}$ satisfy $\left(\frac{\tilde{h}(\tilde{s}_i)}{2\pi} + \frac{1}{p(\tilde{s}_i)}\right) \notin \mathrm{Z}, \forall i \in N$.

From iii) it follows that there exists $r \in \mathrm{N}$, such that

$$-\frac{2\pi}{p(\tilde{s}_k)} < \tilde{h}(\tilde{s}_k) < \frac{2\pi}{q(\tilde{s}_k)}, k = \overline{r,\infty}.$$

Enumerate the elements of the set $\{\tilde{s}_i\}_1^r$ in increasing order and denote it by $\{s_i\}_1^r$:$0 < s_1 < ... < s_r < \pi$. Denote the jumps corresponding to them by $\{h(s_i)\}_1^r$:

$$h(s_i) = \beta(s_i - 0) - \beta(s_i + 0) + \alpha(s_i + 0) - \alpha(s_i - 0), i = \overline{1,r}.$$

Assume that for some n_0 it follows

$$\frac{1}{p(0)} + 2(n_0 - 1) < \frac{\beta(0) - \alpha(0)}{\pi} < \frac{1}{p(0)} + 2n_0. \tag{3}$$

By iv) define the integers $n_i, i = \overline{1,r}$ as follows

$$-\frac{1}{p(s_i)} < \frac{h(s_i)}{2\pi} + n_i - n_{i-1} < \frac{1}{q(s_i)}, i = \overline{1,r}. \tag{4}$$

We have the following main result:

Theorem 1. *Let the coefficient functions $a(\cdot)$ and $b(\cdot)$ satisfy i)-iv), the integers $\{n_i\}_1^r$ are defined as in* (3), (4). *Assume that*

$$\frac{\beta(\pi) - \alpha(\pi)}{2\pi} + \frac{1}{2p(\pi)} \notin Z. \tag{5}$$

If

$$-\frac{1}{p(\pi)}+2n_r<\frac{\beta(\pi)-\alpha(\pi)}{\pi}<-\frac{1}{p(\pi)}+2(n_r+1), \tag{6}$$

then the system $\{v_n\}_{n\in N}$ *forms a basis in* $L_{p(\cdot)}(0,\pi)$. *If*

$$\beta(\pi)-\alpha(\pi)<-\frac{\pi}{p(\pi)}+2n_r\pi,$$

then the system $\{v_n\}_{n\in \mathrm{N}}$ *is not complete but minimal in* $L_{p(\cdot)}(0,\pi)$*; If*

$$\beta(\pi)-\alpha(\pi)>-\frac{\pi}{p(\pi)}+2(n_r+1)\pi,$$

then the system $\{v_n\}_{n\in \mathrm{N}}$ *is complete but is not minimal in* $L_{p(\cdot)}(0,\pi)$.

3. Main Results

Consider the following system of exponentials:

$$\varphi_n(\theta)\equiv \exp\left[i\left(n\theta-sgnn\alpha(\theta)\right)\right],\quad n=\pm1,\pm2,..., \tag{7}$$

where $\alpha(\theta)$ is a piecewise continuous odd function on $[-\pi,\pi]$, that is $\alpha(-\theta)=-\alpha(\theta)$, $\forall\theta\in[-\pi,\pi]$. Let the set $\{t_k\}_1^\infty$ is the set of jump points of the function $\alpha(\theta)$ on $(0,\pi)$, which may have just one limit point $t_0\in(0,\pi)$. Assume that the function $\alpha(\theta)$ has finite left and right limits att_0. Furthermore, let

$$\sum_{k=1}^{\infty}\left|\alpha(t_k+0)-\alpha(t_k-0)\right|<+\infty. \tag{8}$$

Assume that

$$\frac{\alpha(t_i-0)-\alpha(t_i+0)}{\pi}\neq-\frac{1}{p(t_i)}+k,\qquad i=\overline{1,\infty}, \tag{9}$$

for any integer k.

