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On a compactness criteria for multidimensional Hardy type operator in p-convex Banach function spaces

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Abstract. The main goal of this paper is to prove a criteria on compactness of multidimensional Hardy type operator from weighted Lebesgue spaces into p-convex weighted Banach function spaces. Analogously problem for the dual operator is considered.

Key Words and Phrases: Banach function spaces, weights, Hardy type operator, compactness

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1. Introduction

The investigation of Hardy operator in weighted Banach function spaces (BFS) have recently history. The goal of this investigations were closely connected with the found of criterion on the geometry and on the weights of BFS for validity of boundedness of Hardy operator in BFS. Characterization of the mapping properties such as boundedness and compactness were considered in the papers [8], [9], [13], [25] and e.t.c. More precisely, in [8] and [9] were considered the boundedness of certain integral operator in ideal Banach spaces. In [13] was proved the boundedness of Hardy operator in Orlicz spaces. Also, in [25] the compactness and measure of non-compactness of Hardy type operator in Banach function spaces was proved. But in this paper we consider the boundedness of Hardy operator in p-convex Banach function spaces and find a new type criterion on the weights for validity of Hardy inequality. Note that the notion of BFS was introduced in [26]. In particular, the weighted Lebesgue spaces, weighted Lorentz spaces, weighted variable Lebesgue spaces, variable Lebesgue spaces with mixed norm, Musielak-Orlicz spaces and e.t.c. is BFS.

In this paper, we establish an integral-type necessary and sufficient condition on weights, which provides the compactness of the multidimensional Hardy type operator from weighted Lebesgue spaces into p-convex weighted BFS. We also investigate the corresponding problems for the dual operator.

2. Preliminaries

Let $(\Omega,\,\mu)$ be a complete σ-finite measure space. By $L_0=L_0(\Omega,\mu)$ we denote the collection of all real-valued μ-measurable functions on Ω.

Definition 1. *[26, 24, 7] We say that real normed space X is a Banach function space (BFS) if:*

(P1) the norm $\|f\|_X$ is defined for every μ-measurable function f, and $f\in X$ if and only if $\|f\|_X<\infty$; $\|f\|_X=0$ if and only if $f=0$ a.e. ;

(P2) $\|f\|_X=\||f|\|_X$ for all $f\in X$;

(P3) if $0\le f\le g$ a.e., then $\|f\|_X\le\|g\|_X$;

(P4) if $0\le f_n\uparrow f$ a.e., then $\|f_n\|_X\uparrow\|f\|_X$ (Fatou property);

(P5) if E is a measurable subset of Ω such that $\mu(E)<\infty$, then $\|\chi_E\|_X<\infty$, where χ_E is the characteristic function of the set E;

(P6) for every measurable set $E\subset\Omega$ with $\mu(E)<\infty$, there is a constant $C_E>0$ such that $\int_E f(x)\,dx\le C_E\,\|f\|_X$.

Given a BFS X we can always consider its associate space X' consisting of those $g\in L_0$ that $f\cdot g\in L_1$ for every $f\in X$ with the usual order and the norm $\|g\|_{X'}=\sup\left\{\|f\cdot g\|_{L_1}:\ \|g\|_{X'}\le1\right\}$. Note that X' is a BFS in $(\Omega,\,\mu)$ and a closed norming subspaces.

Let X be a BFS and ω be a weight, that is, positive Lebesgue measurable and a.e. finite functions on Ω. Let $X_\omega=\left\{f\in L_0:\ f\omega\in X\right\}$. This space is a weighted BFS equipped with the norm $\|f\|_{X_\omega}=\|f\,\omega\|_X$. (For more detail and proofs of results about BFS we refer the reader to [7] and [24].)

Let us recall the notion of p-convexity and p-concavity of BFS's.

