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Interpolation Theorems for Lizorkin-Triebel-Morrey type Spaces with Many Groups Variables

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Abstract. In this paper, we introduce a new function space $F^{l^{\varrho}}_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau}(G,s)$ with the parameters of many groups of variables of type Lizorkin-Triebel-Morrey. In view of interpolation theorems we study some properties of functions, which are belonging to intersection of these spaces.
Key Words and Phrases: intersection of spaces Lizorkin-Triebel-Morrey type, many groups of variables, integral representation, interpolation theorems.

1. Introduction

In this paper we study interpolation theorems for space

$$F^{l}_{p,\theta,a,\varkappa,\tau}(G,s), \tag{1}$$

that is, with help of theory embedding we study some characterization of function which are belonging to intersection of space $F^{l^{\varrho}}_{p_{\varrho},\theta_{\varrho},a,\varkappa,\tau}(G,s)$ $(\varrho = 1,2,\ldots,N)$, that is, the space Lizorkin-Triebel-Morrey type with many group variables.

Let $G \subset R^n$ be a domain and $1 \le s \le n$; s, n be naturals, in addition $e_n = \{1, 2, ..., n\}$, $n_1 + ... + n_s = n$. Hence we suppose the sufficient smooth function f(x), where the points $x = (x_1, ..., x_s) \in R^n$ have coordinates $x_k = (x_{k.1}; ...; x_{k,n_k}) \in R^{n_k}$ $(k \in e_s = \{1, ..., s\})$. Consequently, $R^n = R^{n_1} \times R^{n_2} \times \cdots \times R^{n_s}$.

Let $l = (l_1, ..., l_s)$ be a given positive vector such that, $l_k = (l_{k.1}; ...; l_{k,n_k})$, $(k \subset e_s)$, that is, $l_{k,j} > 0$, $(j = 1, ..., n_k)$ for every $k \in e_s$ and we shall denote by Q the set of vectors $i = (i_1, ..., i_s)$, where $i_k = 1, 2, ..., n_k$ for all $k \in e_s$. The number of the set Q is equal to: $|Q| = \prod_{k=1}^{s}(1 + n_k)$.

Therefore, to the vector $i = (i_1, ..., i_s) \in Q$, we let correspond the vector $l^i = \left(l_1^{i_1}; ...; l_s^{i_s}\right)$, where vectors $l^i = \left(l_1^{i_1}; ...; l_s^{i_s}\right)$ are coordinates of $l = (l_1, ..., l_s)$ and $l^0 = (0, 0, ..., 0)$, $l_k^1 = (l_{k,1}, 0, ..., 0), ..., l_k^{i_k} = (0, 0, .., l_{k,n_k})$ for all $k \in e_s$. And the the vectors e^i, we correspond the vector $\bar{l}^i = \left(\bar{l}_1^{i_1}, \bar{l}_2^{i_2}, ..., \bar{l}_s^{i_s}\right)$, where $\bar{l}_k^{i_k} = \left(\bar{l}_{k,1}^{i_1}, \bar{l}_{k,2}^{i_2}, ..., \bar{l}_{k,n_k}^{i_k}\right)$ $(k \in e_s)$, and

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the largest number $\overline{l}^{i_k}_{k,j}$ is less than $l^{i_k}_{k,j}$ for every $l^{i_k}_{k,j}$ >0,when $l^{i_k}_{k,j}$=0 then we assume that $\overline{l}^{i_k}_{k,j}=0$ for each $k \in e_s$.

Let $R^{|e^i|}=R^{|e^{i_1}|}\times R^{|e^{i_2}|}\times\cdots\times R^{|e^{i_s}|}$, where $R^{|e^{i_k}|}=R^{\left|e^{i_{k,1}}\right|}\times R^{\left|e^{i_{k,2}}\right|}\times\cdots\times R^{\left|e^{i_{k,n_k}}\right|}$.

Further for every $k\in e_s$, $R^{|e^{i_k}|}=\{t_k=(t_{k,1},...,t_{k,n_k})\in R^{n_k}, t_{k,j}\in R^{k,n_k}, t_{k,j}=0, \forall j\notin e^{i_k}=\sup p\overline{l}^{i_k}\ ,\ k\in e_s\}$.