Let for some integer n_0 it follows

$$\frac{\pi}{2p(0)}+\left(n_0-\frac{1}{2}\right)\pi<\alpha(0)<\frac{\pi}{2p(0)}+n_0\pi. \tag{10}$$

Denote by r the integer, for which

$$-\frac{\pi}{p(t_k)}<\alpha(t_k-0)-\alpha(t_k+0)<\frac{\pi}{q(t_k)},\ k=\overline{r,\infty}. \tag{11}$$

Enumerate the elements of the set $\{t_i\}$, $i=\overline{1,r}$ in increasing order and denote the new set by $\{t_i\}_1^r:\ 0<t_1<...<t_r<\pi$. Define the integers n_i, $i=\overline{1,r}$ as follows:

$$-\frac{1}{p(t_i)}<\frac{\alpha(t_i-0)-\alpha(t_i+0)}{\pi}+n_i-n_{i-1}<\frac{1}{q(t_i)},\quad i=\overline{1,r}. \tag{12}$$

Theorem 2. *Let $\alpha(t)$ be a real, piecewise continuous, odd function on $[-\pi,\pi]$, of which jumps satisfy* (8)-(10). *The integers n_i, $i=\overline{1,r}$ are defined as in* (10) - (12). *In addition, let*

$$\alpha(\pi) \neq -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi.$$

Then to be a basis of the system of exponentials (7) *in $L_{p(\cdot)}(-\pi,\pi)$ it is sufficient that*

$$-\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi < \alpha(\pi) < -\frac{\pi}{2p(\pi)} + (n_r+1)\pi. \tag{13}$$

If $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$ then the system (7) *is not complete, but minimal in $L_{p(\cdot)}(-\pi,\pi)$; if $\alpha(\pi) \geq -\frac{\pi}{2p(\pi)} + (n_r+1)\pi$ then the system* (7) *is complete but is not minimal in $L_{p(\cdot)}(-\pi,\pi)$.*

Before proving the theorem we give some direct consequences of Theorem 1.

Let $\alpha(\cdot)$ be a piecewise continuous function on $[0,\pi]$ of which jumps satisfies the conditions (8), (9).

Corollary 1. *Let for some integer n_0*

$$\frac{\pi}{2p(0)} + (n_0-1)\pi < \alpha(0) < \frac{\pi}{2p(0)} + n_0\pi, \tag{14}$$

holds, the integer n_r is defined as in (14), (12) , *and it is assumed that $\alpha(\pi) \neq -\frac{\pi}{2p(\pi)} + n_r\pi$. If*

$$-\frac{\pi}{2p(\pi)} + n_r\pi < \alpha(\pi) < -\frac{\pi}{2p(\pi)} + (n_r+1)\pi,$$

then the system $\sin(nt-\alpha(t))$, $n=\overline{1,\infty}$, forms a basis in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + n_r\pi$, then the system $\sin(nt-\alpha(t))$, $n=\overline{1,\infty}$ is not complete, but minimal in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi) \geq -\frac{\pi}{2p(\pi)} + (n_r+1)\pi$, then it is complete, but is not minimal in $L_{p(\cdot)}(0,\pi)$.

For the case of cosine system we have the following

Corollary 2. *Let $\alpha(\cdot)$ be a piecewise continuous function on $[0,\pi]$ of which jumps satisfy* (8), (9) , *and for some $n_0 \in Z$ it holds*

$$\frac{\pi}{2p(0)} + \left(n_0 - \frac{1}{2}\right)\pi < \alpha(0) < \frac{\pi}{2p(0)} + \left(n_0 + \frac{1}{2}\right)\pi. \tag{15}$$

The integer n_r is defined as in (15), (12) *and it is assumed that $\alpha(\pi) \neq -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$. If*

$$-\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi < \alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{3}{2}\right)\pi,$$

then the system $\cos(nt-\alpha(t))$, $n=\overline{1,\infty}$, forms a basis in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$, then the system $\cos(nt-\alpha(t))$, $n=\overline{1,\infty}$ is not complete but minimal in $L_{p(\cdot)}(0,\pi)$; if $\alpha(\pi) > -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{3}{2}\right)\pi$, then it is complete but is not minimal in $L_{p(\cdot)}(0,\pi)$.

