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A Rearrangement Estimate for the Generalized Multilinear Anisotropic Fractional Integrals

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Abstract. In this paper, author studies $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}$ boundedness of the generalized multilinear anisotropic fractional integrals. We give a new proof of the Hardy-Littlewood-Sobolev multilinear anisotropic fractional integration theorem, based on a pointwise estimate of the rearrangement multilinear anisotropic fractional type integral.

Key Words and Phrases: Lebesgue space, multilinear anisotropic fractional integral.

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1. Introduction

Fractional maximal function and fractional integral is an important technical tool in harmonic analysis, real analysis and partial differential equations. Multilinear fractional maximal operator and multilinear fractional integral operator and related topics have been areas of research of many mathematicians such as R.Coifman and L. Grafakos [5], L. Grafakos [6, 7], L. Grafakos and N. Kalton [8], C.E. Kenig and E.M. Stein [12], Y. Ding and S. Lu [11] and others.

The purpose of this article is to describe several results about generalized multilinear anisotropic fractional integral operators. We study $L_{p_1} \times L_{p_2} \times \ldots \times L_{p_k}$ boundedness of the generalized multilinear anisotropic fractional integrals. We give a new proof of the Hardy-Littlewood-Sobolev multilinear anisotropic fractional integration theorem, based on a pointwise estimate of the rearrangement generalized multilinear anisotropic fractional integral.

2. Rearrangements of functions

Let $\mathbb{R}^n$ is the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$ with norms $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Let $\lambda > 0$, $a = (a_1, \ldots, a_n)$, $a_1 > 0, \ldots, a_n > 0$, $d = a_1 + \ldots + a_n$, $\delta_\lambda x = (\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n)$.

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Let $\rho(x)$ be a non-isotropic norm on $\mathbb{R}^n$ defined as the unique positive solution of the equation

$$\sum_{j=1}^{n} \frac{x_j^2}{\rho(x)^{2a_j}} = 1.$$

Note that $\rho(x)$ is equivalent to $\sum_{i=1}^{n} |x_i|^{1/a_i}$, i.e.,

$$c_1\rho(x) \le \sum_{i=1}^{n} |x_i|^{1/a_i} \le c_2\rho(x)$$

for certain positive c_1 and c_2 independent of x (see [2]).

It is immediate that $\rho(\delta_\lambda x) = \lambda\ \rho(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^n$. With this norm, $\mathbb{R}^n$ is a space of homogeneous type in the sense of Coifman and Weiss [4] with homogeneous dimension $d = |a|$. In particular, there is a constant $c_0 \ge 1$ such that $\rho(x+y) \le c_0\ (\rho(x)+\rho(y))$ for all $x,\ y \in \mathbb{R}^n$.

One has the polar decomposition

$$x = \delta_\lambda\ \sigma \tag{1}$$

with $\sigma \in S^{n-1}$, $r = \rho(x)$ and $dx = r^{d-1}drJ(\sigma)d\sigma$, where $J(\sigma)$ is a smooth and nonnegative function of $\sigma \in S^{n-1}$ and is even in each of variables $\sigma_1, \ldots, \sigma_n$.

The isotropic and anisotropic balls of radius r and center x are defined

$$B(x,r) = \{y \in \mathbb{R}^n\ :\ |x-y| < r\},$$

$$\mathcal{E}(x,r) = \{y \in \mathbb{R}^n\ :\ \rho(x-y) < r\},$$

respectively.

For $1 \le p < \infty$ let $L_p(\mathbb{R}^n)$ be the space of all measurable functions g on $\mathbb{R}^n$ with finite norm

$$\|g\|_{L_p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |g(x)|^p dx\right)^{1/p}.$$

Let g be a measurable function on $\mathbb{R}^n$. The distribution function of g is defined by the equality

$$\lambda_g(t) = |\{x \in \mathbb{R}^n\ :\ |g(x)| > t\}|, \qquad t \ge 0.$$

We shall denote by $L_0(\mathbb{R}^n)$ the class of all measurable functions g on $\mathbb{R}^n$, which are finite almost everywhere and such that $\lambda_g(t) < \infty$ for all $t > 0$ (see [13]).

