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Growth Of Entire Functions With Respect To The Totality Of Variables

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Abstract. This work is focused on the entire functions of severable variables. A finite set of entire functions is considered. The relationship between the orders of these functions is established under some conditions. The inequalities concerning the upper and lower orders of these functions are obtained.

Key Words and Phrases: entire functions, several complex variables.

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1. On the order of the system of entire functions of several complex variables

We consider entire functions of two complex variables represented by double power series.

Let

$$f\left(z_1, z_2\right) = \sum_{m_1, m_2}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \tag{1}$$

where $f\left(z_1, z_2\right) \in B\left(C^2\right)$ is a function of two complex variables z_1 and z_2.

It is known that

$$M\left(r_1, r_2\right) \equiv M\left(r_1, r_2; f\right) = \max_{|z_1| < r_i} \left|f\left(z_1, z_2\right)\right|, \quad i = 1, 2, \tag{2}$$

is the maximum of the modulus of the function $f\left(z_1, z_2\right)$ and

$$\max_{m_1, m_2} \left\{m_1, m_2\right\} = \nu\left(r_1, r_2\right) = \left(\nu_1\left(r_1, r_2; f\right), \nu_2\left(r_1, r_2; f\right)\right),$$

$$\mu\left(r_1, r_2; f\right) = \left|a_{v(r_1, r_2; f)}\right| r_1^{v_1(r_1, r_2; f)} r_2^{v_2(r_1, r_2; f)}. \tag{3}$$

It is proved in [1] that the functions $v_i\left(r_1, r_2; f\right) \quad (i = 1, 2)$ are increasing and continuous functions with an uncountable set of points of discontinuity with respect to each variable and $\mu\left(r_1, r_2; f\right)$ is an increasing and continuous function.

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Lemma. *(M.M. Djrbashian [2]). In order for the series* (1) *to represent an entire function of variables z_1 and z_2, it is necessary and sufficient that the relation*

$$\lim_{n+m} \sqrt[n+m]{|a_{n,m}|} = 0, \tag{4}$$

hold.

By definition (see [3] and [4]), we have

$$\overline{\lim_{r_1+r_2\to\infty}} \frac{\ln\ln M(r_1,r_2;f)}{\ln(r_1+r_2)} =^{\rho,}_{\lambda,} \tag{5}$$

$$\overline{\lim_{r_1+r_2\to\infty}} \frac{\ln M(r_1,r_2;f)}{r_1^\rho + r_2^\rho} =^{T}_{t}, \quad (0<\rho<\infty). \tag{6}$$

Let

$$\overline{\lim_{m_1+m_2\to\infty}} \frac{1}{e\rho} \left\{m_1^{m_1} m_2^{m_2} |a_{m_1,m_2}| \rho\right\}^{\frac{1}{m_1+m_2}} =^{T_1}_{t_1}, \tag{7}$$

$$\overline{\lim_{m_1+m_2\to\infty}} \frac{\ln(m_1^{m_1} m_2^{m_2})}{\ln|a_{m_1,m_2}|} =^{\rho_1}_{\lambda_1}. \tag{8}$$

It was also proved there that $\rho=\rho_1$, $\lambda=\lambda_1$ and $T_1=T$, $t=t_1$.

Let there be given a function

$$f(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}, \tag{9}$$

and a system of entire functions

$$\left\{ f_k(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} a^{(k)}_{m_1,m_2} z_1^{m_1} z_2^{m_2} \right\}_{k=1}^{n}, \tag{10}$$

where $a_{m_1,m_2}, a^{(k)}_{m_1,m_2}$ $(k=1,2,...n)$ are the complex numbers and $f_k(z_1,z_2)\in B(C^2)$.

Theorem 1.1. *Let every function $f_k(z_1,z_2)\in B(C^2)$ in the system* (10) *be of regular growth. In order for the orders of these functions to be the same, it is necessary and sufficient that the condition*

$$\ln\left\{\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|\right\} = o\left\{\ln(m_1^{m_1} m_2^{m_2})\right\}, \quad (k=1,2,...,n-1)$$

be satisfied as $m_1+m_2\to\infty$.

Proof. If the functions $f_k(z_1,z_2)$ $(k=1,2,....n)$ are of finite regular growth, then

$$\overline{\lim_{m_1+m_2\to\infty}} \frac{\ln(m_1^{m_1} m_2^{m_2})}{\ln\left|a^{(k)}_{m_1,m_2}\right|^{-1}} = \rho_k = \lambda_k = \underline{\lim_{m_1+m_2\to\infty}} \frac{\ln(m_1^{m_1} m_2^{m_2})}{\ln\left|a^{(k)}_{m_1,m_2}\right|^{-1}},$$

where $k=1,2,...n$.

Let the functions $f_k(z_1,z_2)\quad (k=1,2,....n)$ have the same order $\rho=\rho_1=\rho_2=...=\rho_n=\lambda_1=\lambda_2=...=\lambda_n=\lambda$, i.e.

$$\lim_{m_1+m_2\to\infty}\frac{-\ln\left|a^{(k)}_{m_1,m_2}\right|}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=\frac{1}{\rho}=\lim_{m_1+m_2\to\infty}\frac{-\ln\left|a^{(k+1)}_{m_1,m_2}\right|}{\ln\left(m_1^{m_1}m_2^{m_2}\right)},\quad \left(k=\overline{1,n-1}\right).$$

Hence,

$$\lim_{m_1+m_2\to\infty}\frac{-\ln\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=0\qquad \left(k=\overline{1,n-1}\right),$$

or

$$\ln\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|=o\ \left(\ln\left(m_1^{m_1}m_2^{m_2}\right)\right)\qquad \left(k=\overline{1,n-1}\right)$$

as $m_1+m_2\to\infty$.

Now let's prove the converse. Let the functions $f_k(z_1,z_2)\quad \left(k=\overline{1,(n-1)}\right)$ be of order $\rho_k\ \left(k=\overline{1,n}\right)$. Then

$$\frac{1}{\rho_k}-\frac{1}{\rho_{k+1}}=\lim_{m_1+m_2\to\infty}\frac{-\ln\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=0,\qquad \left(k=\overline{1,n-1}\right)$$

Consequently, $\rho_k=\rho_{k+1}\quad \left(k=\overline{1,\ (n-1)}\right)$. The theorem is proved.