Proof of Theorem 3.1. Let us prove the sufficiency. First of all let us show that the system of exponentials (7) is complete in $L_{p(\cdot)}(-\pi,\pi)$ under the conditions of Theorem 2. Assume the contrary. Then there exists a nonzero function $f(\theta) \in L_{q(\cdot)}(-\pi,\pi)$, $\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}=1$, such that

$$\int_{-\pi}^{\pi} f(\theta)\exp\left[i\left(n\theta - sgnn\alpha(\theta)\right)\right]d\theta = 0, \quad n=\pm 1,\pm 2,... \tag{16}$$

From here we have

$$\begin{array}{l}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta + i\int_{-\pi}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0\,,\\ \int_{-\pi}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta - i\int_{-\pi}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0\,.\end{array}$$

By summing up we get

$$\begin{array}{l}0=\int_{-\pi}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta = \int_{0}^{\pi} f(\theta)\cos(n\theta-\alpha(\theta))\,d\theta +\\ +\int_{0}^{\pi} f(-\theta)\cos(-n\theta-\alpha(-\theta))\,d\theta = \int_{0}^{\pi}\left[f(\theta)+f(-\theta)\right]\cos(n\theta-\alpha(\theta))\,d\theta, \quad n=\overline{1,\infty}.\end{array}$$

Since under (13), as it follows from Corollary 2, the cosine system is complete in $L_{p(\cdot)}(0,\pi)$, we get that

$$f(\theta) = -f(-\theta)\,.$$

Since the function $f(\theta)$ is odd , by (16) we have

$$\begin{array}{l}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0, \quad n=\overline{1,\infty}\,,\\ \int_{0}^{\pi} f(\theta)\sin(n\theta-\alpha(\theta))\,d\theta = 0, \quad n=\overline{1,\infty}\,.\end{array}$$

Under the condition (13), as it follows from Corollary 1, the sine system is complete in $L_{p(\cdot)}(0,\pi)$, from here we get that $f(\theta)\equiv 0$, which proves the completeness of the system of exponentials (7) in $L_{p(\cdot)}(-\pi,\pi)$.

Under the conditions of Theorem 2, as it follows from Corollary 1 and 2, the system $\sin(nt-\alpha(t))$ and $\cos(nt-\alpha(t))$ $n=\overline{1,\infty}$, forms a basis in $L_{p(\cdot)}(0,\pi)$. Let $h_m^s(t)$ and $h_m^c(t)$, $m=\overline{1,\infty}$, are biorthogonal with these systems, respectively:

$$\begin{array}{l}\int_{0}^{\pi}\sin(nt-\alpha(t))\,h_m^s(t)\,dt = \delta_{nm}\,,\\ \int_{0}^{\pi}\cos(nt-\alpha(t))\,h_m^c(t)\,dt = \delta_{nm},\end{array}$$

$n,m=\overline{1,\infty}$, δ_{nm} is Kronecker delta. Define the following system of functions:

$$h_n(\theta) = \frac{1}{4}\left[\hat{h}_{|n|}^c(\theta) - isgnn\hat{h}_{|n|}^s(\theta)\right]\,, \quad n=\pm 1,\pm 2,... \tag{17}$$

where

$$\hat{h}_{|n|}^c(\theta) = \left\{\begin{array}{ll}\hat{h}_{|n|}^c(\theta)\,, & \theta\in(0,\pi)\,,\\ \hat{h}_{|n|}^c(-\theta)\,, & \theta\in(-\pi,0)\,,\end{array}\right.$$

$$\hat{h}_{|n|}^s(\theta) = \left\{\begin{array}{ll}h_{|n|}^s(\theta)\,, & \theta\in(0,\pi)\,,\\ -h_{|n|}^s(-\theta)\,, & \theta\in(-\pi,0)\,.\end{array}\right.$$

