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General Minkowski type and Related Inequalities for Semiconormed Fuzzy Integrals

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Abstract. A general Minkowski type inequalities for the semiconormed fuzzy integrals on abstract spaces are studied. Some related inequalities to this type one and Chebyshev type one for the semiconormed fuzzy integral are also discussed. Also a related inequalities for the Minkoski type inequality are obtained.

Key Words and Phrases: fuzzy measure, Sugeno integral, Minkowski's inequality, comonotone function, seminormed fuzzy integrals, semiconormed fuzzy integrals.

2010 Mathematics Subject Classifications: 03E72, 26E50, 28E10

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [15]. The properties and applications of the Sugeno integral have been studied by many authors, including Ralescu and Adams [9] in the study of several equivalent definitions of fuzzy integrals, Román-Flores et al. [13] and Wang and Klir [16], among others. Many authors generalized the Sugeno integral by using some other operators to replace the special operators $\wedge$ and/or $\vee$ [17, 3, 4]. In [14] Suárez and Gil presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals.

The study of inequalities for Sugeno integral was initiated by Román-flores et al. [10, 11, 12] and then followed by the authors [1, 2, 8]. Recently Ouyang et al. [1] proved a general Minkowski type inequality for comonotone functions and arbitrary fuzzy measure-based Sugeno integrals and then they provided the inverse of this inequality for the same conditions [8]. In [7], Chebyshev type inequality for seminormed fuzzy integrals and a related inequality for semiconormed fuzzy integral were proposed in a rather general form by Ouyang and Mesiar.

Theorem 1. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the seminorm T satisfies*

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$T(a \star b, c) \geq (T(a,c) \star b) \vee (a \star T(b,c))$,
then

$$\int_{T,A} f \star g d\mu \geq \int_{T,A} f d\mu \star \int_{T,A} g d\mu$$

holds for any $A \in \mathcal{F}$.

Theorem 2. *Let*$(X, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $f, g : X \to [0,1]$ *two comonotone measurable functions. Let* $\star : [0,1]^2 \to [0,1]$ *be continuous and nondecreasing in both arguments. If the semiconorm* S *satisfies*
$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c))$,
then

$$\int_{S,A} f \star g d\mu \leq \int_{S,A} f d\mu \star \int_{S,A} g d\mu$$

holds for any $A \in \mathcal{F}$.
This paper is organized as follows: In Section 2 some preliminaries and summarization of some previous known results are given. Section 3 proposes general Minkowski type inequalities for semiconormed fuzzy integrals. Section 4 includes a revers inequality for this type of integrals. Section 5 contains a short conclusion.

2. Preliminaries

In this section, we recall some basic definitions and previous results that will be used in the next sections. Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Throughout this paper, all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 1 (Sugeno [15])**.** *A set function* $\mu : \mathcal{F} \to [0,1]$ *is called a fuzzy measure if the following properties are satisfied:*
(FM1) $\mu(\emptyset) = 0$ *and* $\mu(X) = 1$
(FM2) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$
(FM3) $A_n \to A$ *implies* $\mu(A_n) \to \mu(A)$.

When μ is a fuzzy measure, the triple (X,$\mathcal{F}, \mu$) is called a fuzzy measure space.

Let (X,$\mathcal{F}, \mu$) be a fuzzy measure space, and $\mathcal{F}_+(X) = \{f | f : X \to [0,\ 1]$ is measurable with respect to $\mathcal{F}\}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. For any $\alpha \in [0,1]$, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α and $\{x \in X | f(x) > \alpha\}$ by $F_{\bar{\alpha}}$. Clearly, both F_α and $F_{\bar{\alpha}}$ are non-increasing with respect to α, i.e., $\alpha \leq \beta$ implies $F_\alpha \supseteqq F_\beta$ and $F_{\bar{\alpha}} \supseteqq F_{\bar{\beta}}$.