Definition 2. *[33] Let X is a BFS. Then X is called p-convex for $1\le p\le\infty$ if there exists a constant $M>0$ such that for all $f_1,\dots,f_n\in X$*

$$\left\|\left(\sum_{k=1}^n|f_k|^p\right)^{\frac1p}\right\|_X\le M\,\left(\sum_{k=1}^n\|f_k\|_X^p\right)^{\frac1p}\quad \text{if } 1\le p<\infty,$$

or $\left\|\sup\limits_{1\le k\le n}|f_k|\right\|_X\le M\,\max\limits_{1\le k\le n}\|f_k\|_X$ *if $p=\infty$. Similarly X is called p-concave for $1\le p\le\infty$ if there exists a constant $M>0$ such that for all $f_1,\dots,f_n\in X$*

$$\left(\sum_{k=1}^n\|f_k\|_X^p\right)^{\frac1p}\le M\,\left\|\left(\sum_{k=1}^n|f_k|^p\right)^{\frac1p}\right\|_X\quad \text{if } 1\le p<\infty,$$

or $\max\limits_{1\le k\le n}\|f_k\|_X\le M\,\left\|\sup\limits_{1\le k\le n}|f_k|\right\|_X$ *if $p=\infty$.*

Remark 1. *Note that the notions of p-convexity, respectively p-concavity are closely related to the notions of upper p-estimate (strong ℓ_p- composition property), respectively lower p-estimate (strong ℓ_p-decomposition property) as can be found in [24].*

Now we reduce some examples of p-convex and respectively p-concave BFS. Let R^n be the n-dimensional Euclidean space of points $x=(x_1,...,x_n)$ and let Ω be a Lebesgue measurable subset in R^n and $|x|=\left(\sum\limits_{i=1}^{n}x_i^2\right)^{1/2}$. The Lebesgue measure of a set Ω will be denoted by $|\Omega|$. It is well known that $|B(0,1)|=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$, where $B(0,1)=\{x:\,x\in R^n\,;\ |x|<1\}$.

Example 1.1. Let $1\le q\le\infty$ and $X=L_q$. Then the space L_q is p-convex (p-concave) BFS if and only if $1\le p\le q\le\infty$ $(1\le q\le p\le\infty.)$

The proof implies from usual Minkowski inequality in Lebesgue spaces.

Example 1.2. The following Lemma shows that the variable Lebesgue spaces $L_{q(y)}(\Omega)$ is p-convex BFS.

Lemma 1. *[1] Let $1\le p\le q(x)\le\overline{q}<\infty$ for all $y\in\Omega_2\subset R^m$. Then the inequality*

$$\left\|\|f\|_{L_p(\Omega_1)}\right\|_{L_{q(\cdot)}(\Omega_2)}\le C_{p,q}\left\|\|f\|_{L_{q(\cdot)}(\Omega_2)}\right\|_{L_p(\Omega_1)}$$

is valid, where $C_{p,q}=\left(\|\chi_{\Delta_1}\|_\infty+\|\chi_{\Delta_2}\|_\infty+p\left(\frac{1}{\underline{q}}-\frac{1}{\overline{q}}\right)\right)(\|\chi_{\Delta_1}\|_\infty+\|\chi_{\Delta_2}\|_\infty)$, $\underline{q}=ess\inf\limits_{\Omega_2}q(x)$, $\overline{q}=ess\sup\limits_{\Omega_2}q(x)$, $\Delta_1=\{(x,y)\in\Omega_1\times\Omega_2:\,q(y)=p\}$, $\Delta_2=\Omega_1\times\Omega_2\backslash\Delta_1$ *and* $f:\Omega_1\times\Omega_2\to R$ *is any measurable function such that*

$$\left\|\|f\|_{L_p(\Omega_1)}\right\|_{L_{q(\cdot)}(\Omega_2)}=\inf\left\{\mu>0:\int\limits_{\Omega_2}\left(\frac{\|f(\cdot,y)\|_{L_p(\Omega_1)}}{\mu}\right)^{q(y)}dy\le1\right\}<\infty$$

and $\|f(\cdot,y)\|_{L_p(\Omega_1)}=\left(\int_{\Omega_1}|f(x,y)|^p\,dx\right)^{1/p}$.

Analogously, if $1\le q(x)\le p<\infty$, then $L_{q(x)}(\Omega)$ is p-concave BFS.

Definition 3. *[31, 15]. Let $\Omega\subset R^n$ be a Lebesgue measurable set. A real function $\varphi:\Omega\times[0,\infty)\mapsto[0,\infty)$ is called a generalized φ-function if it satisfies:*

a) $\varphi(x,\cdot)$ is a φ-function for all $x\in\Omega$, i.e., $\varphi(x,\cdot):[0,\infty)\mapsto[0,\infty)$ is convex and satisfies $\varphi(x,0)=0$, $\lim\limits_{t\to+0}\varphi(x,t)=0$;

b) $\psi:x\mapsto\varphi(x,t)$ is measurable for all $t\ge0$.