Now we show that this system is biorthogonal with the system (7). Indeed, we have

$$(\varphi_m, h_n) =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\left[\hat{h}^c_{|n|}(\theta) - isgnn\hat{h}^s_{|n|}(\theta)\right]\left[\cos\left(m\theta - sgnm\alpha(\theta)\right) + i\sin\left(m\theta - sgnm\alpha(\theta)\right)\right]d\theta =$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\cos\left(m\theta - sgnm\alpha(\theta)\right)\hat{h}^c_{|n|}(\theta)\,d\theta - \frac{i}{4}sgnn\int_{-\pi}^{\pi}\cos\left(m\theta - sgnm\alpha(\theta)\right)\hat{h}^s_{|n|}(\theta)\,d\theta +$$

$$+\frac{i}{4}\int_{-\pi}^{\pi}\sin\left(m\theta - sgnm\alpha(\theta)\right)\hat{h}^c_{|n|}(\theta)\,d\theta + \frac{sgnn}{4}\int_{-\pi}^{\pi}\sin\left(m\theta - sgnm\alpha(\theta)\right)\hat{h}^s_{|n|}(\theta)\,d\theta =$$

$$= I_1(n,m) + I_2(n,m) + I_3(n,m) + I_4(n,m).$$

Since $\hat{h}^c_{|n|}(\theta)$, $\cos(m\theta - \alpha(\theta))$ are even functions, and $\hat{h}^c_{|n|}(\theta)$, $\sin(m\theta - \alpha(\theta))$ are odd functions on $(-\pi, \pi)$, from here we get that $I_2(n,m) = I_3(n,m) = 0, \quad n, m = \pm 1, \pm 2, ...$ So, we have

$$(\varphi_m, h_n) = I_1(n,m) + I_4(n,m) = \frac{1}{2}\int_0^{\pi}\cos\left(m\theta - sgnm\alpha(\theta)\right)h^c_{|n|}(\theta)\,d\theta +$$

$$+\frac{sgnn}{2}\int_0^{\pi}\sin\left(m\theta - sgnm\alpha(\theta)\right)h^s_{|n|}(\theta)\,d\theta = \delta_{nm}.$$

It is clear that $h_n(\theta) \in L_{q(\cdot)}(-\pi, \pi)$, $\quad n = \pm 1, \pm 2, ...$ Hence the minimality of the system (7) was proved.

Now take any function $\psi(t) \in L_{p(\cdot)}(-\pi, \pi)$. Consider the following series:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\int_{-\pi}^{\pi}\psi(t)\,h_n(t)\,dt e^{i[n\theta - sgnn\alpha(\theta)]}. \tag{18}$$

We show that this series converges to the function $\psi(t)$ in $L_{p(\cdot)}(-\pi, \pi)$. Let $S_N(\theta)$ be truncated sum of the series (18). Then

$$\|\psi(\theta) - S_N(\theta)\|_{L_{p(\cdot)}} = \left\|\psi(\theta) - \sum_{\substack{n=-N \\ n\neq 0}}^{N}\int_{-\pi}^{\pi}\psi(t)\,h_n(t)\,dt\exp\left[i\left(n\theta - sgnn\alpha(\theta)\right)\right]\right\|_{L_{p(\cdot)}} =$$

$$= \left\|\psi(\theta) - \sum_{\substack{n=-N \\ n\neq 0}}^{N}\frac{1}{4}\left\{\int_{-\pi}^{\pi}\psi(t)\,\hat{h}^c_{|n|}(t)\,dt\,\cos\left(n\theta - sgnn\alpha(\theta)\right) + \right.\right.$$

$$+i\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{|n|}^{c}(t)\,dt\sin(n\theta-sgnn\alpha(\theta))-isgnn\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{|n|}^{s}(t)\,dt\cos(n\theta-sgnn\alpha(\theta))+$$

$$+sgnn\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{|n|}^{s}(t)\,dt\sin(n\theta-sgnn\alpha(\theta))\Big\}\Big\|_{L_{p(\cdot)}}=$$