If a function g belongs to $L_0(\mathbb{R}^n)$, then its non-increasing rearrangement is defined to be the function g^* which is non-increasing on $]0,\infty[$ equimeasurable with $|g(x)|$:

$$|\{t > 0\ :\ g^*(t) > s\}| = \lambda_g(s)$$

for all $s \ge 0$.

Set
$$g^{**}(t)=\frac{1}{t}\int_0^t g^*(s)\ ds.$$

Moreover, by the Hardy-Littlewood theorem (see [3], p. 44), for every $f_1,\ f_2\in L_0(\mathbb{R}^n)$,
$$\int_{\mathbb{R}^n}|f_1(x)f_2(x)|\ dx\le\int_0^\infty f_1^*(t)f_2^*(t)\ dt.$$

It is well known that if $p>1$, then $\left(\int_0^\infty (g^{**}(t))^p\ \ dt\right)^{1/p}$ is comparable with the $L_p(\mathbb{R}^n)$ norm of g.

For $1\le p<\infty$ the weak L_p space $WL_p(\mathbb{R}^n)$ is the set of all locally integrable functions g on $\mathbb{R}^n$ with finite norm
$$\|g\|_{WL_p(\mathbb{R}^n)}=\sup_{t>0}\ t\ \lambda_g(t)^{1/p}.$$

Equimeasurable rearrangements of functions play an important role in various fields of mathematics. Note some properties of the rearrangement (see, for example [3]):

1) if $0<t<t+s$, then
$$(g+h)^*\,(t+s)\le g^*(t)+h^*(s),$$

2) if $0<p<\infty$, then
$$\int\limits_{\mathbb{R}^n}|g(x)|^p\ dx=\int_0^\infty (g^*(t))^p\ \ dt,$$

3) for any $t>0$
$$\sup_{|E|=t}\int_E|g(x)|\ dx=\int_0^t g^*(s)\ ds.$$

Let $k\ge 2$ be an integer and θ_j $(j=1,2,\cdots,k)$ be a fixed, distinct and nonzero real numbers.

Lemma 1. *[9] Let $f_1,f_2,\ldots,f_k\in L_0(\mathbb{R}^n)$, $k\ge 2$. Then for all $x\in\mathbb{R}^n$ and nonzero real numbers $\theta_1,\ldots,\theta_k$*
$$\int_{\mathbb{R}^n}|f_1(x-\theta_1y)f_2(x-\theta_2y)\cdots f_k(x-\theta_ky)|dy\le C_\theta\int_0^\infty f_1^*(t)f_2^*(t)\cdots f_k^*(t)dt,\qquad(2)$$
where $C_\theta=|\theta_1\ldots\theta_k|^{-n}$.

3. A rearrangement estimate for the generalized multilinear fractional integrals

By $\mathbf{f}$ we denote $(f_1,f_2,\cdots,f_k)$ and define
$$\mathbf{f}^*(t)=f_1^*(t)\ldots f_k^*(t),$$

$$\mathbf{f}^{**}(t) = \frac{1}{t}\int_0^t f_1^*(s)\dots f_k^*(s)\ ds, \qquad t>0.$$

Let $k \geq 2$ be an integer and θ_j $(j = 1, 2, \cdots, k)$ be fixed and nonzero real numbers. The analogy of O'Neil inequality (see, [14]) for k-linear integral operator by

$$(\mathbf{f}, g)(x) = \int_{\mathbb{R}^n} g(y) f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\, dy,$$

is correct

Lemma 2. *[9] Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then for all $0 < t < \infty$, the following inequality holds*

$$(\mathbf{f}, g)^{**}(t) \leq C_\theta \left(t\ \mathbf{f}^{**}(t)\ g^{**}(t) + \int_t^\infty \mathbf{f}^*(s)\ g^*(s)\ ds \right). \tag{3}$$