If φ is a generalized φ-function on Ω, we shortly write $\varphi\in\Phi$.

Definition 4. *[31, 15]. Let $\varphi \in \Phi$ and be ρ_φ defined by the expression*

$$\rho_\varphi(f) := \int\limits_\Omega \varphi(y, |f(y)|)\, dy \quad \textit{for all} \quad f \in L_0(\Omega).$$

We put $L_\varphi = \{f \in L_0(\Omega) : \ \rho_\varphi(\lambda_0 f) < \infty \quad \text{for some} \quad \lambda_0 > 0\}$ and

$$\|f\|_{L_\varphi} = \inf\left\{\lambda > 0 : \ \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The space L_φ is called Musielak-Orlicz space.

Let ω be a weight function on Ω, i.e., ω is a non-negative, almost everywhere positive function on Ω. In this work we considered the weighted Musielak-Orlicz spaces. We denote

$$L_{\varphi,\,\omega} = \{f \in L_0(\Omega) : \ \ f\omega \in L_\varphi\}.$$

It is obvious that the norm in this spaces is given by

$$\|f\|_{L_{\varphi,\,\omega}} = \|f\omega\|_{L_\varphi}.$$

Remark 2. *Let $\varphi(x,t) = t^{q(x)}$ in the Definition 4, where $1 \le q(x) < \infty$ and $x \in \Omega$. Then we have the definition of variable exponent weighted Lebesgue spaces $L_{q(x)}(\Omega)$ (see [15]).*

Example 1.3. The following Lemma shows that the Musielak-Orlicz spaces L_φ is p-convex BFS.

Lemma 2. *[4] Let $\Omega_1 \subset R^n$ and $\Omega_2 \subset R^m$. Let $(x,t) \in \Omega_1 \times [0,\infty)$, and $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \le p < \infty$. Suppose $f : \Omega_1 \times \Omega_2 \mapsto R$. Then the inequality*

$$\left\|\|f(x,\cdot)\|_{L_p(\Omega_2)}\right\|_{L_\varphi} \le 2^{1/p}\ \left\|\|f(\cdot,y)\|_{L_\varphi}\right\|_{L_p(\Omega_2)}$$

is valid.

We note that the Lebesgue spaces with mixed norm, weighted Lorentz spaces and e.t.c. is p-convex (p-concave) BFS. Now we reduce more general result connected with Minkowski's integral inequality.

Let X and Y be BFSs on (Ω_1, μ) and (Ω_2, ν), respectively. By $X[Y]$ and $Y[X]$ we denote the spaces with mixed norm and consisting of all functions $g \in L_0(\Omega_1 \times \Omega_2,\, \mu \times \nu)$ such that $\|g(x,\,\cdot)\|_Y \in X$ and $\|g(\cdot,\,y)\|_X \in Y$. The norms in these spaces is defined as

$$\|g\|_{X[Y]} = \|\|g(x,\,\cdot)\|_Y\|_X\,, \quad \|g\|_{Y[X]} = \|\|g(\cdot,\,y)\|_X\|_Y\,.$$

Theorem 1. *[33] Let X and Y be BFSs with the Fatou property. Then the generalized Minkowski integral inequality*

$$\|f\|_{X[Y]} \le M\,\|f\|_{Y[X]}$$

holds for all measurable functions $f(x,y)$ if and only if there exists $1 \le p \le \infty$ such that X is p-convex and Y is p-concave.

It is known that $X[Y]$ and $Y[X]$ are BFSs on $\Omega_1 \times \Omega_2$ (see [24].)

3. Main results

We consider the multidimensional Hardy type operator and its dual operator

$$Hf(x) = \int\limits_{|y|<|x|} f(y)\,dy \quad \text{and} \quad H^{*}f(x) = \int\limits_{|y|>|x|} f(y)\,dy,$$

where $f \geq 0$ and $x \in R^n$.

Now we prove a two-weight criterion for multidimensional Hardy type operator acting from the p-concave weighted BFS to weighted Lebesgue spaces.