Definition 1. *We denote by $F^{<l>}_{p,\theta,a,\varkappa,\tau}(s,G)$ normed Lizorkin-Triebel-Morrey space of function f on G, with many groups variables, with finite norm*

$$\|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau}}(G,s)=\sum_{i\in Q}\|f\|_{L^{<l^i>}_{p,\theta,a,\varkappa,\tau}(G)}, \tag{2}$$

$$\|f\|_{L^{<l^i>}_{p,\theta,a,\varkappa,\tau}(G)}=\left\|\left\{\int_0^{t^i_{0,1}}\int_0^{t^i_{0,s}}\left[\frac{\Delta^{2\omega}(t,G)\,D^{\overline{l}^i}f}{\prod_{k\in e^i}t_k^{\left|\beta^{i_k}_k\right|}}\right]^{\theta}\prod_{k\in e^i}\frac{dt_k}{t_k}\right\}^{1/\theta}\right\|_{p,a,\varkappa,\tau}, \tag{3}$$

and

$$\|f\|_{p,a,\varkappa,\tau:\ G}=sup_{x\in G}\left\{\int_0^{\infty}\cdots\int_0^{\infty}\left[\prod_{k\in e_s}[t_k]_1^{\frac{-|\varkappa_k|a}{p}}\|f\|_{p,G_{t^{\varkappa}}(x)}\right]^{\tau}\prod_{k\in e_s}\frac{dt_k}{t_k}\right\}^{1/\tau}, \tag{4}$$

Further it means that, $D^{\overline{l}^i}f=D_1^{\overline{l}^{i_1}_1}\cdots D_s^{\overline{l}^{i_s}_s}f$, $D_s^{\overline{l}^{i_k}_k}f=D_{k,1}^{\overline{l}^{i_k}_k}\cdots D_{k,n_k}^{\overline{l}^{i_k}_k}f$; $G_{t^{\varkappa}}(x)=G\cap I_{t^{\varkappa}}(x)$; $I_{t^{\varkappa}}(x)=I_{t_1^{\varkappa_1}}(x_1)\times I_{t_2^{\varkappa_2}}(x_2)\times\cdots\times I_{t_s^{\varkappa_s}}(x_s)$; $I_{t_k^{\varkappa_k}}(x_k)=$

$\left\{y_k:|y_k-x_k|<\frac{1}{2}t_k^{|\varkappa_k|},\,k\in e_s\right\}$, $|\beta_k|=\sum_{j=1}^{n_k}\beta^{i_k}_{k,j}$; $\frac{dt_k}{t_k}=\prod_{j\in e^i_k}\frac{dt_{k,j}}{t_{k,j}}$, where $0<\beta^{i_k}_{k,j}=l^{i_k}_{k,j}-\overline{l}^{i_k}_{k,j}\le 1$ for $l^{i_k}_k>0$, but when $l^{i_k}_{k,j}=0$, $\beta^{i_k}_{k,j}=0$; $t=(t_1,...,t_s)$, $t_k=(t_{k,1},...,t_{k,n_k})$, $\omega=(\omega_1,...,\omega_s)$, $\omega_k=(\omega_{k,\,1},...,\omega_{k,n_k})$ and in addition $\omega_{k,j}=1$ or $\omega_{k,\,j}=0$, $k\in e_s$, $e^i=\sup p\overline{l}^i=\sup p\omega$, $1<\theta<\infty$; $(1\le p<\infty)$; $t_0=(t_{0,1},...,t_{0,s})$, $t_{0,k}=(t_{0,k,1},...,t_{0,k,n_k})$ be a fixed vector and $\varkappa\in(0,\infty)^n$, $a\in[0,1]$, $\tau\in[1,\ \infty]$, $[t_k]_1=\min\{1,t_k\}$, $k\in e_s$.

When $s=1$ then space (1) is equivalent to the space Lizorkin-Triebel-Morrey type $F^{<l>}_{p,\theta,a,\varkappa,\tau}(G)$, which was investigated in [1, 4, 9], when s=n then the space (1) is equivalent to the space Lizorkin-Triebel-Morrey type with mixed derivatives, $S^{<l>}_{p,\theta,a,\varkappa,\tau}F(G)$ which was studied in [5, 6], when $a=0,\tau=\infty$ $s=1$, $N=1$, then this space is equivalent to the space $F^l_{p,\theta}(G)$, which was developed in [2, 13, 14].

Similarly results for the Morrey spaces was investigated in [3, 12, 13].

It is clear, that $V(\sigma)\subset I_{T^{\sigma}}, U-$ is an open set, which belonging to the domain G and $U+V\subset G$. Here it is said that, the subdomain $U\subset G\subset R^n$ calls domain satisfying the condition "$\sigma-semi-horn''$, if the vector $\sigma=(\sigma_1,...,\sigma_s)$ is such that, $x+V(\sigma)\subset G$ for all $x\subset U$. It is said that, the domain $G\subset E_n$ satisfying the condition "$\sigma-semi-horn''$, that is, $G \subset A(T^{\sigma})$, if we have finite sub domains $G_1, ..., G_N \subset G$, satisfying the condition "$\sigma - semi - horn''$ and surfacing the domain G, that is,

$$G = \bigcup_{j=1}^{N} G_j. \tag{5}$$

But we suppose $G \in A_{\epsilon}\ (T^{\sigma})\ (\epsilon > 0)$, if we substitute the condition $G = \bigcup_{j=1}^{N} G_{j,\epsilon}$ in the condition (5). Note that $G_{j,\epsilon} = \{x : x \in G_j :\ \rho(x, G/G_j) > \epsilon\}$.