Theorem 1.2. *Let the functions $\{f_k(z_1,z_2)\}_{k=1}^{n}\in B\left(C^2\right)$ be of regular growth. In order for the types of these functions to be the same, it is necessary and sufficient that the condition*

$$\ln\left\{\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|=o\ (m_1+m_2),\right.$$

be satisfied as $m_1+m_2\to\infty$.

Proof. $f_k(z_1,z_2)\quad \left(k=\overline{1,n}\right)$ are the regular functions, therefore

$$\varliminf_{m_1+m_2\to\infty}\frac{1}{e\rho}\left\{m_1^{m_1}m_2^{m_2}\left|a^{(k)}_{m_1,m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}}=t_k=T_k=$$

$$=\varlimsup_{m_1+m_2\to\infty}\frac{1}{e\rho}\left\{m_1^{m_1}m_2^{m_2}\left|a^{(k)}_{m_1,m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}}.$$

where $k=\overline{1,n}$.

Let the functions $\{f_k(z_1,z_2)\}$ be of the same type, i.e.

$$\lim_{m_1+m_2\to\infty}\frac{1}{e\rho}\left\{m_1^{m_1}m_2^{m_2}\left|a^{(k)}_{m_1+m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}}=T=\lim_{m_1+m_2\to\infty}\frac{1}{e\rho}\left\{m_1^{m_1}m_2^{m_2}\left|a^{(k+1)}_{m_1+m_2}\right|^{\rho}\right\}^{\frac{1}{m_1+m_2}}.$$

Hence,

$$\lim_{m_1+m_2\to\infty} \frac{\rho}{m_1+m_2}\left\{\ln\left|a^{(k)}_{m_1,m_2}\right| - \ln\left|a^{(k+1)}_{m_1,m_2}\right|\right\} = 0,$$

or

$$\ln\left\{\left|\frac{a^{(k)}_{m_1,m_2}}{a^{(k+1)}_{m_1,m_2}}\right|\right. = o\left(m_1+m_2\right),$$

as $m_1+m_2\to\infty$.

Let the functions $\{f_k(z_1,z_2)\}$ be of type T_k $\left(k=\overline{1,n}\right)$. Then

$$\ln T_k - \ln T_{k+1} = \frac{1}{\rho}\lim_{m_1,\,m_2\to\infty}\frac{1}{m_1+m_2}\ln\left|\frac{a^{(k)}_{m_1,\,m_2}}{a^{(k+1)}_{m_1,m_2}}\right| = 0.$$

Hence $T_k = T_{k+1}$ $\left(k=\overline{1,\,n}\right)$.

Theorem 1.3. *Let every function* $f_k(z_1,\,z_2)\in B\left(C^2\right)$ *in the system* (10) *be of order* ρ_k $\left(k=\overline{1,\,n}\right)$. *If*

$$\ln\left|a_{m_1,\,m_2}\right|^{-1} \sim \ln\prod_{k=1}^{n}\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}, \quad m_1+m_2\to\infty, \tag{11}$$

then the function (1) *is an entire function of order* ρ *such that*

$$\frac{1}{\rho}\geq\sum_{k=1}^{n}\frac{1}{\rho_k}. \tag{12}$$

Proof. First, let's prove that the function (1) is an entire function. By the condition of the theorem, the functions $f_k(z_1,\,z_2)$, $k=\overline{1,\,n}$, are entire functions. Therefore, by Lemma [2] we have

$$\lim_{m_1+m_2}\left|a^{(k)}_{m_1,\,m_2}\right|^{-\frac{1}{m_1+m_2}} = \infty,\ \ k=\overline{1,\,n}\,.$$

Hence for sufficiently large $R>0$ and sufficiently small $\varepsilon>0$, for $m_1+m_2>N_k$ and for fixed n we have

$$\left(R-\varepsilon\right)^{\frac{1}{n}} < \left|a^{(k)}_{m_1,\,m_2}\right|^{-\frac{1}{m_1+m_2}}, \quad k=\overline{1,\,n}\,.$$

Taking logarithms of this last relation, we have

$$\frac{m_1+m_2}{n}\ln\left(R-\varepsilon\right) < \ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1},\ \ k=\overline{1,\,n}\,.$$

Assigning values $1,\,2,\,\ldots,\,n$ to k and then summing up the resulting inequalities, we obtain

$$\left(m_1+m_2\right)\ln\left(R-\varepsilon\right) < \ln\prod_{k=1}^{n}\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}.$$

Taking into account (11), we have

$$R-\varepsilon<|a_{m_1,\,m_2}|^{-\frac{1}{m_1+m_2}}$$

for $R>0$ and $\varepsilon>0$ as $m_1+m_2>N=\max(N_1,\,N_2,\,\ldots,\,N_n)$. This means that $f(z_1,\,z_2)$ is an entire function.
Hence, we have

$$\frac{1}{\rho_k}-\frac{\varepsilon}{n}<\frac{\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=\frac{1}{\rho_k},\quad \left(k=\overline{1,\,n}\right)$$

as $m_1+m_2>N_k\ \left(k=\overline{1,\,n}\right)$ for any $\varepsilon>0$.

Assigning values 1, 2, …, n to k and summing up the resulting inequalities, we obtain

$$\sum_{k=1}^{n}\frac{1}{\rho_k}-\varepsilon<\frac{\ln\prod_{k=1}^{n}|a_{m_1,\,m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}.$$

Taking into account the condition (11), we have

$$\sum_{k=1}^{n}\frac{1}{\rho_k}-\varepsilon<\frac{\ln|a_{m_1,\,m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)},$$

as $m_1+m_2>N=\max(N_1,\,N_2,\,\ldots,\,N_n)$ for any $\varepsilon>0$. Passing to the limit as $m_1+m_2\to\infty$, we obtain

$$\sum_{k=1}^{n}\frac{1}{\rho_k}\leq\frac{1}{\rho}=\varliminf_{m_1+m_2\to\infty}\frac{\ln|a_{m_1,m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}.$$

Hence, it follows that the inequality (12) is valid.