Definition 2 (Sugeno [15])**.** *Let (X,*$\mathcal{F}, \mu$*) be a fuzzy measure space and* $A \in \mathcal{F}$*. The Sugeno integral of* f *over* A *with respect to the fuzzy measure* μ*, is defined by*

$$⨍_A f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When A=X, then

$$\int_X f d\mu = \int f d\mu = \bigvee_{\alpha\in[0,1]} (\alpha \wedge \mu(F_\alpha)).$$

Notice that Ralescu and Adams (see [9]) extended the range of fuzzy measures and the Sugeno integrals from [0, 1] to [0, ∞]. But we only deal with the original fuzzy measure and the Sugeno integrals which was introduced by Sugeno in 1974.

Note that in the above definition, $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum. A t-conorm [6] is a function $S : [0,1] \times [0,1] \to [0,1]$ satisfying the following condition:

(S1) $S(x,\ 0) \ = \ S(0,\ x) \ = \ x \quad \forall x \in [0,\ 1]$.

(S2) $\forall x_1, x_2, y_1, y_2$ in [0, 1], if $x_1 \le x_2,\ y_1 \le y_2$ then $S(x_1,\ y_1) \le S(x_2,\ y_2)$.

(S3) $S(x,\ y) \ = \ S(y,\ x)$.

(S4) $S(S(x,\ y),\ z) \ = \ S(x,\ S(y,\ z))$.

A t-norm [6] is a function $T : [0,\ 1] \times [0,1] \to [0,\ 1]$ satisfying the following condition:

(T1) $T(x,\ 1) \ = \ T(1,\ x) \ = \ x \quad \forall x \in [0,\ 1]$.

(T2) $\forall x_1, x_2, y_1, y_2$ in [0, 1], if $x_1 \le x_2,\ y_1 \le y_2$ then $T(x_1,\ y_1) \le T(x_2,\ y_2)$.

(T3) $T(x,\ y) \ = \ T(y,\ x)$.

(T4) $T(T(x,\ y), z) \ = \ T(x,\ T(y,\ z))$.

A binary operator S (T) on [0, 1] is called a t-semiconorm (t-seminorm) [14] if it satisfies the above conditions (S1) and(S2) ((T1) and (T2)). Using the concepts of t-seminorm and t-semiconorm, Suárez and Gil proposed two families of fuzzy integrals:

Definition 3. *Let S be a t-semiconorm, then the semiconormed fuzzy integral of f over A with respect to S and the fuzzy measure μ is defined by*

$$\int_{S,A} f d\mu = \bigwedge_{\alpha\in[0,\ 1]} S(\alpha, \mu(A \cap F_{\bar{\alpha}})).$$

Definition 4. *Let T be a t-seminorm, then the seminormed fuzzy integral of f over A with respect to T and the fuzzy measure μ is defined by*

$$\int_{T,A} f d\mu = \bigvee_{\alpha\in[0,1]} T(\alpha, \mu(A \cap F_\alpha)).$$

It is easy to see that the Sugeno integral is a special seminormed fuzzy integral. Moreover, Kandel and Byatt (see [5]) showed another expression of the Sugeno integral as follows:

$$\int_A f d\mu = \bigwedge_{\alpha\in[0,\ 1]} (\alpha \vee \mu(A \cap F_{\bar{\alpha}})).$$

So the semiconormed fuzzy integrals also generalized the concept of the Sugeno integral. Note that if $\int_{S,A} f d\mu = a$, then $S(\alpha, \mu(A \cap F_{\bar{\alpha}})) \ge a$ for all $\alpha \in [0,\ 1]$ and, for $\varepsilon > 0$ there exists α_ε such that $S(\alpha_\varepsilon, \mu(A \cap F_{\bar{\alpha}_\varepsilon})) \le a + \varepsilon$.

In [1] Agahi et al. proved the following inequality for the Sugeno integral (with respect to a fuzzy measure in the sense of Ralescu and Adams [9]):

Theorem 3. *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $⨍_A f \star g d\mu$ is finite. Let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality*

$$\Big(⨍_A (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}}. \tag{1}$$

holds for all $0 < s < \infty$.