2. Preliminaries

Let $\Psi_i \in C_0^{\infty}(R^n)$ be such, that their carries belonging to $I_1 = \left\{x : |x_j| < \frac{1}{2}; j = 1, .., n_k\right\}$. Then we put

$$V(\sigma) = \bigcup_{\substack{0 < t_j \le T_j; \\ j \in e_n}} \left\{y : \left(\frac{y}{t^{\sigma}}\right) \in S(\Psi_i)\right\},$$

where $0 < T_j \le 1$, $j \in e_n$. U is an open set which belonging to the domain G. Furthermore we assume that $U + V \subset G$, for $T = (T_1, ..., T_s)$, $T_k = (T_{k,1}, ..., T_{k,n_k})$, $0 < T_{k,j} \le 1$, $k \in e_s, j = 1, .., n_k$, $(t^{\sigma} + T^{\sigma})^i = t_k^{\sigma_k}$, $(k \in e^i)$; $(t^{\sigma} + T^{\sigma})^i = T^{\sigma}$, $(k \in e_s/e^i)$, $\sigma = (\sigma_1, ..., \sigma_s)$, $\sigma_j > 0$, $j = 1, .., n_k$. Let $G_{(t^{\sigma}+T^{\sigma})^i}(U) = \left(U + I_{(t^{\sigma}+T^{\sigma})^i}(x)\right) \bigcap G = Z$, $p_{\varrho} = (p_{\varrho 1}, ..., p_{\varrho n})$, $q_{\varrho} = (q_{\varrho 1}, ..., q_{\varrho n})$, $\alpha_{\varrho} \ge 0$, $\sum_{\varrho=1}^{N} \alpha_{\varrho} = 1$, $\frac{1}{p} = \sum_{\varrho=1}^{N} \frac{\alpha_{\varrho}}{p_{\varrho}}$, $\frac{1}{q} = \sum_{\varrho=1}^{N} \frac{\alpha_{\varrho}}{q_{\varrho}}$, $\frac{1}{\theta} = \sum_{\varrho=1}^{N} \frac{\alpha_{\varrho}}{\theta_{\varrho}}$, $l = \sum_{\varrho=1}^{N} l^{\varrho} \alpha_{\varrho}$.

Lemma 1. *Let* $1 \le p_{\varrho} \le q_{\varrho} \le r_{\varrho} \le \infty; \varrho = 1, 2, .., N;\ 0 < |\varkappa_k| < |\sigma_k|; 0 \le \eta_{k,j} \le T_{k,j} \le 1; \eta = (\eta_1, ..., \eta_n), 0 < \eta_{k,j} \cdot t_{k,j} \le T_{k,j} \le 1;\ (k \in e_s,\ j = 1, 2, .., n_k)$, $1 \le \tau \le \infty$; $v = (v_1, ..., v_s)$, $v_{k,j} \ge 0$ *are integers,* $0 < \rho_{k,j} < \infty$; $j = 1, ..., n_k; k \in e_s$; *and* $\Delta^{2\omega}(t) D^{\bar{l}^i} f \in L_{p_{\varrho}, a, \varkappa, \tau}(G)$,

$$\mu_{k,i_k} = \sum_{\varrho=1}^{N} l_{k,i_k}^{\varrho} \alpha_{\varrho} \sigma_k - (v_k, \sigma_k) - (|\sigma_k| - |\varkappa_k| a) \left(\frac{1}{p} - \frac{1}{q}\right),$$

$$(v_k, \sigma_k) = \sum_{j=1}^{n_k} \sigma_{k,j} v_{k,j}, \quad |\sigma_k| = \sum_{j=1}^{n_k} \sigma_{k,j}, |\varkappa_k| = \sum_{j=1}^{n_k} \varkappa_{k,j},$$

$$F_{\eta}^i(x) = \prod_{k \in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k} \bar{l}_{k,i_k} - (v_k, \sigma_k)} \int_0^{\eta^i} \cdots \int_0^{\eta^i} \varphi_i(x, t,\ T)$$

$$\times \prod_{k \in e^i} \frac{dt_k}{t_k^{1 + |\sigma_k| - \sigma_{k,i_k} \bar{l}_{k,i_k} + (v_k, \sigma_k)}}, \tag{6}$$

$$F^i_{\eta T}(x) = \prod_{k\in e_s/e^i} T_k^{-|\sigma_k|+\sigma_{ki_k},\bar{l}_{k,i_k}-(v_k,\sigma_k)} \int_{\eta^i}^{T^i}\cdots\int_{\eta^i}^{T^i} \varphi_i(x,t,T)$$

$$\times \prod_{k\in e^i} \frac{dt_k}{t_k^{1+|\sigma_k|-\sigma_{k,i_k}\bar{l}_{k,i_k}+(v_k,\sigma_k)}} \qquad (7)$$