Theorem 1.4. *Let every function $f_k(z_1,\,z_2)\in B\left(C^2\right)$ in the system* (10) *be of order $\rho_k\left(k=\overline{1,\,n}\right)$. If*

$$\ln|a_{m_1,\,m_2}|^{-1}\sim\ln\prod_{k=1}^{n}\left(\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}\right)^{\alpha_k},\tag{13}$$

$0<\alpha_k<1,\ \sum_{k=1}^{n}\alpha_k=1$, *then the function* (1) *is an entire function of order ρ such that*

$$\rho\leq\prod_{k=1}^{n}\rho_k^{\alpha_k}.\tag{14}$$

Proof. The entireness of the function $f_k(z_1,\,z_2)$ is easy to prove. By the condition of the theorem, $f_k(z_1,\,z_2)\,,\ \left(k=\overline{1,\,n}\right)$ are the entire functions. Then each of them is of order $\rho_k\,(0<\rho_k<\infty)\,,\ \left(k=\overline{1,\,n}\right)$. Therefore, we have

$$\left(\frac{1}{\rho_k}-\varepsilon\right)^{\alpha_k}<\left\{\frac{\ln\left|a^{(k)}_{m_1,\,m_2}\right|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}\right\}^{\alpha_k},\quad k=\overline{1,\,n},$$

for any $\varepsilon > 0$ and $\sum_{k=1}^{n} \alpha_k = 1,\ 0 < \alpha_k < 1$ as $m_1 + m_2 > N_k$.

Assigning values 1, 2, …, n to k in the last inequality and multiplying the resulting inequalities, we get

$$\prod_{k=1}^{n} \left(\frac{1}{\rho_k} - \varepsilon \right)^{\alpha_k} < \frac{\prod_{k=1}^{n} \left(\ln \left| a_{m_1,\, m_2}^{(k)} \right|^{-1} \right)^{\alpha_k}}{\ln \left(m_1^{m_1} m_2^{m_2} \right)}.$$

Taking into account the condition (13) of the theorem, we have

$$\prod_{k=1}^{n} \frac{1}{\rho_k^{\alpha_k}} \leq \frac{1}{\rho} \Rightarrow \rho \leq \prod_{k=1}^{n} \rho_k^{\alpha_k}.$$

Remark. *If* $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$, *then the inequality* (14) *takes the form*

$$\rho \leq \sqrt[n]{\rho_1 \rho_2 \ldots \rho_n}.$$

Theorem 1.5. *Let every function* $f_k\left(z_1,\, z_2\right) \in B\left(C^2\right)$ *in the system* (10) *be of order* ρ_k, $\left(k = \overline{1,\, n}\right)$. *If*

$$n \left(\ln \left| a_{m_1,\, m_2} \right|^{-1} \right)^{-1} \sim \sum_{k=1}^{n} \left(\ln \left| a_{m_1,\, m_2}^{(k)} \right|^{-1} \right)^{-1}, \tag{15}$$

then the function (1) *is an entire function of order* ρ *such that*

$$\rho \leq \frac{1}{n} \sum_{k=1}^{n} \rho_k. \tag{16}$$

Furthermore, if $\lambda_k \quad \left(k = \overline{1,\, n}\right)$ is the lower order of function $f_k\left(z_1,\, z_2\right),\ \left(k = \overline{1,\, n}\right)$, then λ is the lower order of function (1) such that

$$\lambda \geq \frac{1}{n} \sum_{k=1}^{n} \lambda_k. \tag{17}$$

Proof. As $f_k\left(z_1,\, z_2\right),\ \left(k = \overline{1,\, n}\right)$ are entire functions, we have

$$\frac{\ln \left(m_1^{m_1} m_2^{m_2} \right)}{\ln \left| a_{m_1,\, m_2}^{(k)} \right|^{-1}} < \rho_k + \varepsilon,\ \ m_1 + m_2 > N_k,\ \ k = \overline{1,\, n},\ \ \varepsilon > 0.$$

For $k = 1, 2, \ldots, n$ summing the last inequalities we obtain

$$\sum_{k=1}^{n} \ln \left(m_1^{m_1} m_2^{m_2} \right) \left(\ln \left| a_{m_1,\, m_2}^{(k)} \right|^{-1} \right)^{-1} < \sum_{k=1}^{n} \rho_k + \varepsilon n,$$

or

$$\ln\left(m_1^{m_1}m_2^{m_2}\right)\sum_{k=1}^{n}\left(\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}\right)^{-1}<\sum_{k=1}^{n}\rho_k+\varepsilon n.$$

Taking into account the condition (15) of the theorem, we have

$$n\ln\left(m_1^{m_1}m_2^{m_2}\right)\left(\ln\left|a_{m_1,\,m_2}\right|^{-1}\right)^{-1}<\sum_{k=1}^{n}\rho_k+\varepsilon n.$$

Passing to the limit as $m_1+m_2\to\infty$

$$n\rho\le\sum_{k=1}^{n}\rho_k\Rightarrow\rho\le\frac{1}{n}\sum_{k=1}^{n}\rho_k,$$

we can easily prove the inequality (17).