Theorem 2. *[5] Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1 \leq p < \infty$ on R^n. Then the inequality*

$$\|Hf\|_{X_w} \leq C\,\|f\|_{L_{p,\,v}} \tag{3.1}$$

holds for every $f \geq 0$ if and only if there is a $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\,dy \right)^{\frac{\alpha}{p'}} \left\| \chi_{\{|z|>t\}}(\cdot) \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\,dy \right)^{\frac{1-\alpha}{p'}} \right\|_{X_w} < \infty. \tag{3.2}$$

Moreover, if $C > 0$ is the best possible constant in (3.1), then

$$\sup_{0<\alpha<1} \frac{p'\,A(\alpha)}{(1-\alpha)\,\left[\left(\frac{p'}{1-\alpha}\right)^{p} + \frac{1}{\alpha(p-1)}\right]^{1/p}} \leq C \leq M \inf_{0<\alpha<1} \frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

Example 3.1. *Let $n = 2$, $q(x) = q = const$, $x = (x_1, x_2) \in R^2$ and $1 < p \leq q < \infty$. Suppose that $v(x) = \dfrac{|x_1|^{\beta}}{|x|}$, $w(x) = |x|^{\gamma}$ and $\beta < \dfrac{1}{p'}$, and $\gamma = \beta - 1 - 2\left(\dfrac{1}{p'} + \dfrac{1}{q}\right)$. Then the condition of Theorem 2 is satisfied.*

For the dual operator, the below stated theorem is proved analogously.

Theorem 3. *[5] Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1 \leq p < \infty$ on R^n. Then the inequality*

$$\|H^{*}f\|_{X_w} \leq C\,\|f\|_{L_{p,\,v}} \tag{3.3}$$

holds for every $f \geq 0$ if and only if there is a $\gamma \in (0,1)$ such that

$$B(\gamma) = \sup_{t>0} \left(\int\limits_{|y|>t} [v(y)]^{-p'}\,dy \right)^{\frac{\gamma}{p'}} \left\| \chi_{\{|z|<t\}}(\cdot) \left(\int\limits_{|y|>|\cdot|} [v(y)]^{-p'}\,dy \right)^{\frac{1-\gamma}{p'}} \right\|_{X_w} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (3.3) then

$$\sup_{0<\gamma<1} \frac{p'\,B(\gamma)}{(1-\gamma)\,\left[\left(\frac{p'}{1-\gamma}\right)^p + \frac{1}{\gamma(p-1)}\right]^{1/p}} \le C \le M \inf_{0<\gamma<1} \frac{B(\gamma)}{(1-\gamma)^{1/p'}}.$$

Corollary 1. *Note that Theorem 2 and Theorem 3 in the case $X_w = L_{\varphi,w}$, $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \le p < \infty$, $x \in R^n$ was proved in [4]. In the case $X_w = L_{q,w}$, $1 < p \le q < \infty$, for $x \in (0,\infty)$, $\alpha = \dfrac{s-1}{p-1}$ and $s \in (1, p)$ Theorem 2 and Theorem 3 was proved in [35]. For $x \in R^n$ in the case $X_w = L_{q(x),w}$ and $1 < p \le q(x) \le ess \sup\limits_{x\in R^n} q(x) < \infty$ Theorem 2 and Theorem 3 was proved in [3] (see also [2]). Also, in [6] the embeddings theorems between different variable Lebesgue spaces with measures was proved.*

Remark 3. *In the case $n = 1$, $X_w = L_{q,w}$, $1 < p \le q \le \infty$, at $x \in (0,\infty)$, for classical Lebesgue spaces the various variants of Theorem 2 and Theorem 3 were proved in [19], [11], [22], [23], [29], [30], [34] and etc. In particular, in the Lebesgue spaces with variable exponent the boundedness of Hardy type operator was proved in [14], [16], [18], [20], [21], [27], [28] and etc. For $X_w = L_{q(x),w}$, $1 < p \le q(x) \le ess \sup\limits_{x\in[0,1]} q(x) < \infty$ and $x \in [0,1]$ the two-weighted criterion for one-dimensional Hardy operator was proved in [21]. Also, other type two-weighted criterion for multidimensional Hardy type operator in the case $X_w = L_{q(x),w}$, $1 < p \le q(x) \le ess \sup\limits_{x\in R^n} q(x) < \infty$ and $x \in R^n$ was proved in [27] (see also [28]). In the papers [10] and [32] the inequalities of modular type for more general operators was proved. Also, in [12] the Hardy type inequalities with special power-type weights in Orlicz spaces was proved.*

Now we reduce a compactness criteria for multidimensional Hardy type operator from weighted Lebesgue spaces into p-convex weighted Banach function spaces.