$$=\Bigg\|\psi(t)-\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{1}{4}\left\{\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{|n|}^{c}(t)\,dt\,\cos(n\theta-sgnn\alpha(\theta))+\right.$$

$$+sgnn\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{|n|}^{s}(t)\,dt\sin(n\theta-sgnn\alpha(\theta))\Big\}\Big\|_{L_{p(\cdot)}}=$$

$$=\Bigg\|\psi(\theta)-\sum_{n=1}^{N}\frac{1}{2}\left\{\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{n}^{c}(t)\,dt\,\cos(n\theta-\alpha(\theta))+\right.$$

$$+\int_{-\pi}^{\pi}\psi(t)\,\hat{h}_{n}^{s}(t)\,dt\sin(n\theta-\alpha(\theta))\Big\}\Big\|_{L_{p(\cdot)}}=\Bigg\|\psi(\theta)-\frac{1}{2}\psi(-\theta)+\frac{1}{2}\psi(-\theta)-$$

$$-\sum_{n=1}^{N}\frac{1}{2}\left\{\int_{0}^{\pi}(\psi(t)+\psi(-t))\,h_{n}^{c}(t)\,dt\,\cos(n\theta-\alpha(\theta))+\right.$$

$$+\int_{0}^{\pi}(\psi(t)+\psi(-t))\,h_{|n|}^{s}(t)\,dt\,\sin(n\theta-\alpha(\theta))\Big\}\Big\|_{L_{p(\cdot)}}\leq$$

$$\leq\left\|\frac{1}{2}(\psi(\theta)+\psi(-\theta))-\sum_{n=1}^{N}\int_{0}^{\pi}\frac{1}{2}(\psi(t)+\psi(-t))\,h_{n}^{c}(t)\,dt\cos(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(-\pi,\pi)}+$$

$$+\left\|\frac{1}{2}(\psi(\theta)-\psi(-\theta))-\sum_{n=1}^{N}\int_{0}^{\pi}\frac{1}{2}(\psi(t)-\psi(-t))\,h_{n}^{s}(t)\,dt\sin(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(-\pi,\pi)}=$$

$$=\left\|\psi(\theta)+\psi(-\theta)-\sum_{n=1}^{N}\int_{0}^{\pi}(\psi(t)+\psi(-t))\,h_{n}^{c}(t)\,dt\cos(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(0,\pi)}+$$

$$+\left\|\psi(\theta)-\psi(-\theta)-\sum_{n=1}^{N}\int_{0}^{\pi}(\psi(t)-\psi(-t))\,h_{n}^{s}(t)\,dt\sin(n\theta-\alpha(\theta))\right\|_{L_{p(\cdot)}(0,\pi)}\to 0,$$

as $N\to\infty$.

It proves that the series (18) converges to the function $\psi(\theta)$ in $L_{p(\cdot)}(-\pi,\pi)$. Hence, under the condition (13) the system (7) forms a basis in $L_{p(\cdot)}(-\pi,\pi)$.

Now let us prove the necessity part of the theorem. Consider the case $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$. As in this case the system $\cos(n\theta - \alpha(\theta))$ $n = \overline{1,\infty}$, is not complete in $L_{p(\cdot)}(0,\pi)$, there exists a nontrivial function $\psi(\theta) \in L_{q(\cdot)}(0,\pi)$, $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$, such that

$$\int_0^\pi \psi(\theta)\cos(n\theta - \alpha(\theta))\,d\theta = 0, \quad n = \overline{1,\infty}.$$

Introduce the following function:

$$f(\theta) = \begin{cases} \psi(\theta), & \theta \in (0,\pi), \\ \psi(-\theta), & \theta \in (-\pi, 0). \end{cases}$$

Since $\cos(nt - \alpha(t))$, is odd function on $(-\pi,\pi)$, but $f(\theta)$ is even, we get that

$$\int_{-\pi}^{\pi} f(\theta)\, e^{i[n\theta - sgnn\alpha(\theta)]}d\theta = 0, \ \ n = \pm 1, \pm 2, ...,$$

which shows that the system (7) is not complete in $L_{p(\cdot)}(-\pi,\pi)$.