Theorem 1.6. *Let every function $f_k\left(z_1,\,z_2\right)\in B\left(C^2\right)$ in the system (10) be of order ρ_k, of type $T_k\left(0<T_k<\infty\right)$ and of lower type t_k $\left(0<t_k<\infty\right)$, $k=\overline{1,\,n}$, and let*

$$n\left(\ln\left|a_{m_1,\,m_2}\right|^{-1}\right)^{-1}\sim\sum_{k=1}^{n}\left(\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}\right)^{-1}. \tag{18}$$

Then the function (1) is an entire function of order ρ such that

$$\rho=\frac{1}{n}\sum_{k=1}^{n}\rho_k. \tag{19}$$

Proof. According to (7), the type of an entire function is calculated by the formula

$$\overline{\underline{\lim}}_{m_1+m_2}\frac{1}{e\rho_k}\left\{m_1^{m_1}m_2^{m_2}\left|a_{m_1,\,m_2}^{(k)}\right|^{\rho_k}\right\}^{\frac{1}{m_1+m_2}}=^{T_k}_{t_k},\quad k=\overline{1,\,n}\,.$$

Hence, for any $\varepsilon>0$ and $m_1+m_2>N_k$, we have

$$\left(m_1^{m_1}m_2^{m_2}\right)\left|a_{m_1,\,m_2}^{(k)}\right|^{\rho_k}<\left[\left(T_k+\varepsilon\right)e\rho_k\right]^{m_1+m_2},\quad k=\overline{1,\,n}\,.$$

Taking logarithms of this inequality, we have

$$\ln\left(m_1^{m_1}m_2^{m_2}\right)<\rho_k\ln\left|a_{m_1,m_2}^{(k)}\right|+\left(m_1+m_2\right)\ln\left[\left(T_k+\varepsilon\right)e\rho_k\right].$$

Consequently

$$\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}}<\rho_k+\frac{m_1+m_2}{\ln\left|a_{m_1,\,m_2}^{(k)}\right|}\ln\left[\left(T_k+\varepsilon\right)e\rho_k\right],$$

or

$$\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,\,m_2}^{(k)}\right|}<\rho_k+o\left(1\right),\quad k=\overline{1,\,n}\,.$$

Summing this inequality with respect to k, we obtain

$$\ln\left(m_1^{m_1}m_2^{m_2}\right)\sum_{k=1}^{n}\left(\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}\right)^{-1}<\sum_{k=1}^{n}\rho_k+o\left(1\right),\quad k=\overline{1,\,n}\,.$$

Taking into account here the condition (18) of the theorem and passing to the limit as $m_1+m_2\to\infty$, we get

$$n\rho=n\lim_{m_1+m_2}\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,\,m_2}\right|}\leq\sum_{k=1}^{n}\rho_k,$$

or

$$\rho\leq\frac{1}{n}\sum_{k=1}^{n}\rho_k. \tag{20}$$

Similarly we can prove that

$$\rho>\lambda\geq\frac{1}{n}\sum_{k=1}^{n}\rho_k. \tag{21}$$

(20) and (21) imply (19).

Theorem 1.7. *Let every function $f_k\left(z_1,\,z_2\right)\in B\left(C^2\right)$ in the system* (10) *be of regular growth of order ρ_k $\left(k=\overline{1,\,n}\right)$. If*

$$\ln\left|a_{m_1,\,m_2}\right|^{-1}\sim\prod_{k=1}^{n}\left(\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}\right)^{\alpha_k},\ \ 0<\alpha_k<1, \tag{22}$$

$\left(k=\overline{1,\,n}\right)$, $\sum_{k=1}^{n}\alpha_k=1$, *then the function* (1) *is an entire function of regular growth of order ρ such that*

$$\rho=\prod_{k=1}^{n}\rho_k^{\alpha_k}. \tag{23}$$

Proof. Using the definition of the order of an entire function, we have

$$\underline{\lim}_{m_1+m_2\to\infty}\frac{\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=\frac{1}{\rho_k},\ \ k=\overline{1,\,n},$$

or

$$\underline{\lim}_{m_1+m_2\to\infty}\left(\frac{\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}\right)^{\alpha_k}=\frac{1}{\rho_k^{(\alpha_k)}},\ \ k=\overline{1,\,n}\,.$$

Assigning values 1, 2, …, n to k and then multiplying the resulting equalities, we have

$$\underline{\lim}_{m_1+m_2\to\infty}\frac{\prod_{k=1}^{n}\left(\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}\right)^{\alpha_k}}{\left(\ln\left(m_1^{m_1}m_2^{m_2}\right)\right)^{\sum_{k=1}^{n}\alpha_k}}=\prod_{k=1}^{n}\frac{1}{\rho_k^{(\alpha_k)}}. \tag{24}$$

Taking into account the condition (22) of the theorem in (24), we have

$$\frac{1}{\rho}=\varliminf_{m_1+m_2\to\infty}\frac{\ln|a_{m_1m_2}|^{-1}}{\ln\left(m_1^{m_1}m_2^{m_2}\right)}=\frac{1}{\prod_{k=1}^{n}\rho_k^{\alpha_k}},$$

or

$$\rho=\prod_{k=1}^{n}\rho_k^{\alpha_k},$$

which completes the proof.

Corollary. *If* $\alpha_1=\alpha_2=\ldots=\alpha_n=\frac{1}{n}$, *then the equality* (23) *takes the form* $\rho=\sqrt[n]{\rho_1\rho_2\ldots\rho_n}$.

Theorem 1.8. *Let every function* $f_k\left(z_1,\,z_2\right)\in B\left(C^2\right)$ *in the system* (10) *be of order* ρ_k *and of lower order* λ_k $\left(0<\lambda_k\le\rho_k<\infty\right),\ k=\overline{1,\,n}$. *If*

$$n\left(\ln|a_{m_1,m_2}|^{-1}\right)^{-1}\sim\sum_{k=1}^{n}\left(\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}\right)^{-1},\tag{25}$$

then $f\left(z_1,\,z_2\right)$ *is an entire function of order* ρ *and of lower order* λ *such that*

$$\sum_{k=1}^{n-1}\lambda_k\le\left(n\lambda-\lambda_n;\,n\rho-\rho_n\right)\le\sum_{k=1}^{n-1}\rho_k.\tag{26}$$

Proof. The entireness of the function $f\left(z_1,\,z_2\right)$ is easy to prove. According to (8), we have

$$\overline{\underline{\lim}}_{m_1+m_2\to\infty}\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}}=\begin{matrix}\rho_k\\ \lambda_k\end{matrix}\ ,\quad k=\overline{1,\,n}\,.$$