Here $\left|\beta^{\varrho}_k\right| = \sum_{j=1}^{n_k}\beta^{i_k,\ \varrho}_{k,j}$, $(v_k,\sigma_k)=\sum_{j=1}^{n_k}\sigma_{k,j}v_{k,j}$, $|\sigma_k|=\sum_{j=1}^{n_k}\sigma_{k,j}$, $|\varkappa_k|=\sum_{j=1}^{n_k}\varkappa_{k,j}$,

$$\varphi_i(x,t,T)$$

$$= \int_{R^{|e^i|}}\int_{R^n}\left\{\Delta^{2\omega}(u)\,D^{\bar{l}^i}f(x+y)\,\Psi_i^{(v)}\left(\frac{y}{(t^\sigma+T^\sigma)^i},\ \frac{u}{(t^\sigma+T^\sigma)^i}\right)\right\}dydu, \qquad (8)$$

where $\Psi_i\in C^\infty(R^n\times R^n)$, and $\Psi_i(\cdot,z)\in C_0^\infty$.

Then the following inequalities hold:

$$\sup_{\overline{x}\in U}\left\|F^i_\eta\right\|_{q,U_{\rho^\varkappa}(\overline{x})} \le C_1\prod_{\varrho=1}^{N}\left\{\left\|\prod_{k\epsilon e^i} t_k^{-|\beta_k{}^\varrho|}\Delta^{2\omega}(t)\,D^{\bar{l}^{i,\varrho}}f\right\|_{p_\varrho,a,\varkappa,\tau}\right\}^{\alpha_\varrho}$$

$$\times\prod_{k\in e_s}[\rho_k]_1^{\frac{|\varkappa_k|a}{p}}\prod_{k\in e_s/e^i}T_k^{\mu_{k,i_k}}\prod_{k\epsilon e^i}t_k^{\mu_{k,i_k}};(\mu_{k,i_k}>0), \qquad (9)$$

$$\sup_{\overline{x}\in U}\left\|F^i_{\eta T}\right\|_{q,U_{\rho^\varkappa}(\overline{x})} \le C_2\prod_{\varrho=1}^{N}\left(\left\|\prod_{k\epsilon e^i} t_k^{-|\beta_k{}^\varrho|}\Delta^{2\omega}(t)\,D^{\bar{l}^{i,\varrho}}f\right\|_{p_\varrho,a,\ \varkappa,\ \tau}\right)^{\alpha_\varrho}$$

$$\times\begin{cases}\prod_{k\epsilon e^i}T_k^{\mu_{k,i_k}}; & \mu_{k,i_k}>0,\\ \prod_{k\in e^i}ln\frac{T_k}{\eta_k}; & \mu_{k,i_k}=0,\\ \prod_{k\epsilon e^i}\eta_k^{\mu_{k,i_k}}; & \mu_{k,i_k}<0,\end{cases}\times\prod_{k\in e_s}[\rho_k]_1^{\frac{|\varkappa_k|a}{p}}. \qquad (10)$$

Where C_1, and C_2 are constants independent of f, ρ, η and T.

Proof. Using Minkowski's inequality for any $\overline{x}\in U$, we have:

$$\sup_{\overline{x}\in U}\left\|F^i_\eta\right\|_{q,U_{\rho^\varkappa}(\overline{x})} \le C\prod_{k\in e_s/e^i}T_k^{-|\sigma_k|+\sigma_{k,i_k}\bar{l}_{k,i_k}-(v_k,\sigma_k)}$$

$$\times\int_{0^i}^{\eta^i}\|\varphi_i(\cdot;t;T)\|_{q,U_{\rho^\varkappa}(\overline{x})}\prod_{k\in e^i}t_k^{-1+|\sigma_k|+\sigma_{k,i_k}\bar{l}_{k,i_k}-(v_k,\sigma_k)}dt_k \qquad (11)$$