Lemma 3. *[9] Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$. Then for any $t > 0$*

$$(\mathbf{f}, g)^{**}(t) \leq C_\theta \int_t^\infty \mathbf{f}^{**}(t) g^{**}(t) dt. \tag{4}$$

In the following we define the k-sublinear anisotropic fractional maximal operator by

$$\mathcal{M}_{\Omega,\alpha}\mathbf{f}(x) = \sup_{r>0} \frac{1}{r^{d-\alpha}} \int_{\mathcal{E}(0,r)} |\Omega(y)| |f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)|\ dy,$$

the k-linear anisotropic fractional integral operator by

$$R_{\Omega,\alpha}\mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{\rho(y)^{d-\alpha}} f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\, dy$$

and the generalized k-linear anisotropic fractional integral operator by

$$K_\alpha\mathbf{f}(x) = \int_{\mathbb{R}^n} K_\alpha(y) f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\, dy,$$

where $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$.

Note that, if $K_\alpha(x) = \frac{\Omega(x)}{\rho(x)^{d-\alpha}}$, $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, then $K_\alpha^*(t) = \left(\frac{A}{nt}\right)^{(d-\alpha)/d}$, $K_\alpha^{**}(t) = \frac{d}{\alpha} K_\alpha^*(t)$, where $A = \|\Omega\|^{d/(d-\alpha)}_{L_{d/(d-\alpha)}(S^{n-1})}$ and therefore $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$. And also, if $K_\alpha(x) = \frac{\Omega(x)}{\rho(x)^{d-\alpha}}$, $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, then $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$.

The following lemma in the isotropic case was proved in [11]. In the anisotropic case it is proved analogously.

Lemma 4. *Suppose that $0 < \alpha < d$, $\Omega \in L_s(S^{n-1})$, $s \geq 1$. Then*

$$\mathcal{M}_{\Omega,\alpha}\mathbf{f}(x) \leq R_{|\Omega|,\alpha}(|\mathbf{f}|)(x), \tag{5}$$

where $|\mathbf{f}| = (|f_1|, \dots, |f_k|)$.

Proof. Indeed, for all $r > 0$, we have

$$R_{|\Omega|,\alpha}(|\mathbf{f}|)(x) \geq \int_{\mathcal{E}(0,r)} \frac{\Omega(y)}{\rho(y)^{d-\alpha}} f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\,dy$$
$$\geq \frac{1}{r^{d-\alpha}} \int_{\mathcal{E}(0,r)} |\Omega(y)||f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)|\,dy,$$

where $\mathcal{E}(0,r)$ is the anisotropic ball centered at the origin of radius r. Taking supremum over all $r > 0$, we get (5).

For the generalized multilinear fractional integrals $K_\alpha \mathbf{f}$ the following theorem is valid:

Theorem 1. *Let $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$, $0 < \alpha < d$. Then*

$$(K_\alpha \mathbf{f})^*(t) \leq (K_\alpha \mathbf{f})^{**}(t) \leq C_1 \left(t^{\frac{\alpha}{d}-1} \int_0^t \mathbf{f}^*(s)\,ds + \int_t^\infty s^{\frac{\alpha}{d}-1}\,\mathbf{f}^*(s)\,ds \right), \quad (6)$$

where $C_1 = \left(\frac{d}{\alpha}\right)^2 C_\theta \, \|K_\alpha\|_{WL_{d/(d-\alpha)}}$.

Proof. Let $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$, then

$$K_\alpha^*(t) \leq \|K_\alpha\|_{WL_{d/(d-\alpha)}} \, t^{\frac{\alpha}{d}-1}, \qquad K_\alpha^{**}(t) \leq \frac{d}{\alpha}\, K_\alpha^*(t).$$

Taking into account inequality (3) we have (6).