Hence, for every $\varepsilon>0$ and $m_1+m_2>N_k$, we have

$$\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,\,m_2}^{(k)}\right|^{-1}}<\rho_k+\varepsilon,\quad k=\overline{1,\,n}\,.\tag{27}$$

Summing up the inequalities (27) for $k=1,\,2,\,\ldots,\,n$, we obtain

$$\ln\left(m_1^{m_1}m_2^{m_2}\right)\sum_{k=1}^{n}\left(\ln\left|a_{m_1,m_2}^{(k)}\right|^{-1}\right)^{-1}<\sum_{k=1}^{n}\rho_k+\varepsilon n.$$

Taking into account the condition (25) of the theorem in last inequality, we obtain

$$n\frac{\ln\left(m_1^{m_1}m_2^{m_2}\right)}{\ln\left|a_{m_1,\,m_2}\right|^{-1}}<\sum_{k=1}^{n}\rho_k+\varepsilon n.$$

Passing to the limit as $m_1+m_2\to\infty$, we obtain

$$n\rho\le\sum_{k=1}^{n}\rho_k=\sum_{k=1}^{n-1}\rho_k+\rho_n,\tag{28}$$

or

$$n\rho - \rho_n \le \sum_{k=1}^{n-1} \rho_k.$$

We can easily prove that

$$n\lambda - \lambda_n \le \sum_{k=1}^{n-1} \rho_k, \quad n\lambda - \lambda_n \ge \sum_{k=1}^{n-1} \lambda_k, \quad n\rho - \rho_n \ge \sum_{k=1}^{n-1} \lambda_k. \tag{29}$$

(28) and (29) imply the validity of (26).

Theorem 1.9. *Let every function $f_k(z_1, z_2) \in B(C^2)$ in the system (10) be of order ρ_k and of lower order λ_k $(0 < \lambda_k \le \rho_k < \infty)$, $k = \overline{1, n}$, and let*

$$\ln |a_{m_1,m_2}|^{-1} \sim \prod_{k=1}^{n} \left(\ln \left| a_{m_1,m_2}^{(k)} \right|^{-1} \right)^{\alpha_k}, \tag{30}$$

where $0 < \alpha_k < 1$, $\sum_{k=1}^{n} \alpha_k = 1$. Then the function $f(z_1, z_2)$ is an entire function of order ρ and of lower order λ, such that

$$\prod_{k=1}^{n-1} \lambda_k^{\alpha_k} = \left\{ \frac{\lambda}{\lambda_n^{\alpha_n}}, \frac{\rho}{\rho_n^{\alpha_n}} \right\} \le \prod_{k=1}^{n-1} \rho_k^{\alpha_k}. \tag{31}$$

The proof is similar to that of Theorem 1.8.

Corollary. *If $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \frac{1}{n}$, then the relation (31) takes the form*

$$\sqrt[n-1]{\lambda_1 \lambda_2 \ldots \lambda_{n-1}} \le \frac{\lambda}{\sqrt[n]{\lambda_n}}, \frac{\rho}{\sqrt[n]{\rho_n}} \le \sqrt[n-1]{\rho_1 \rho_2 \ldots \rho_{n-1}}.$$

Theorem 1.10. *Let every function $f_k(z_1, z_2) \in B(C^2)$ in the system (10) be of order ρ_k and of lower order λ_k $(0 < \lambda_k < \infty)$. If*

$$\ln |a_{m_1,m_2}| \sim \ln \left(\prod_{k=1}^{n} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} \right), \tag{32}$$

where $\alpha_k = const$ $\left(k = \overline{1, n}\right)$, then the function $f(z_1, z_2)$ is an entire function of order ρ and of lower order λ with

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k} \le \left\{ \frac{1}{\rho} - \frac{\alpha_n}{\rho_n}; \frac{1}{\lambda} - \frac{\alpha_n}{\lambda_n} \right\} \le \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda_k}. \tag{33}$$

Proof. The entireness of the function $f(z_1, z_2)$ is proved as in Theorem 1.3 using condition (32).

According to (8), we have

$$\ln \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\frac{\alpha_k \ln\left(m_1^{m_1} m_2^{m_2}\right)}{\rho_k + \varepsilon}, \quad k = \overline{1, n}, \tag{34}$$

for any $\varepsilon > 0$ as $m_1 + m_2 > N_k$.

Summing this inequality for $k = 1, 2, \ldots, n$, we have

$$\ln \prod_{k=1}^{n} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \sum_{k=1}^{n} \frac{\alpha_k}{\rho_1 + \varepsilon}.$$

Taking into account the condition (32) of the theorem, we get

$$\ln \left| a_{m_1,m_2} \right| < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \sum_{k=1}^{n} \frac{\alpha_k}{\rho_k + \varepsilon},$$

or

$$\frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln \left| a_{m_1,m_2} \right|^{-1}} < \frac{1}{\sum_{k=1}^{n} \frac{\alpha_k}{\rho_k+\varepsilon}}.$$

Passing to the limit as $m_1 + m_2 \to \infty$, we obtain

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k} \leq \frac{1}{\rho} - \frac{\alpha_n}{\rho_n}. \tag{35}$$

From (34) we have

$$\ln \prod_{k=1}^{n-1} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k + \varepsilon}. \tag{36}$$

For subsequence $\left\{ m_1 = m_1^{(i)},\, m_2 = m_2^{(i)} \right\}$, we have

$$\ln \left| a_{m_1,m_2}^{(n)} \right|^{\alpha_n} < -\frac{\alpha_n \ln\left(m_1^{m_1} m_2^{m_2}\right)}{\lambda_n + \varepsilon}. \tag{37}$$

Summing up the inequalities (36) and (37), we obtain

$$\ln \prod_{k=1}^{n-1} \left| a_{m_1,m_2}^{(k)} \right|^{\alpha_k} < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \left\{ \sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k + \varepsilon} + \frac{\alpha_n}{\lambda_n + \varepsilon} \right\}.$$

Taking into account the condition (32) of the theorem, we have

$$\ln \left| a_{m_1,m_2} \right| < -\left(\ln\left(m_1^{m_1} m_2^{m_2}\right)\right) \left\{ \sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k + \varepsilon} + \frac{\alpha_n}{\lambda_n + \varepsilon} \right\}.$$

Passing to the limit as $m_1 + m_2 \to \infty$, we obtain

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{\rho_k} \leq \frac{1}{\lambda} - \frac{\alpha_n}{\lambda_n}. \tag{38}$$

It is easy to prove that

$$\frac{1}{\lambda} - \frac{\alpha_n}{\lambda_n} \leq \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda_k}, \quad \frac{1}{\rho} - \frac{\alpha_n}{\rho_n} \leq \sum_{k=1}^{n-1} \frac{\alpha_k}{\lambda_k}. \tag{39}$$

From (35), (38) and (39) we get the validity of (33).