We must estimate $\|\varphi_i(\cdot,t,\ T)\|_{q,U_{\rho^\varkappa(\overline{x})}}$ from the Holder's inequality $(q\le r)$ we get:

$$\|\varphi_i(\cdot,t,T)\|_{q,U_{\rho^\varkappa}(\overline{x})} \le C_1\left(\int_{U_{\rho^\varkappa}(\overline{x})}\prod_{\varrho=1}^{N}\{|\varphi_i(x\cdot,t,T)|\}^{\alpha_\varrho q}dx\right)^{1/q}.$$

Using the Holder's inequality into right part with the indication $\lambda_\varrho = \frac{q_\varrho}{q\alpha_\varrho}$, $\varrho = 1, 2, \dots, N$, $\left(\sum_{\varrho=1}^{N} \frac{1}{o_\varrho} = q \sum_{\varrho=1}^{N} \frac{\alpha_\varrho}{q_\varrho} = 1\right)$. Then we have

$$\|\varphi_i(\cdot, t)\|_{q_\varrho, U_{\rho^\varkappa}(\overline{x})} \le C_2 \prod_{\varrho=1}^{N} \left\{ \|\varphi_i(\cdot, t,\ T)\|_{r_\varrho, U_{\rho^\varkappa}(\overline{x})} \right\}^{\alpha_\varrho}. \tag{12}$$

Once again, using Holder's inequality ($q_\varrho \le r_\varrho$) we have

$$\|\varphi_i(\cdot, t)\|_{q_\varrho, U_{\rho^\varkappa}(\overline{x})} \le \|\varphi_i(\cdot, t, T)\|_{r_\varrho, U_{\rho^\varkappa}(\overline{x})}$$

$$\times \prod_{j \in e_s} \rho_j^{|\varkappa_j| \left(\frac{1}{q_\varrho} - \frac{1}{r_\varrho}\right)}. \tag{13}$$

Let X be a characterization function of the set $S(\Psi_i)$. Noting that, $1 \le p_\varrho \le r_\varrho \le \infty$; $s_\varrho \le r_\varrho$ $\left(\frac{1}{s_\varrho} = 1 - \frac{1}{p_\varrho} + \frac{1}{r_\varrho}\right)$ we get

$$\left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f \Psi_i\right| = \left(\left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f\right|^{p_\varrho} |\Psi_i|^{s_\varrho}\right)^{\frac{1}{r_\varrho}} \left(\left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f\right|^{p_\varrho} X\right)^{\frac{1}{p_\varrho} - \frac{1}{r_\varrho}} \left(|\Psi_i|^{s_\varrho}\right)^{\frac{1}{r_\varrho}}$$

and using for $|\varphi_i|$ Holder's inequality $\left(\frac{1}{r_\varrho} + \left(\frac{1}{p_\varrho} - \frac{1}{r_\varrho}\right) + \left(\frac{1}{s_\varrho} + \frac{1}{r_\varrho}\right) = 1\right)$, then we have

$$\|\varphi_i(\cdot, t, T)\|_{r_\varrho, U_{\rho^\varkappa}(\overline{x})} \le \sup_{x \in U_{\rho^\varkappa}(\overline{x})}$$

$$\left(\int_{R^{|e^i|}} \int_{R^n} \left|\Delta^{2\omega} D^{\overline{l}^{i,\varrho}} f(x+y)\right|^{p_\varrho} X\left(\frac{y}{(t^\sigma + T^\sigma)^i}\right) dudy\right)^{\frac{1}{p_\varrho} - \frac{1}{r_\varrho}}$$

$$\times \sup_{x \in V} \left(\int_{R^{|e^i|}} \int_{R^n} \left|\Delta^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\right|^{p_\varrho} dudy\right)^{\frac{1}{r_\varrho}}$$

$$\times \left(\int_{R^{|e^i|}} \int_{R^n} \left|\Psi_i\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i}\right)\right|^{s_\varrho} dudy\right)^{\frac{1}{s_\varrho}}. \tag{14}$$

Because of $U + V \subset Z$, and $Z_{(t^\sigma + T^\sigma)^i}(x) \subset Z_{(t^\varkappa + T^\varkappa)^i}(x)$, for all $x \in U$ and $0 < t_j \le T_j \le 1$, $|\varkappa_k| \le |\sigma_k|$, $k \in e_n$ we find:

$$\int_{R^n} \left|\int_{R^{|e^i|}} \Delta_u^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\, du\right|^{p_\varrho} X\left(\frac{y}{(t^\sigma + T^\sigma)^i}\right) dy$$

$$\le \int\limits_{Z_{(t^\sigma + T^\sigma)^i}(\overline{x})} \left|\int_{R^{|e^i|}} \Delta^{2\omega}(u) D^{\overline{l}^{i,\varrho}} f(x+y)\right|^{p_\varrho} dudy$$

$$\times \left\| \prod_{k\in e^i} t_k^{\left|\beta_k^{\varrho}\right|} \Delta^{2\omega}(t) D^{\bar{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\varkappa}^{p_\varrho} \prod_{k\in e^i} t_k^{|\varkappa_k|a} \prod_{k\in e_s/e^i} T_k^{|\varkappa_k|a}. \quad (15)$$