Now, consider the case of $\alpha(\pi) > -\frac{\pi}{2p(\pi)} + (n_r + 1)\pi$. In this case by Corollary 1 the system $\sin(n\theta - \alpha(\theta))$, $n = \overline{1,\infty}$, is not minimal. We show that the system of exponentials (7) is not minimal as well. If it is not, there is a system of functions $h_n(\theta) \in L_q(-\pi,\pi)$, $n = \pm 1, \pm 2, ...$, such that

$$\int_{-\pi}^{\pi} h_m(\theta)\, e^{i(n\theta - sgnn\alpha(\theta))}d\theta = \delta_{nm}, \ \ n, m = \pm 1, \pm 2, ...$$

Define the following system of functions:

$$h_n^s(\theta) = \frac{2}{i}\left[h_n(-\theta) - h_n(\theta)\right], \quad n \ge 1.$$

We have for any integers $n, m \ge 1$:

$$I(n,m) = \int_0^\pi h_m^s(\theta)(n\theta - \alpha(\theta))\,d\theta = -\frac{1}{2i}\int_0^\pi h_m^s(\theta)\, e^{-i(n\theta - \alpha(\theta))}d\theta +$$

$$+\frac{1}{2i}\int_0^\pi h_m^s(\theta)\exp\left[i(n\theta - \alpha(\theta))\right]d\theta = -\frac{1}{2i}\int_{-\pi}^0 h_m^s(-\theta)\, e^{-i(n\theta - \alpha(-\theta))}d\theta +$$

$$+\frac{1}{2i}\int_0^\pi h_m^s(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta = \int_{-\pi}^0 h_m(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta - \int_{-\pi}^0 h_m(-\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta -$$

$$-\int_0^\pi h_m(-\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta + \int_0^\pi h_m(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta = \int_{-\pi}^\pi h_m(\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta -$$

$$-\int_0^\pi h_m(-\theta)\, e^{i(n\theta - \alpha(\theta))}d\theta = \delta_{nm} + \int_{-\pi}^\pi h_m(\theta)\cdot e^{-i(n\theta - \alpha(\theta))}d\theta = \delta_{nm}.$$

Hence we got a contradiction. This completes the proof of Theorem 2.

Acknowledgments

The authors would like to express their deep gratitude to Corresponding member of NAS of Azerbaijan, prof. Bilal T. Bilalov for his attention to this work.

References

[1] S.M. Ponomarev, *On the theory of boundary value problems for equations of mixed type in three-dimensional domains*, Dokl. Akad. Nauk SSSR, **246(6)**, 1979, 1303-1304.

[2] E.I. Moiseev, *On some boundary value problems for mixed type equations*, Diff Uravn., **28(1)**, 1992, 123-132.

[3] E.I. Moiseev, *On solution of Frankle's problem in special domain*, Diff. Uravn., **28(4)**, 1992, 682-692.

[4] E.I. Moiseev, *On existence and uniqueness of solution a classical problem*, Dokl. RAN, **336(4)**, 1994, 448–450.

[5] A.M. Sedletskii, *Biorthogonal expansions in series of exponents on intervals of real axis*, Usp. Mat. Nauk, **37(5)**, 1982, 51-95.

[6] E.I. Moiseev, *On basicity of systems of sines and cosines*, DAN SSSR, **275(4)**, 1984, 794–798.

[7] E.I. Moiseev, *On basicity of a system of sines*, Diff. Uravn., **23(1)**, 1987, 177–179.

[8] B.T. Bilalov, *Basicity of some systems of exponents, cosines and sines*, Diff. Uravn., **26(1)**, 1990, 10–16.

[9] B.T. Bilalov, *Basis properties of some systems of exponents, cosines and sines*, Sibirskiy Matem. Jurnal, **45(2)**, 2004, 264–273.

[10] E.I. Moiseev, *On basicity of systems of sines and cosines in a weighted space*, Diff. Uravn., **34(1)**, 1998, 40-44.