Theorem 4 (Ouyang et al. [8])**.** *Let $f, g \in \mathcal{F}_+(X)$ and μ be an arbitrary fuzzy measure such that $⨍_A f d\mu$ and $⨍_A g d\mu$ are finite. Let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$\Big(⨍_A (f \star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}}. \tag{2}$$

holds for all $0 < s < \infty$.
It should be pointed out that Inequalities (1) *and* (2) *also hold for the original Sugeno integral.*

3. Minkowski type inequality

In this section, we prove the Minkowski type inequality for the semiconormed fuzzy integrals.

Theorem 5. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments. If the semiconorm S satisfies*

$$S(a \star b, c) \leq (S(a,c) \star b) \wedge (a \star S(b,c)), \tag{3}$$

then the inequality

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}} \tag{4}$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Let $\int_{S,A} f^s d\mu = a$ and $\int_{S,A} g^s d\mu = b$, then for any $\varepsilon > 0$, there exist a_ε and b_ε such that $\mu(A \cap F_{\overline{(a_\varepsilon)^{\frac{1}{s}}}}) = a_1$ and $\mu(A \cap G_{\overline{(b_\varepsilon)^{\frac{1}{s}}}}) = b_1$, where $S(a_\varepsilon, a_1) \leq a + \varepsilon$ and $S(b_\varepsilon, b_1) \leq b + \varepsilon$. The fact of $H_{\overline{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}} \subset F_{\overline{(a_\varepsilon)^{\frac{1}{s}}}} \cup G_{\overline{(b_\varepsilon)^{\frac{1}{s}}}}$ and the comonotonicity of f, g imply that $\mu(A \cap H_{\overline{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}}) \leq a_1 \vee b_1$, where $F_{\overline{(a_\varepsilon)^{\frac{1}{s}}}} = \{x \mid f(x) > a_\varepsilon^{\frac{1}{s}}\} = \{x \mid$

$f^s(x) > a_\varepsilon\}$, $G_{\overline{(b_\varepsilon)^{\frac{1}{s}}}} = \{x \mid g(x) > b_\varepsilon^{\frac{1}{s}}\} = \{x \mid g^s(x) > b_\varepsilon\}$ and $H_{\overline{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}} = \{x \mid f \star g > a_\varepsilon^{\frac{1}{s}} \star b_\varepsilon^{\frac{1}{s}}\} = \{x \mid (f \star g)^s(x) > a_\varepsilon \star b_\varepsilon\}$. Hence

$$\begin{aligned}
\int_{S,A} (f \star g)^s d\mu &= \inf_{\alpha \in [0,1]} S(\alpha, \mu(A \cap H_{\overline{\alpha^{\frac{1}{s}}}})) \\
&\leq S(a_\varepsilon \star b_\varepsilon, a_1 \vee b_1) \\
&= S(a_\varepsilon \star b_\varepsilon, a_1) \vee S(a_\varepsilon \star b_\varepsilon, b_1) \\
&\leq [S(a_\varepsilon, a_1) \star b_\varepsilon] \vee [a_\varepsilon \star S(b_\varepsilon, b_1)] \\
&\leq [(a+\varepsilon) \star b_\varepsilon] \vee [a_\varepsilon \star (b+\varepsilon)] \\
&\leq (a+\varepsilon) \star (b+\varepsilon),
\end{aligned}$$

whence $\int_{S,A}(f \star g)^s d\mu \leq (a^{\frac{1}{s}} \star b^{\frac{1}{s}})^s$ follows from the continuity of $\star$ and the arbitrariness of ε. It follows that

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \leq a^{\frac{1}{s}} \star b^{\frac{1}{s}} = \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}}.$$

Example 1. *Let $X = [0,1]$ and the fuzzy measure μ be the Lebesgue measure. Let $\star$ and S be defined as $\star(x,y) = x + y - xy$ and $S(x,y) = x + y - xy$. Let $f(x) = x$, $g(x) = \frac{1}{2}$ and $s = 2$. A straightforward calculus shows that*

$$\begin{aligned}
\int_{S,X} f^2 d\mu = \int_{S,X} x^2 d\mu &= \inf_{\alpha \in [0,1]} S(\alpha, \mu(\{x \in [0,1] \mid x^2 > \alpha\})) \\
&= \inf_{\alpha \in [0,1]} S(\alpha, (1 - \sqrt{\alpha})) \\
&= 0.6151.
\end{aligned}$$