Next for $y \in V$

$$\int_{U_{\rho^\varkappa}(\bar{x})} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u) D^{\bar{l}^{i,\varrho}} f(x+y)\, du \right|^{p_\varrho} dx$$

$$\le \int_{Z_{\rho^\varkappa}(\bar{x}+y)} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u) D^{\bar{l}^{i,\varrho}} f(x)\, du \right|^{p_\varrho} dx$$

$$\le \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^{\varrho}\right|} \Delta^{2\omega}(t) D^{\bar{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\varkappa}^{p_\varrho}$$

$$\times \prod_{k\in e^i} t_k^{\left|\beta_k^{\varrho}\right| p_\varrho} \prod_{k\in e_s} [\rho_k]_1^{|\varkappa_k|a}, \quad (16)$$

$$\int_{R^{|e^i|}} \int_{R^n} \left| \Psi_i \left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i} \right) \right|^s dudy$$

$$= \prod_{k\in e^i} t_k^{|\sigma_k|} \prod_{k\in e_s/e^i} T_k^{|\sigma_k|} \|\Psi_i\|_{s_\varrho}^{s_\varrho}. \quad (17)$$

From (12)-(17) we get

$$\|\varphi_i(\cdot,t,T)\|_{q,U_{\rho^\varkappa}(\bar{x})} \le C \prod_{\varrho=1}^{N} \left\{ \left\| \prod_{k\epsilon e^i} t_k^{-|\beta_k{}^\varrho|} \Delta^{2\omega}(t) D^{\bar{l}^{i,\varrho}} f \right\|_{p_\varrho,a,\ \varkappa} \right\}^{\alpha_\varrho}$$

$$\times \prod_{k\in e_s/e^i} T^{|\sigma_k| - (|\sigma_k| - |\varkappa_k|a)\left(\frac{1}{p} - \frac{1}{q}\right)}{}_k \prod_{k\epsilon e^i} t_k^{|\sigma_k| - (|\sigma_k| - |\varkappa_k|a)\left(\frac{1}{p} - \frac{1}{q}\right)}$$

$$\times \prod_{k\in e_n} [\rho_k]_1^{\frac{|\varkappa_k|a}{r}} \prod_{k\in e_n} \rho_k^{|\varkappa_k|\left(\frac{1}{q} - \frac{1}{r}\right)}. \quad (18)$$

Taking consideration $\|\cdot\|_{p,a,\varkappa} \le \|\cdot\|_{p,a,\varkappa,\tau}$ for $1 \le \tau \le \infty$ and putting (18) into (11) for $r = q$, then we get the inequality (9). Similarly, we can prove the inequality (10). ◄

Lemma 2. *Let* $1 \le p_\varrho \le q_\varrho < \infty; \varrho = 1,2,..,N;\ 0 < \ |\varkappa_k| \le |\sigma_k|; 0 \le T_k \le 1; (k \in e_s, j = 1,2,..,n_k),\ \ 1 \le \tau_1 \le \tau_2 \le \infty;\ \ \mu_{k,i_k} > 0$ *and* $\Delta^{2\omega}(u) D^{\bar{l}^i} \in L_{p_\varrho,a,\varkappa,\tau}$

$$\mu_{k,i_k,0} = \sigma_{k,i_k} \sum_{\varrho=1}^{N} l_{k,i_k}^{\varrho} \alpha_{\varrho} - (v_k, \sigma_k) - (|\sigma_k| - |\varkappa_k|\, a) \frac{1}{p}.$$

Then the following inequality holds for the function $B^i_\eta(x)$:

$$\left\|F^i_\eta\right\|_{q,b,\varkappa,\tau_2;U} \leq$$

$$\times C^1 \prod_{\varrho=1}^{N} \left\{ \left\| \prod_{k\epsilon e^i} t_k^{-|\beta_k{}^\varrho|} \Delta^{2\omega}(t)\, D^{\bar{l}^i,\ \varrho} f \right\|_{p_\varrho,a,\ \varkappa,\tau_1} \right\}^{\alpha_\varrho}, \tag{19}$$

where b, is an arbitrary number satisfying the following condition:

$$\begin{gathered} 0 \leq b \leq 1, \ \text{ if}\, \mu_{k,i_k,0} > 0, \\ 0 \leq b < 1, \text{ if}\, \mu_{k,i_k,0} = 0, \\ 0 \leq b < 1 + \frac{\mu_{k,i_k,0}q\,(1-a)}{|\sigma_k| - |\varkappa_k|\,a}, \text{ if}\, \mu_{k,i_k,0} < 0. \end{gathered} \tag{20}$$

The proof of this lemma is similarly 1.