2. On the type of the system of entire functions of several complex variables

Let there be given the functions $f(z_1,\, z_2) \in B\left(C^2\right)$,

$$f(z_1,\, z_2) = \sum_{m_1,m_2}^{\infty} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}, \tag{40}$$

$$\left\{ f_k(z_1,\, z_2) = \sum_{m_1,m_2}^{\infty} a_{m_1,m_2}^{(k)} z_1^{m_1} z_2^{m_2} \right\}_{k=1}^{n}, \tag{41}$$

It is known from [4] and [5] that

$$\overline{\lim_{r_1+r_2\to\infty}} \frac{\ln\ln M(r_1,\, r_2;\, f)}{\ln(r_1+r_2)} \overset{\rho}{\underset{\lambda}{=}} \overline{\lim_{m_1+m_2\to\infty}} \frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)}{\ln|a_{m_1,\, m_2}|^{-1}}, \tag{42}$$

$$\overline{\lim_{r_1+r_2\to\infty}} \frac{\ln M(r_1,\, r_2;\, f)}{r_1^{\rho}+r_2^{\rho}} \overset{T}{\underset{t}{=}} \frac{1}{e\rho} \overline{\lim_{m_1+m_2\to\infty}} \left\{ m_1^{m_1} m_2^{m_2} |a_{m_1,m_2}|^{\rho} \right\}^{\frac{1}{m_1+m_2}}. \tag{43}$$

Theorem 2.1. *Let every function $f_k(z_1,\, z_2)$ in the system (41) be an entire function of order ρ_k and of type T_k $(0 < T_k < +\infty)$, $\left(k = \overline{1,\, n}\right)$. If*

$$|a_{m_1,\, m_2}| \sim \prod_{k=1}^{n} \left| a_{m_1,\, m_2}^{(k)} \right|, \tag{44}$$

then the function (40) is an entire function of order ρ and of type T with

$$\left(\frac{T}{\alpha}\right)^{\alpha} \le \prod_{k=1}^{n} \left(\frac{T_k}{\alpha_k}\right)^{\alpha_k}, \tag{45}$$

where $\alpha = \frac{1}{\rho}$, $\alpha_k = \frac{1}{\rho_k}$ and $\alpha = \sum_{k=1}^{n} \alpha_k$.

Proof. By the condition of the theorem, the functions in the system (41) are entire functions. Then

$$\lim_{m_1+m_2\to\infty} \left| a_n^{(k)} \right|^{-\frac{1}{m_1+m_2}} = +\infty, \quad k = \overline{1,\, n}.$$

Considering condition (44) of the theorem, we obtain

$$\lim_{m_1+m_2\to\infty} |a_{m_1,\, m_2}|^{-\frac{1}{m_1+m_2}} \ge \prod_{k=1}^{n} \lim_{m_1+m_2} \left| a_{m_1,\, m_2}^{(k)} \right|^{-\frac{1}{m_1+m_2}} = +\infty.$$

Hence, the function $f(z_1,\, z_2)$ is entire. Next, by virtue of condition (44) of the theorem, we have

$$\left(m_1^{m_1} m_2^{m_2} |a_{m_1,\, m_2}|^{\rho}\right)^{\frac{1}{\rho(m_1+m_2)}} = \left\{ m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right\}^{\frac{1}{\rho}} |a_{m_1,\, m.2}|^{\frac{1}{m_1+m_2}} \sim$$

$$\sim \left(m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right)^{\sum_{k=1}^n \frac{1}{\rho_k}} \left(\prod_{k=1}^n \left| a_{m_1,\, m_2}^{(k)} \right| \right)^{\frac{1}{m_1+m_2}} .$$

Hence, for any $\varepsilon > 0$ and $m_1 + m_2 > N = \max(N_1 \dots N_n)$, we have

$$\{ m_1^{m_1} m_2^{m_2} |a_{m_1,\, m_2}|^{\rho} \}^{\frac{1}{\rho(m_1+m_2)}} < (1+\varepsilon) \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,m_2}^{(1)} \right|^{\rho} \right\}^{\frac{1}{\rho(m_1+m_2)}} \times$$

$$\times \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(2)} \right|^{\rho_2} \right\}^{\frac{1}{\rho_2(m_1+m_2)}} \dots \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(n)} \right|^{\rho_n} \right\}^{\frac{1}{\rho_n(m_1+m_2)}} .$$

Passing to the limit as $m_1 + m_2 \to \infty$, we have

$$(\rho T)^{\frac{1}{\rho}} \le (\rho_1 T_1)^{\frac{1}{\rho_1}} \dots (\rho_n T_n)^{\frac{1}{\rho_n}} ,$$

which proves (45).