Theorem 4. *Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1 \le p < \infty$ on R^n. Then H is compact from $L_{p,v}$ to X_w if and only if the following two conditions are satisfied:*

(a) There exists an $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}} \left\| \chi_{\{|z|>t\}}(\cdot) \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}} \right\|_{X_w} < \infty;$$

(b) $$\lim_{\gamma\to+0} \sup_{0<t<\gamma} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}} \left\| \chi_{\{t<|z|<\gamma\}}(\cdot) \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}} \right\|_{X_w} = 0 \text{ and}$$

$$\lim_{\delta\to\infty}\sup_{\delta<t<\infty}\left(\int\limits_{\delta<|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0;$$

(c) for every $\varepsilon\in(0,\infty)$ the following two alternatives hold:

$$\lim_{\beta\to\varepsilon+0}\left\|\chi_{\{\varepsilon<|z|<\beta\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0\ \ \textit{and}$$

$$\lim_{\beta\to\varepsilon-0}\left\|\chi_{\{\beta<|z|<\varepsilon\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0.$$

The proof of Theorem 4 follows from the general result of paper [17].

Now suppose that the space X_w is a BFS with absolute continuous norm. Then the condition (c) of Theorem is satisfied automatically. On the other words, we have the following Corollary.

Corollary 2. *Let $v(x)$ and $w(x)$ are weights on R^n. Suppose that X_w be a p-convex weighted BFSs for $1\le p<\infty$ on R^n. Then H is compact from $L_{p,v}$ to X_w if and only if the following two conditions are satisfied:*

(a) There exists an $\alpha\in(0,1)$ such that

$$A(\alpha)=\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}<\infty;$$

$$(b)\ \lim_{\gamma\to+0}\sup_{0<t<\gamma}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{t<|z|<\gamma\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0\ \ \textit{and}$$

$$\lim_{\delta\to\infty}\sup_{\delta<t<\infty}\left(\int\limits_{\delta<|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\chi_{\{|z|>t\}}(\cdot)\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{X_w}=0.$$

Corollary 3. *Let $1<p\le q(x)\le\overline{q}<\infty$ and $v(x)$ and $w(x)$ are weights on R^n. Then H is compact from $L_{p,v}$ to $L_{q(x),w}$ if and only if the following two conditions are satisfied:*

(a) There exists an $\alpha\in(0,1)$ such that

$$A(\alpha)=\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{L_{q(\cdot),w}(|\cdot|>t)}<\infty;$$

$$(b)\ \lim_{\gamma\to+0}\sup_{0<t<\gamma}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{L_{q(\cdot),w}(|\cdot|>t)}=0\ \ and$$

$$\lim_{\delta\to\infty}\sup_{\delta<t<\infty}\left(\int\limits_{\delta<|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{L_{q(\cdot),w}(|\cdot|>t)}=0.$$

Example 3.2. *Let $q(x)=q=const$ and $1<p\le q<\infty$. Suppose that $v(x)=|x|^{\beta}$ and $w(x)=\begin{cases}|x|^{\gamma_1}, & for\ |x|<\frac12\\ |x|^{\gamma_2}, & for\ |x|\ge\frac12,\end{cases}$ and $\gamma_2+n\left(\frac{1}{p'}+\frac{1}{q}\right)<\beta\le\min\left\{\frac{n}{p'},\ \gamma_1+n\left(\frac{1}{p'}+\frac{1}{q}\right)\right\}$. Then the conditions of Corollary 3 are satisfied.*

Example 3.3. *Let $q(x)=q=const$, $x\in B(0,1)$ and $1<p\le q<\infty$. Suppose that $v(x)=|x|^{\beta}$, $w(x)=|x|^{\gamma}$ and $\beta\le\min\left\{\frac{n}{p'},\ \gamma+n\left(\frac{1}{p'}+\frac{1}{q}\right)\right\}$ or $\gamma+n\left(\frac{1}{p'}+\frac{1}{q}\right)<\beta<\frac{n}{p'}$. Then the conditions of Corollary 3 are satisfied.*

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