Using these facts, we can show the general theorems, which give us the structure of such space $F^{l\varrho}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,\ s)\ (\varrho = 1, 2, .., N)$.

3. Embedding theorems

Using these facts, we can show the general theorems, which give us the structure of such space $F^{l\varrho}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,\ s)\ (\varrho = 1, 2, .., N)$.

Theorem 1. *Let $G \in A(T^\sigma)$ be a domain , $1 \leq p_\varrho \leq q_\varrho \leq \infty$, $(\varrho = 1, 2, .., N)$; $v = (v_1, \ldots, v_n)$; $v_j \geq 0$ are integers, (j=1,2,.., n) and in addition*

1) $v_{k,j} \geq l^0_{k,j}\ \ (j = 1, 2, .., n_k; k \in e_s)$;

2) $v_{k,j} \geq l^{i_k}_{k,j} + 1$, $v_{k,i_k} < l^{i_k}_{k,i_k} + 1$, $0 < \varkappa_k < \sigma_k\ \ (k \in e_s)$; $1 \leq \tau_1 \leq \tau_2 \leq \infty$, $f \in \bigcap_{\varrho=1}^{N} F^{<l\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G, s)$ and let $\mu_{k,i_k} > 0$, $(i_k = 1, 2, ..., n_k, k \in e_s)$.

Then following inequality holds:

$$\|D^v f\|_{q,G} \leq C^1 B_1(T) \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,s)} \right\}^{\alpha_\varrho}, \tag{21}$$

$$\begin{gathered} \|D^v f\|_{p,b,\varkappa,\tau_2;G} \leq C^2 \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,\ s)} \right\}^{\alpha_\varrho}, \\ (p_{\varrho,j} \leq q_{\varrho,j} < \infty, j \in e_n). \end{gathered} \tag{22}$$

where $B_1(T) = \sum_{i\in Q} \prod_{k\in e_s} T_k^{\mu_{k,i_k}}$.

Particular, if $\mu_{k,i_k,0} > 0$, $(i_k = 1, 2, ..., n_k,\ k \in e_s)$ then the function $D^v f$ is continuous on G and

$$\sup_{x\in G} |D^v f| \leq C^3 {B_1}^0(T) \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,\ s)} \right\}^{\alpha_\varrho}, \tag{23}$$

where

$$B_1{}^0(T) = \sum_{i=(i_1,\ldots,i_s)\in Q} \prod_{j\in e_s} T_j^{\mu_{k,i_k},0},$$

and $T_k \in (0, \min(1, T_{0,k})]$, $(k \in e_s)$, $T_0 = (T_{0,1}, ..., T_{0,k})$ is a fixed positive vector, b is an arbitrary number satisfying condition (**??**), C^1 and C^2 C^1,C^3 are constants independent of f, and C^1 dependent of the vector T.

The proof of Theorem 1. Obviously, in this case for $f \in F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau}(G,s)$ generalized derivatives D^{vf} exit. It means that, if $\mu_{k,i_k} > 0 (k \in e_s)$, because of $p_\varrho \le q_\varrho, |\varkappa_k| < |\sigma_k| \, (k \in e_s)$, $a \in [0,1]^n$, $f \in F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau}(G,s) \to F^{<l^\varrho>}_{p_\varrho,\theta_\varrho}(G,s)$, $\varrho = 1,2,..,N)$

It means that, for almost every point of $x \in G$, there exits generalized derivatives D^{vf} with the same carries [3]:

$$D^v f(x) = \sum_{i=(i_1,\ldots,i_s)\in Q} (-1)^{\left|\bar{l}^i - v\right|} C_i$$

$$\prod_{k\in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k}\bar{l}_{k,i_k} - (v_k,\sigma_k)}$$

$$\times \int_0^{T_1^i} \cdots \int_0^{T_n^i} \prod_{k\in e^i} t_k^{-1-|\sigma_k| + \sigma_{k,i_k}\bar{l}_{k,i_k} - (v_k,\sigma_k)} dt_k$$

$$\times \int_{R^{|e^i|}} \int_{R^n} \{\Delta^{2\omega}(u)\, D^{\bar{l}^{i,\varrho}} f(x+y)$$

$$\times \Psi_i^{(v)}\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i}\right)\} \, dydu. \tag{24}$$