Also we have

$$\begin{aligned}
\int_{S,X} g^2 d\mu = \int_{S,X} \frac{1}{4} d\mu &= \inf_{\alpha \in [0,1]} S(\alpha, \mu([0, \frac{1}{4}))) \\
&= \inf_{\alpha \in [0,1]} S(\alpha, \frac{1}{4}) \\
&= 0.25
\end{aligned}$$

$$\begin{aligned}
\int_{S,X} (f \star g)^2 d\mu &= \int_{S,X} \frac{1}{4}(x+1)^2 d\mu \\
&= \inf_{\alpha \in [\frac{1}{4},1]} S(\alpha, \mu(\{x \in [0,1] \mid \frac{1}{4}(x+1)^2 > \alpha\})) \\
&= \inf_{\alpha \in [\frac{1}{4},1]} S(\alpha, \mu(\{x \in [0,1] \mid x > 2\sqrt{\alpha} - 1\}))
\end{aligned}$$

$$= \inf_{\alpha\in[\frac{1}{4},1]} S(\alpha, 2-2\sqrt{\alpha})$$
$$= 0.780145.$$

Therefore,

$$0.883259 = \Big(\int_{S,X} (f\star g)^2 d\mu\Big)^{\frac{1}{2}} \le \Big(\int_{S,X} f^2 d\mu\Big)^{\frac{1}{2}} \star \Big(\int_{S,X} g^2 d\mu\Big)^{\frac{1}{2}}$$
$$= 0.784283 \star 0.5 = 0.8921415$$

Notice that if the semiconormed S is maximum (i.e. for the sugeno integral) and $\star$ is bounded from below by maximum. Then S dominated by $\star$. Thus the following result holds.

Corollary 1. *Let $f, g : X \to [0,1]$ be two comonotone measurable functions. And let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments and bounded from below by maximum. Then the inequality*

$$\Big(⨍_A (f\star g)^s d\mu\Big)^{\frac{1}{s}} \le \Big(⨍_A f^s d\mu\Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu\Big)^{\frac{1}{s}}.$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Corollary 2 (Ouyang and Mesiar [7])**.** *Let$(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments. If the semiconorm S satisfies*

$$S(a\star b, c) \le (S(a,c)\star b) \wedge (a \star S(b,c))$$

then

$$\int_{S,A} f\star g d\mu \le \Big(\int_{S,A} f d\mu\Big) \star \Big(\int_{S,A} g d\mu\Big)$$

for any $A \in \mathcal{F}$.

Theorem 6. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measures space and $f, g : X \to [0,1]$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and non-decreasing in both arguments and $\varphi : [0,1] \to [0,1]$ be a continuous and strictly increasing function such that φ commutes whit $\star$. If the semiconorm S satisfies*

$$S(a\star b, c) \le (S(a,c)\star b) \wedge (a \star S(b,c)),$$

then

$$\varphi^{-1}\big(\int_{S,A} \varphi(f\star g) d\mu\big) \le \varphi^{-1}\big(\int_{S,A} \varphi(f) d\mu\big) \star\ \varphi^{-1}\big(\int_{S,A} \varphi(g) d\mu\big) \tag{5}$$

holds for any $A \in \mathcal{F}$.

Proof. Since φ commutes with $\star$, so we have

$$\int_{S,A} \varphi(f \star g)d\mu = \int_{S,A} (\varphi(f) \star \varphi(g))d\mu. \tag{6}$$

If f, g are comonotone functions and φ is continuous and strictly increasing function, then $\varphi(f)$ and $\varphi(g)$ are also comonotone. From (6) and using the Corollary 2, we have

$$\begin{aligned}\int_{S,A} (\varphi(f) \star \varphi(g))d\mu &\leq (\int_{S,A} \varphi(f)d\mu) \star (\int_{S,A} \varphi(f)d\mu)\\ &= \varphi[\varphi^{-1}(\int_{S,A} \varphi(f)d\mu) \star \varphi^{-1}(\int_{S,A} \varphi(g)d\mu)]\end{aligned}$$

where φ commutes with $\star$. Hence (5) is valid.