[11] E.I. Moiseev, *Basicity of a system of eigenfunctions of the differential operator in the weighted space*, Diff. Uravn., **35(2)**, 1999, 200-205.

[12] S.S. Pukhov, A.M. Sedletskii, *Bases of exponents, sines and cosines in weight spaces on finite interval*, Dokl. RAN, **425(4)**, 2009, 452-455.

[13] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Springer, 2013.

[14] I.I. Sharapudinov, *Some problems of approximation theory in spaces* $L^{p(x)}(E)$, Anal. Math., **33(2)**, 2007, 135–153.

[15] B.T. Bilalov, Z.G. Guseynov, *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math., **9(3)**, 2012, 487–498.

[16] B.T. Bilalov, Z.G. Guseynov, *Basicity criterion for perturbed systems of exponents in Lebesgue spaces with variable summability*, Dokl. RAN, **436(5)**, 2011, 586–589.

[17] B.T. Bilalov, Z.G. Guseynov, *Bases from exponents in Lebesgue spaces of functions with variable summability exponent*, Trans. of NAS of Az., **XXVIII(1)**, 2008, 43-48.

[18] B.T. Bilalov, Z.G. Guseynov, *On the basicity from exponents in Lebesgue spaces with variable exponents*, TWMS J. Pure Appl. Math., **1(1)**, 2010, 14-23.

[19] T.I. Najafov, N.P. Nasibova, *On the Noetherness of the Riemann problem in a generalized weighted Hardy classes*, Azerb. J. of Math., **5(2)**, 2015, 109-139.

[20] B.T. Bilalov, *On uniform convergence of series with regard to some system of sines*, Diff. uravn., **24(1)**, 1988, 175-177.

[21] B.T. Bilalov, *Basicity of some systems of functions*, Diff. uravn., **25**, 1989, 163-164.

[22] G.G. Devdariani, *On basicity of a system of functions*, Diff. Uravn., **22(1)**, 1986, 170-171.

[23] G.G. Devdariani, *On basicity of a trigonometric system of functions*, Diff. Uravn., **22(1)**, 1986, 168-170.

[24] G.G. Devdariani, *Basicity of a system of sines*, Proc. of I.N. Venua Institute of Applied Mathematics, **19**, 1987, 21-27.

[25] I.I. Danilyuk, *Irregular boundary value problems in the plane*, Nauka, Moscow 1975.

[26] B.T. Bilalov, F.I. Mamedov, R.A. Bandaliev, *On classes of harmonic functions with variable exponent*, Dokl NAN Azerb., **LXIII(5)**, 2007, 16–21.

[27] B.T. Bilalov, *Necessary and sufficient condition for completeness of some system of functions*, Diff. Uravn., **27(1)**, 1991, 158-161.

[28] B.T. Bilalov, *Completeness and minimality of some trigonometric systems*, Diff. Uravn., **28(1)**, 1992, 170-173.

[29] B.T. Bilalov, *Basis properties of eigen functions of some not self-adjoint differential operators*, Diff. Uravn., **30(1)**, 1994, 16-21.

[30] B.T. Bilalov, *On Bases for Some Systems of Exponentials, Cosines and Sines in L_p*, Doklady Mathematics, **379(2)**, 2001, 7-9.

[31] B.T. Bilalov, *Bases of Exponentials, Sines and Cosines*, Diff. Uravn., **39(5)**, 2003, 619-622.

[32] B.T. Bilalov, A.A. Huseynli, M.I. Aleskerov, *On the basicity of unitary system of exponents in the variable exponent Lebesgue spaces*, Transactions of NAS of Azerbaijan, Issue Mathematics, **37(1)**, 2017, 63-76.

Miran I. Aleskerov
Ganja State University, Baku, Azerbaijan
E-mail: miran.alesgerov@mail.ru

Xayala M. Gadirova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9. B.Vahabzade St, Az1141, Baku, Azerbaijan
E-mail: memmedagaqizi@gmail.com

Received 15 August 2017
Accepted 24 October 2017