Using the Minkowski's inequality, then we have:

$$\|D^v f\|_{q,\ G} \le C_1 \sum_{i=(i_1,\ldots,i_s)\in Q} \left\|F_T^i\right\|_{q;G}. \tag{25}$$

From (10) for $U = G$, $\eta = T$, $\varrho \to \infty$ we get

$$\left\|F_T^i\right\|_{q;G} \le$$

$$\times C_2 \prod_{k\in e_s} T_k^{\mu_{k,i_k}} \prod_{\varrho=1}^{N} \left\{ \left\| \prod_{k\in e^i} t_k^{-|\beta_k{}^\varrho|} \Delta^{2\omega}(t)\, D^{\bar{l}^i,\varrho} f \right\|_{p_\varrho,a,\varkappa} \right\}^{\alpha_\varrho}.$$

Using it for (25), and taking consideration $p_\varrho \le \theta_\varrho$ and $1 < \theta_\varrho < \infty$, $\varrho = 1,2,..,N$, we get (21).

Using (19) we can proof (22).

Next we suppose $\mu_{k,i_k0} > 0$, $k \in e_s$. We must show that, the function D^{vf} is continuous on G. From (24) and (25) for $q_j \equiv \infty, j \in e_n$, $\mu_{k,i_k} = \mu_{k,i_k,0}$, $k \in e_s$ we have:

$$\|D^v f - D^v f_{T^\sigma}\|_{\infty,G} \leq \sum_{i \in Q} \prod_{k\in e_s/e^i} T_k^{\mu_{k,i_k}}$$

$$\times \prod_{\varrho=1}^{N} \left(\left\| \int_0^{h_0^i} \cdots \int_0^{h_{0n}^i} \left(\left(\prod_{k\in e^i} t_k^{-|\beta^{\varrho}{}_k|} \Delta^{2\omega} (\cdot) D^{\bar{l}^i} f \right)^{\theta_\varrho} \prod_{k\in e^i} \frac{dt_k}{t_k} \right)^{1/\theta_\varrho} \right\|_{p_\varrho,a,\varkappa,\tau} \right)^{\alpha_\varrho}.$$

$\lim_{T\to 0}\|D^v f - D^v f_{T^\sigma}\|_{\infty,G} = 0$. Because of $D^v f_{T^\sigma}$ is continuous on G, then convergence of $L_\infty(G)$ coincides with the absolutely convergence. Consequently, it is continuous on G. This completes the proof.

Let γ be a n dimensional vector.

Theorem 2. *Let all conditions of Theorem 1 be satisfied.In addition, $G \in A_\in(T^\sigma)$. Then for $\mu_{k,i_k} > 0$, ($i_k = 1, 2, ..., n_k$, $k \in e_s$) the derivative D^{vf} satisfies condition the Holder on the domain G, for metric L_q with indication ε. More precisely,*

$$\|\Delta(\gamma, G) D^v f\|_{q,G} \leq C \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\varkappa,\tau_1}(G,s)} \right\}^{\alpha_\varrho}$$

$$\times \prod_{k\in e^i} |\gamma_k|^{\varepsilon_k}, \tag{26}$$

where $\varepsilon = (\varepsilon_1, ..., \varepsilon_s)$, $\varepsilon_k = (\varepsilon_{k,1}, ..., \varepsilon_{k,n_k})$, and ε_k is an arbitrary number satisfying the condition:

$$0 < \varepsilon_k \leq 1, \; if \; \frac{\mu_{k,i_k}}{\sigma_0} > 1,$$

$$0 < \varepsilon_k < 1, \; if \; \frac{\mu_{k,i_k}}{\sigma_0} = 1,$$

$$0 < \varepsilon_k \leq \frac{\mu_{k,i_k}}{\sigma_0}, \; if \; \frac{\mu_{k,i_k}}{\sigma_0} < 1. \tag{27}$$

where $\mu_k = \min \mu_{k,i_k}$, $\sigma_0 = \max|\sigma_k|$ ($i_k = 1, 2, \ldots, n_k$, $k \in e_s$). If $\mu_{k,i_k,0} > 0$, ($i_k = 1, 2, \ldots, n_k$, $k \in e_s$) then

$$\sup_{x\in G} |\Delta(\gamma, G) D^v f(x)| \leq C \prod_{\varrho=1}^{N} \left\{ \|f\|_{F^{<l^\varrho>}_{p_\varrho,\theta_\varrho,a,\tau_1}(G,s)} \right\}^{\alpha_\varrho} \prod_{k\in e^i} |\gamma_k|^{\varepsilon_k{}^0}, \tag{28}$$

where ε_k^0 satisfies the same condition, but we must substitute $\mu_{k,i_k,0}$ into μ_k and C is a constant independent of f and γ.

The proof of this theorem is similarly 1.

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