Corollary 1. *Suppose that $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. Then the following inequality*

$$(R_{\Omega,\alpha}\mathbf{f})^*(t) \leq (R_{\Omega,\alpha}\mathbf{f})^{**}(t) \leq C_2 \left(t^{\frac{\alpha}{d}-1} \int_0^t \mathbf{f}^*(s)\,ds + \int_t^\infty s^{\frac{\alpha}{d}-1}\,\mathbf{f}^*(s)\,ds \right),$$

holds, where $C_2 = \left(\frac{d}{\alpha}\right) C_\theta \left(\frac{A}{d}\right)^{(d-\alpha)/d}$, $A = \|\Omega\|^{d/(d-\alpha)}_{L_{d/(d-\alpha)}(S^{n-1})}$.

From Corollary 1 and Lemma 4 we get

Corollary 2. *Suppose that $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. Then the following inequality*

$$(\mathcal{M}_{\Omega,\alpha}\mathbf{f})^*(t) \leq (\mathcal{M}_{\Omega,\alpha}\mathbf{f})^{**}(t) \leq C_2 \left(t^{\frac{\alpha}{d}-1} \int_0^t \mathbf{f}^*(s)\,ds + \int_t^\infty s^{\frac{\alpha}{d}-1}\,\mathbf{f}^*(s)\,ds \right),$$

holds.

Analogously we have

Theorem 2. *Let $K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$, $0 < \alpha < d$. Then*

$$(K_\alpha \mathbf{f})^*(t) \leq (K_\alpha \mathbf{f})^{**}(t) \leq C_1 \int_t^\infty s^{\frac{\alpha}{d}-1}\,\mathbf{f}^{**}(s)\,ds. \quad (7)$$

Corollary 3. *Suppose that* $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. *Then the following inequality*

$$(R_{\Omega,\alpha}\mathbf{f})^{*}(t) \le (R_{\Omega,\alpha}\mathbf{f})^{**}(t) \le C_2 \int_t^{\infty} s^{\frac{\alpha}{d}-1}\mathbf{f}^{**}(s)\; ds$$

holds.

Corollary 4. *Suppose that* $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$. *Then the following inequality*

$$(\mathcal{M}_{\Omega,\alpha}\mathbf{f})^{*}(t) \le (\mathcal{M}_{\Omega,\alpha}\mathbf{f})^{**}(t) \le C_2 \int_t^{\infty} s^{\frac{\alpha}{d}-1}\mathbf{f}^{**}(s)\; ds$$

holds.

4. $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}$ boundedness of generalized multilinear fractional integral operators

In the sequel we shall use the following Lemma, which was proved in [1].

Lemma 5. *[1] Let* $0 < p \le 1$, $p \le q < \infty$ *and* k *be a non-negative measurable functions and* u, v *be weight functions on* $(0,\infty)$ *and*

$$T\varphi(t) = \int_0^{\infty} k(t,\tau)\varphi(\tau)d\tau.$$

Then the inequality

$$\left(\int_0^{\infty} (T\varphi(t))^q\, u(t)dt\right)^{1/q} \le C\left(\int_0^{\infty} \varphi(t)^p v(t)dt\right)^{1/p} \tag{8}$$

holds for all non-negative non-increasing functions φ *if and only if*

$$C_0 = \sup_{r>0}\left(\int_0^{\infty}\left(\int_0^{r} k(t,\tau)d\tau\right)^q u(t)dt\right)^{1/q}\left(\int_0^{r} v(t)dt\right)^{-1/p} < \infty.$$

The constant $C = C_0$ *is the best constant in* (8).

Corollary 5. *Let* $0 < p \le 1$, $p \le q < \infty$, $0 < \alpha < d$.

Then the inequality

$$\left(\int_0^{\infty}\left(\int_t^{\infty} \tau^{\frac{\alpha}{d}-1}\varphi(\tau)\; d\tau\right)^q\; dt\right)^{1/q} \le \mathcal{C}_0\left(\int_0^{\infty} \varphi(t)^p\; dt\right)^{1/p}$$

holds for all non-negative non-increasing functions φ *if and only if*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}, \tag{9}$$

where $\mathcal{C}_0 = \left(\frac{d}{\alpha}\right)^{1+\frac{1}{q'}} B\left(\frac{d}{\alpha}, q+1\right)^{\frac{1}{q}}$, $B(s,r) = \int_0^1 (1-\tau)^{s-1}\tau^{r-1}d\tau$ *is the Beta function.*

It is said that p is the harmonic mean of $p_1, p_2, \ldots, p_k > 1$, if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k}$. If $f_j \in L_{p_j}(\mathbb{R}^n)$, $j = 1, 2, \ldots, k$, then we say that $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$.