4. A related inequality

In this section, we prove a related inequality for the Minkowski's inequality in semiconormed fuzzy integrals.

Theorem 7. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,\ 1]$ be two comonotone measurable functions. Let $\star : [0,\ 1]^2 \to [0,\ 1]$ be continuous and nondecreasing in both arguments. If the semiconorm S satisfies*

$$S(a \star b, c) \geq (S(a, c) \star b) \vee (a \star S(b, c)), \tag{7}$$

then the inequality

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}} \tag{8}$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

Proof. Let $\int_{S,A} f^s d\mu = a,\ \int_{S,A} g^s d\mu = b$ and $\int_{S,A} (f \star g)^s d\mu = c$. Then for all $\alpha \in [0,\ 1]$ we have $S(\alpha, \mu(A \cap F_{\alpha^{\frac{1}{s}}})) \geq a,\ S(\alpha, \mu(A \cap G_{\alpha^{\frac{1}{s}}})) \geq b$ and $S(\alpha, \mu(A \cap H_{\alpha^{\frac{1}{s}}})) \geq c$, where $H_{\alpha^{\frac{1}{s}}} = \{x \mid (f \star g)(x) > \alpha^{\frac{1}{s}}\}$. Hence for any $\varepsilon > 0$, there exist $a_\varepsilon,\ b_\varepsilon$ and $c_\varepsilon = a_\varepsilon \star b_\varepsilon$ such that $\mu(A \cap F_{(a_\varepsilon)^{\frac{1}{s}}}) = a_1,\ \mu(A \cap G_{(b_\varepsilon)^{\frac{1}{s}}}) = b_1$ and $\mu(A \cap H_{(c_\varepsilon)^{\frac{1}{s}}}) = c_1$, where $S(c_\varepsilon, c_1) \leq c + \varepsilon$. (Thus $a_\varepsilon \geq a,\ b_\varepsilon \geq b,\ S(a_\varepsilon, a_1) \geq a$ and $S(b_\varepsilon, b_1) \geq b$). The fact of $H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}} \supset F_{(a_\varepsilon)^{\frac{1}{s}}} \cap G_{(b_\varepsilon)^{\frac{1}{s}}}$ and the comonotonicity of f, g imply that $\mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}) \geq a_1 \wedge b_1$. Hence

$$\begin{aligned}c + \varepsilon &\geq S(c_\varepsilon,\ c_1)\\ &= S(a_\varepsilon \star b_\varepsilon, \mu(A \cap H_{(a_\varepsilon)^{\frac{1}{s}} \star (b_\varepsilon)^{\frac{1}{s}}}))\end{aligned}$$

$$\begin{aligned}
&\geq S(a_\varepsilon \star b_\varepsilon, a_1 \wedge b_1 \\
&= S(a_\varepsilon \star b_\varepsilon, a_1) \wedge S(a_\varepsilon \star b_\varepsilon, b_1) \\
&\geq (S(a_\varepsilon, a_1) \star b_\varepsilon) \wedge (a_\varepsilon \star S(b_\varepsilon, b_1)) \\
&\geq (a \star b_\varepsilon) \wedge (a_\varepsilon \star b) \\
&\geq (a \star b) \wedge (a \star b) \\
&= a \star b.
\end{aligned}$$

Hence $c \geq a \star b$ follows from the arbitrariness of ε. Consequently from the continuity of $\star$ we have $c \geq (a^{\frac{1}{s}} \star b^{\frac{1}{s}})^s$. This implies

$$\Big(\int_{S,A} (f \star g)^s d\mu\Big)^{\frac{1}{s}} \geq \Big(\int_{S,A} f^s d\mu\Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu\Big)^{\frac{1}{s}}.$$