Theorem 2.2. *Let the functions in the system* (41) *be of order* ρ_k, *of type* T_k $(0 < T_k < +\infty)$ *and of lower type* t_k $(0 < t_k < +\infty)$, $k = \overline{1, n}$. *If*

$$|a_{m_1,\, m_2}| \sim \prod_{k=1}^n \left| a_{m_1,\, m_2}^{(k)} \right| , \tag{46}$$

then the function (40) *is an entire function of order* ρ, *of type* T *and of lower type* t *with*

$$(\rho t)^{\frac{1}{\rho}} \le \left\{ \begin{matrix} (\rho t_k)^{\frac{1}{\rho_k}} , \prod_{i=1, i\neq k}^n (\rho_i T_i)^{\frac{1}{\rho_i}} \\ (\rho T_k)^{\frac{1}{\rho_k}} , \prod_{i=1, i\neq k}^n (\rho_i t_i)^{\frac{1}{\rho_i}} \end{matrix} \right\} \le (\rho T)^{\frac{1}{\rho}} . \tag{47}$$

Proof. The proof of the entireness of function (40) is carried out as in Theorem 2.1. Let $\psi_k(x, y) \ge 0$, $k = \overline{1, n}$. Then

$$\underline{\lim} \prod_{k=1}^n \psi_k(x, y) \le \left\{ \begin{matrix} \underline{\lim} \psi_i(x,y), \ \overline{\lim} \prod_{k=1, k\neq i}^n \psi_k(x,y) \\ \overline{\lim} \psi_i(x,y), \ \underline{\lim} \prod_{k=1, k\neq n}^n \psi_k(x,y) \end{matrix} \right\} \le \overline{\lim} \prod_{k=1}^n \psi_k(x, y) . \tag{48}$$

From (43) it follows that

$$\left\{ m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right\}^{\frac{1}{\rho}} |a_{m_1,\, m_2}|^{\frac{1}{m_1+m_2}} \sim \left\{ m_1^{\frac{m_1}{m_1+m_2}} m_2^{\frac{m_2}{m_1+m_2}} \right\}^{\sum_{k=1}^n \frac{1}{\rho_k}} \times$$

$$\times \left(\prod_{k=1}^n \left| a_{m_1,\, m_2}^{(k)} \right| \right)^{\frac{1}{m_1+m_2}} = \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(1)} \right|^{\rho_1} \right\}^{\frac{1}{\rho_1(m_1+m_2)}} \times$$

$$\times \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(2)} \right|^{\rho_2} \right\}^{\frac{1}{\rho_2(m_1+m_2)}} \dots \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\, m_2}^{(n)} \right|^{\rho_n} \right\}^{\frac{1}{\rho_n(m_1+m_2)}} =$$

$$= \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\,m_2}^{(k)} \right|^{\rho_k} \right\}^{\frac{1}{\rho_k(m_1+m_2)}} \times \left\{ m_1^{m_1} m_2^{m_2} \prod_{\substack{i=1 \\ i\neq k}}^{n} \left| a_{m_1,\,m_2}^{(i)} \right|^{\rho_i} \right\}^{\frac{1}{\rho_i(m_1+m_2)}} .$$

Using (49), we have

$$(\rho t)^{\frac{1}{\rho}} \leq \left\{ \begin{matrix} (\rho_k t_k)^{\frac{1}{\rho_k}} ,\prod_{i=1,i\neq k}^{n} (\rho_i T_i)^{\frac{1}{\rho_i}} \\ (\rho_k T_k)^{\frac{1}{\rho_k}} ,\prod_{i=1,i\neq k}^{n} (\rho_i t_i)^{\frac{1}{\rho_i}} \end{matrix} \right\} \leq (\rho T)^{\frac{1}{\rho}} .$$

Note that in case of one variable Theorem 2.2 was proved in [7].

In particular, for two functions $f_1(z_1,\,z_2)$ and $f_2(z_1,\,z_2)$ we have a relation

$$(\rho t)^{\frac{1}{\rho}} \leq \left\{ \begin{matrix} (\rho_1 t_1)^{\frac{1}{\rho_1}} & , (\rho_2 T_2)^{\frac{1}{\rho_2}} \\ (\rho_1 T_1)^{\frac{1}{\rho_1}} & , (\rho_2 T_2)^{\frac{1}{\rho_2}} \end{matrix} \right\} \leq (\rho T)^{\frac{1}{\rho}} .$$

In case of one variable this last relation was proved in [8].

Theorem 2.3. *Let every function $f_k(z_1,\,z_2)$ in the system (41) be of regular order ρ_k, $k=\overline{1,\,n}$, of type T_k and of lower type t_k. If*

$$\ln\left|a_{m_1,\,m_2}\right| \sim \left\{ \left| a_{m_1,\,m_2}^{(1)} \right|^{\alpha_1} \left| a_{m_1,\,m_2}^{(2)} \right|^{\alpha_2} \ldots \left| a_{m_1,\,m_2}^{(k)} \right|^{\alpha_k} \right\}, \tag{49}$$

where α_k is a constant $(k=1,\,2,\,\ldots,\,n)$, *then the function (40) is an entire function of order ρ, of type T and of lower type t such that*

$$\prod_{k=1}^{n-1} (t_k\rho_k)^{\frac{\alpha_k}{\rho_k}} \leq \left\{ \frac{(\rho t)^{\frac{1}{\rho}}}{(\rho_n t_n)^{\frac{\alpha_n}{\rho_n}}},\ \frac{(\rho T)^{\frac{1}{\rho}}}{(\rho_n T_n)^{\frac{\alpha_n}{\rho_n}}} \right\} \leq \prod_{k=1}^{n-1} (T_k\rho_k)^{\frac{\alpha_k}{\rho_k}} . \tag{50}$$

Proof. According to (43), we have

$$\overline{\lim_{m_1+m_2\to\infty}} \frac{1}{e\rho_k} \left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\,m_2}^{(k)} \right|^{\rho_k} \right\}^{\frac{1}{m_1+m_2}} =_{t_k}^{T_k}, \tag{51}$$

$$k=(1,\,2,\,\ldots,\,n)\,.$$

From (51), for any $\varepsilon>0$ and $m_1+m_2>N_k$, we have

$$\left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\,m_2}^{(k)} \right|^{\rho_k} \right\}^{\frac{1}{m_1+m_2}} < e\rho_k\,(T_k+\varepsilon)\,,\ \ k=\overline{1,\,n},$$

and for the subsequence $\left\{ m_1=m_1^{(i)},\, m_2=m_2^{(i)} \right\}$

$$\left\{ m_1^{m_1} m_2^{m_2} \left| a_{m_1,\,m_2}^{(n)} \right|^{\rho_n} \right\}^{\frac{1}{m_1+m_2}} < e\rho_n\,(t_n+\varepsilon)\,,$$