Theorem 3. *Suppose that* $0 < \alpha < d, K_\alpha \in WL_{d/(d-\alpha)}(\mathbb{R}^n)$. *Let* p *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$ *and* q *satisfy* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. *Then* $K_\alpha \mathbf{f}$ *is bounded operator from* $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $d/(d+\alpha) \le p < d/\alpha$ *(equivalently* $1 \le q < \infty$*) and*

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C \prod_{j=1}^{k} \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C > 0$ *independent of* f.

Proof. Case I. $1 < p < \frac{d}{\alpha}$ (equivalently $\frac{d}{d-\alpha} < q < \infty$). Let us first prove Theorem 3 in this case.

Taking into account equality (2) and inequality (6) we have

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} = \|(K_\alpha \mathbf{f})^*\|_{L_q(0,\infty)}$$
$$\le C_1 \left(\int_0^\infty t^{q(\alpha/d-1)} \left(\int_0^t \mathbf{f}^*(s)\ ds \right)^q dt \right)^{1/q} + C_1 \left(\int_0^\infty \left(\int_t^\infty s^{\alpha/d-1} \mathbf{f}^*(s)\ ds \right)^q dt \right)^{1/q},$$

where $C > 0$ independent of f.

Applying Hardy inequality we obtain, that for the validity of the following inequality

$$\left(\int_0^\infty t^{q(\alpha/d-1)} \left(\int_0^t \mathbf{f}^*(s) ds \right)^q dt \right)^{1/q} \le C_3 \left(\int_0^\infty \mathbf{f}^*(s)^p ds \right)^{1/p}$$

it is necessary and sufficient that the following condition is satisfied

$$\sup_{t>0} \left(\int_t^\infty s^{q(\alpha/d-1)} ds \right)^{1/q} \left(\int_0^t ds \right)^{1/p'}$$
$$= C_4 \sup_{t>0} t^{\frac{\alpha}{d} - \left(\frac{1}{p} - \frac{1}{q}\right)} < \infty \Leftrightarrow 1/p - 1/q = \alpha/d,$$

where $p' = \frac{p}{p-1}$.

For the validity of the following inequality

$$\left(\int_0^\infty \left(\int_t^\infty s^{\frac{\alpha-d}{d}} \mathbf{f}^*(s)\ ds \right)^q dt \right)^{1/q} \le C_5 \left(\int_0^\infty \mathbf{f}^*(s)^p ds \right)^{1/p}$$

it is necessary and sufficient satisfying the following condition

$$\sup_{t>0} \left(\int_0^t ds \right)^{1/q} \left(\int_t^\infty s^{(\alpha/d-1)(1-p')} ds \right)^{1/p'}$$
$$= C_6 \sup_{t>0} t^{\frac{\alpha}{d} - \left(\frac{1}{p} + \frac{1}{q}\right)} < \infty \Leftrightarrow 1/p - 1/q = \alpha/d.$$

Consequently applying equality (2) we obtain

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C_1(C_3+C_5)\,\|\mathbf{f}^*\|_{L_p(0,\infty)}$$
$$\le C_1(C_3+C_5)\prod_{j=1}^{k}\|f_j^*\|_{L_{p_j}(0,\infty)} = C_1(C_3+C_5)\prod_{j=1}^{k}\|f_j\|_{L_{p_j}(\mathbb{R}^n)}.$$

Case II. $\frac{d}{d+\alpha} \le p \le 1$ (equivalently $1 \le q \le \frac{d}{d-\alpha}$). Now let's prove Theorem 3 for this case.