Example 2. *Let $A = [0,1]$ and μ be the Lebesgue measure. Let $\star$ be the usual product and S be the maximum. Let $f(x) = x^2$, $g(x) = \frac{1}{4}$ and $s = \frac{1}{2}$. A straightforward calculus shows that*

$$\begin{aligned}
\int_{S,A} f^{\frac{1}{2}} d\mu = \int_{S,A} x d\mu &= \inf_{\alpha \in [0,1]} (\alpha \vee \mu(A \cap \{x \mid x > \alpha\})) \\
&= \inf_{\alpha \in [0,1]} (\alpha \vee (1 - \alpha)) \\
&= 0.5,
\end{aligned}$$

$$\begin{aligned}
\int_{S,A} g^{\frac{1}{2}} d\mu = \int_{S,A} \frac{1}{2} d\mu &= \inf_{\alpha \in [0,1]} (\alpha \vee \mu(A \cap [0, \frac{1}{2}))) \\
&= 0.5
\end{aligned}$$

and we have

$$\begin{aligned}
\int_{S,A} (f \star g)^{\frac{1}{2}} d\mu &= \int_{S,A} \frac{1}{2} x d\mu \\
&= \inf_{\alpha \in [0,1]} (\alpha \vee \mu(A \cap \{x \in [0,1] \mid \frac{1}{2} x > \alpha\})) \\
&= \inf_{\alpha \in [0,1]} (\alpha \vee (1 - 2\alpha)) \\
&= 0.33333.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0.11111 = \Big(\int_{S,X} (f \star g)^{\frac{1}{2}} d\mu\Big)^2 &\geq \Big(\int_{S,X} f^{\frac{1}{2}} d\mu\Big)^2 \star \Big(\int_{S,X} g^{\frac{1}{2}} d\mu\Big)^2 \\
&= 0.25 \ . \ 0.25 = 0.0625.
\end{aligned}$$

If in the Theorem 7 we assume $s = 1$, then we get the Chebyshev type inequality for the semiconormed fuzzy integrals.

Corollary 3. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $f, g : X \to [0,\ 1]$ be two comonotone measurable functions. Let $\star : [0,\ 1]^2 \to [0,\ 1]$ be continuous and non-decreasing in both arguments. If the semiconorm S satisfies*

$$S(a \star b, c) \geq (S(a, c) \star b) \vee (a \star S(b, c)),$$

then the inequality

$$\int_{S,A} (f \star g) d\mu \geq \int_{S,A} f d\mu \star \int_{S,A} g d\mu$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

If $\star$ bounded from above by maximum, then $\star$ is dominated by maximum. Thus Corollary 4.4 gives us a general Minkowski type inequality for the Sugeno integral which appears in [8].

Corollary 4. *Let $f, g : X \to [0, 1]$ be two comonotone measurable functions. And let $\star : [0, 1]^2 \to [0, 1]$ be continuous and non-decreasing in both arguments and bounded from above by minimum. Then the inequality*

$$\Big(⨍_A (f \star g)^s d\mu \Big)^{\frac{1}{s}} \geq \Big(⨍_A f^s d\mu \Big)^{\frac{1}{s}} \star \Big(⨍_A g^s d\mu \Big)^{\frac{1}{s}}$$

holds for any $A \in \mathcal{F}$ and for all $0 < s < \infty$.

5. Conclusion

We have proved general Minkowski type inequalities for semiconormed fuzzy integrals on an abstract fuzzy measure space $(X, \mathcal{F}, \mu)$ based on a product like operator $\star$ and an inequality relevant for it. The semiconormed fuzzy integrals generalize the Sugeno integral, so it remains that when the following inequality

$$\Big(\int_{S,A} (f \star g)^s d\mu \Big)^{\frac{1}{s}} = \Big(\int_{S,A} f^s d\mu \Big)^{\frac{1}{s}} \star \Big(\int_{S,A} g^s d\mu \Big)^{\frac{1}{s}}$$

holds.

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