or

$$\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{\alpha_k}{\rho_k}} \left|a_{m_1,\, m_2}^{(k)}\right|^{\alpha_k} < \left[\left(T_k+\varepsilon\right) e\rho_k\right]^{\frac{\alpha_k\left(m_1+m_2\right)}{\rho_k}}, \tag{52}$$

$$k = (1,\, 2,\, \ldots,\, n)\,.$$

Taking logarithms of these inequalities and then summing them up for $k = 1,\, 2,\, \ldots,\, n-1$, we obtain

$$\ln \prod_{k=1}^{n} \left|a_{m_1,\, m_2}^{(k)}\right|^{\alpha_k} <$$

$$< \ln \left\{ \frac{\prod_{k=1}^{n-1} \left[\left(T_k+\varepsilon\right) e\rho_k\right]^{\frac{\alpha_k\left(m_1+m_2\right)}{\rho_k}}}{\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{\alpha_k}{\rho_k}}} \times \frac{\left[\left(t_k+\varepsilon\right) e\rho_k\right]^{\frac{\alpha_n\left(m_1+m_2\right)}{\rho_n}}}{\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{\alpha_n}{\rho_n}}} \right\}.$$

Taking into account the condition (49), we obtain

$$\ln\left|a_{m_1,\, m_2}\right| < \sum_{k=1}^{n-1} \frac{\alpha_k\left(m_1+m_2\right)}{\rho_k} \ln\left[\left(T_k+\varepsilon\right) e\rho_k\right] +$$

$$+\frac{\alpha_n\left(m_1+m_2\right)}{\rho_n} \ln\left[\left(t_n+\varepsilon\right) e\rho_n\right] - \left(\frac{\alpha_1}{\rho_1}+\frac{\alpha_2}{\rho_2}+\ldots+\frac{\alpha_n}{\rho_n}\right) \ln\left(m_1^{m_1} m_2^{m_2}\right).$$

According to the Theorem 2.9.12 [3], we have

$$\frac{1}{\rho} = \frac{\alpha_1}{\rho_1}+\frac{\alpha_2}{\rho_2}+\ldots+\frac{\alpha_n}{\rho_n}.$$

Then we get

$$\frac{\ln\left(m_1^{m_1} m_2^{m_2}\right)^{\frac{1}{\rho}} + \ln\left|a_{m_1,\, m_2}\right|}{m_1+m_2} < \sum_{k=1}^{n} \frac{\alpha_k}{\rho_k} \ln\left[\left(T_k+\varepsilon\right) e\rho_k\right] + \frac{\alpha_n}{\rho_n} \ln\left[\left(t_n+\varepsilon\right) e\rho_n\right].$$

Hence, we obtain

$$\ln\left\{m_1^{m_1} m_2^{m_2} \left|a_{m_1,\, m_2}\right|^{\rho}\right\}^{\frac{1}{\rho\left(m_1+m_2\right)}} < \ln \prod_{k=1}^{n-1} \left[\left(T_k+\varepsilon\right) e\rho_k\right]^{\frac{\alpha_k}{\rho_k}} \left[\left(t_n+\varepsilon\right) e\rho_n\right]^{\frac{\alpha_n}{\rho_n}}.$$

Passing to the limit as $m_1+m_2 \to \infty$, we have

$$\left(te\rho\right)^{\frac{1}{\rho}} \le \prod_{k=1}^{n-1} \left(T_k e\rho_k\right)^{\frac{\alpha_k}{\rho_k}} \quad \left(t_n e\rho_n\right)^{\frac{\alpha_n}{\rho_n}},$$

or

$$\frac{\left(\rho t\right)^{\frac{1}{\rho}}}{\left(\rho_n t_n\right)^{\frac{\alpha_n}{\rho_n}}} \le \prod_{k=1}^{n-1} \left(\rho_k T_k\right)^{\frac{\alpha_k}{\rho_k}}. \tag{53}$$

Note that

$$\frac{(T\rho)^{\frac{1}{\rho}}}{(T_n\rho_n)^{\frac{\alpha_n}{\rho_n}}} < \prod_{k=1}^{n-1} (T_k\rho_k)^{\frac{\alpha_k}{\rho_k}}\,, \tag{54}$$

and

$$\frac{(\rho t)^{\frac{1}{\rho}}}{(\rho_n t_n)^{\frac{\alpha_n}{\rho_n}}} \geq \prod_{k=1}^{n-1} (\rho_k t_k)^{\frac{\alpha_k}{\rho_k}}\ , \quad \prod_{k=1}^{n} (t_k\rho_k)^{\frac{\alpha_k}{\rho_k}} \leq \frac{(T\rho)^{\frac{1}{\rho}}}{(T_n\rho_n)^{\frac{\alpha_n}{\rho_n}}}. \tag{55}$$

From (53), (54) and (55) we get the validity of the theorem.

References

[1] J. Gapala Krishna, Maximum term of a power series in one and several complex variables, Pacific Journal of Mat. vol. 29, No. 3, 1969, pp. 609-612.

[2] M.M. Djrbashian. On the theory of some classes of entire functions of several variables. Izv. AN Arm. SSR, seriya phis.-mat. estestv. i tekhnich. nauk, vol. 8, 1955, pp. 1-23.

[3] F. Salimov, R. Abbasov. The growth of entire functions of several complex variables I. Baku-Elm-2006, 507 pages

[4] F. Salimov, R. Abbasov, On lower order and type of entire functions of twocomplex variables. Mathematics, 2004, No.1

[5] F.Q. Salimov. On the order of entire functions of several complex variables. Izv. Vuzov SSSR, 1972, No. 5, pp. 74-79.

[6] S.N. Srivastava, On the order of integral functions. Riv. Math. Univ. Parma, (2), 6(1961).

[7] F. Salimov. The entire functions I, Baku-Elm-2003, 300 pages.

[8] S.N. Srivastava, On the order of integral functions Riv. Math. Univ. Parma, (2), 2(1961).

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