Taking into account equality (2) and inequality (7) we have

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} = \|(K_\alpha \mathbf{f})^*\|_{L_q(0,\infty)} \le \|(K_\alpha \mathbf{f})^{**}\|_{L_q(0,\infty)}$$
$$\le C_1\left(\int_0^\infty \left(\int_t^\infty s^{\alpha/d-1}\mathbf{f}^{**}(s)ds\right)^q dt\right)^{1/q}.$$

By virtue of Lemma 2 for the validity of the following inequality

$$\left(\int_0^\infty \left(\int_t^\infty s^{\alpha/d-1}\mathbf{f}^{**}(s)ds\right)^q dt\right)^{1/q} \le C_6\left(\int_0^\infty \mathbf{f}^{**}(s)^p ds\right)^{1/p}$$

it is necessary and sufficient satisfying the condition (9).

Consequently applying equality (2), Hardy inequality for monotonic functions and Holder inequality we obtain

$$\|K_\alpha \mathbf{f}\|_{L_q(\mathbb{R}^n)} = \|(K_\alpha \mathbf{f})^*\|_{L_q(0,\infty)}$$
$$\le C_8\,\|\mathbf{f}^{**}\|_{L_p(0,\infty)} \le C_9\,\|\mathbf{f}^*\|_{L_p(0,\infty)}$$
$$\le C_9\prod_{j=1}^{k}\|f_j^*\|_{L_{p_j}(0,\infty)} = C_9\prod_{j=1}^{k}\|f_j\|_{L_{p_j}(\mathbb{R}^n)}.$$

Corollary 6. *Let $0<\alpha<d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, p be the harmonic mean of $p_1, p_2, \ldots, p_k > 1$ and q satisfy $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{d}$. Then $R_{\Omega,\alpha}\mathbf{f}$ is a bounded operator from $L_{p_1}\times L_{p_2}\times\cdots\times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $d/(d+\alpha)\le p<d/\alpha$ (equivalently $1\le q<\infty$) and*

$$\|R_{\Omega,\alpha}\;\mathbf{f}\|_{L_q(\mathbb{R}^n)} \le C\prod_{j=1}^{k}\|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C>0$ independent of f.

Corollary 7. *Let $0<\alpha<n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p be the harmonic mean of $p_1, p_2, \ldots, p_k > 1$ and q satisfy $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Then the k-linear fractional integral operator*

$$I_{\Omega,\alpha}\mathbf{f}(x) = \int_{\mathbb{R}^n}\frac{\Omega(y)}{|y|^{n-\alpha}}f_1(x-\theta_1 y)\ldots f_k(x-\theta_k y)\,dy$$

is a bounded operator from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $n/(n+\alpha) \leq p < n/\alpha$ *(equivalently* $1 \leq q < \infty$*) and*

$$\|I_{\Omega,\alpha}\, \mathbf{f}\|_{L_q(\mathbb{R}^n)} \leq C \prod_{j=1}^{k} \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C > 0$ *independent of* f*.*

Corollary 8. *Let* $0 < \alpha < d$, $\Omega \in L_{d/(d-\alpha)}(S^{n-1})$, *p be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$ *and q satisfy* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$*. Then* $\mathcal{M}_{\Omega,\alpha}\mathbf{f}$ *is a bounded operator from* $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$ *for* $d/(d+\alpha) \leq p < d/\alpha$ *(equivalently* $1 \leq q < \infty$*) and*

$$\|\mathcal{M}_{\Omega,\alpha}\, \mathbf{f}\|_{L_q(\mathbb{R}^n)} \leq C \prod_{j=1}^{k} \|f_j\|_{L_{p_j}(\mathbb{R}^n)},$$

where $C > 0$ *independent of* f*.*

Remark 1. *Note that, Corollary 7 proved in [6], if* $\Omega \equiv 1$ *and in [11], if* $\Omega \in L_s(S^{n-1})$, $s > n/(n-\alpha)$ *and in [9, 10], if* $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$*.*

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