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On frame properties of degenerate system of exponents in Hardy classes

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Abstract. Part of the classical system of exponents with degenerate coefficient is considered. Frame properties of this system in Hardy classes are studied in case when the coefficient may not satisfy the Muckenhoupt condition.

Key Words and Phrases: system of exponents, degeneration, frames, Hardy class, the Muckenhoupt condition

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1. Introduction

In [1], N.K.Bari raised the issue of the existence of normalized basis for L_2 which is not a Riesz basis. The first example was given by K.I.Babenko [2]. He proved that the degenerate system of exponents $\left\{ |x|^{\alpha} e^{int} \right\}_{n \in \mathbb{Z}}$ with $|\alpha| < \frac{1}{2}$ forms a basis for $L_2(-\pi, \pi)$ but is not a Riesz basis when $\alpha \neq 0$. This result has been generalized by V.F.Gaposhkin [3]. In [4], the condition on the weight ρ was found which makes the system $\left\{ e^{int} \right\}_{n \in \mathbb{Z}}$ form a basis for the weight space $L_{2,\rho}(-\pi, \pi)$ with a norm $\|f\|_{2,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^2 \rho(t)\, dt \right)^{\frac{1}{2}}$. All the above-mentioned works consider the cases when the weight or the degenerate coefficient satisfies the Muckenhoupt condition (see, for example, [5]). Basis properties of the linear phase systems of exponents and sines in weighted Lebesgue spaces have been studied in [6-8], while those of the systems of exponents with degenerate coefficients have been studied in [9;10]. In case when the Muckenhoupt condition does not hold, then these systems have finite defects: some parts of them are complete and minimal, but they (i.e. these parts) do not form a basis. The question then arises: are these systems frames?

Note that when solving many problems in mathematical physics and mechanics by Fourier method (see, e.g. [11-13]), one has to deal with the systems of the form $\{\rho(t) \sin[(n+\alpha)t + \beta]\}_{n \in \mathbb{N}}$, where $\rho(t)$ is a degenerate coefficient, $\alpha, \beta \in \mathbb{R}$ are real parameters, and $\mathbb{N}$ is the set of natural numbers. Hence there comes a necessity to study the frame properties of these systems in various function spaces. This is directly related to the study of the same issue for a part of the system of exponents $\left\{ \rho(t)\, e^{int}] \right\}_{n \geq 0}$

($\{\rho(t)e^{int}]\}_{n\leq m}$) in Hardy classes. Our work is dedicated to the study of frame properties of these systems in Hardy classes in the case when the weight $\rho(t)$ is given in the form of power function. Note that the similar issues concerning degenerate systems of sines, cosines and exponents in Lebesgue spaces have been studied in [14;15].

2. Needful information

We will use some concepts and facts from the theory of frames and the standard notation. $\exists$ will mean "there exist(s)"; $\mathbb{Z}_+ = \{0\}\bigcup\mathbb{N}$; $\mathbb{Z} = -\mathbb{N}\bigcup\mathbb{Z}_+$; δ_{nk} will be the Kronecker symbol, and $\overline{(\cdot)}$ will stand for conjugation.

Let X be some Banach space with a norm $\|\cdot\|_X$, and X^* denote its conjugate with the corresponding norm $\|\cdot\|_{X^*}$ By $L[M]$ we denote the linear span of the set $M\subset X$, and $\overline{M}$ will stand for the closure of M.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ *is said to be complete in* X *if* $\overline{L\left[\{x_n\}_{n\in\mathbb{N}}\right]} = X$. *It is called minimal in* X *if* $x_k\notin\overline{L\left[\{x_n\}_{n\neq k}\right]}$, $\forall k\in\mathbb{N}$.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ *is said to be uniformly minimal in* X *if* $\exists\delta>0$: $\inf\limits_{\forall u\in L\left[\{x_n\}_{n\neq k}\right]}\|x_k-u\|_X\geq\delta\,\|x_k\|_X$, $\forall k\in\mathbb{N}$.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ *is said to be a basis for* X *if* $\forall x\in X$, $\exists!\,\{\lambda_n\}_{n\in\mathbb{N}}\subset K$: $x=\sum_{n=1}^{\infty}\lambda_n x_n$.

If system $\{x_n\}_{n\in\mathbb{N}}\subset X$ *forms a basis for* X, *then it is uniformly minimal.*

We will also need some facts about frames.

Definition 1 *Let* X *be a Banach space and* $\mathcal{K}$ *be a Banach sequence space indexed by* $\mathbb{N}$. *Let* $\{f_k\}_{k\in\mathbb{N}}\subset X$, $\{g_k\}_{k\in\mathbb{N}}\subset X^*$. *Then* $\left(\{g_k\}_{k\in\mathbb{N}}\,,\ \{f_k\}_{k\in\mathbb{N}}\right)$ *is an atomic decomposition of* X *with respect to* $\mathcal{K}$ *if :*
(i) $\{g_k(f)\}_{k\in\mathbb{N}}\in\mathcal{K}$, $\forall f\in X$;
(ii) $\exists A,B>0{:}A\|f\|_X\leq\left\|\{g_k(f)\}_{k\in\mathbb{N}}\right\|_{\mathcal{K}}\leq B\|f\|_X$, $\forall f\in X$;
(iii) $f=\sum_{k=1}^{\infty}g_k(f)\ f_k$, $\forall f\in X$.

Definition 2. *Let* X *be a Banach space and* $\mathcal{K}$ *be a Banach sequence space indexed by* $\mathbb{N}$. *Let* $\{g_k\}_{k\in\mathbb{N}}\subset X^*$ *and* $S:\mathcal{K}\to X$ *be a bounded operator. Then* $\left(\{g_k\}_{k\in\mathbb{N}}\ ,\ S\right)$ *is a Banach frame for* X *with respect to* $\mathcal{K}$ *if :*
(i) $\{g_k(f)\}_{k\in\mathbb{N}}\in\mathcal{K}$, $\forall f\in X$;
(ii) $\exists A,B>0{:}A\|f\|_X\leq\left\|\{g_k(f)\}_{k\in\mathbb{N}}\right\|_{\mathcal{K}}\leq B\|f\|_X$, $\forall f\in X$;
(iii) $S\ \left[\{g_k(f)\}_{k\in\mathbb{N}}\right]=f$, $\forall f\in X$.

Proposition 1. *Let* X *be a Banach space and* $\mathcal{K}$ *a Banach sequence space indexed by* $\mathbb{N}$. *Assume that the canonical unit vectors* $\{\delta_k\}_{k\in\mathbb{N}}$ *constitute a basis for* $\mathcal{K}$ *and let* $\{g_k\}_{k\in\mathbb{N}}\subset X^*$ *and* $S:\mathcal{K}\to X$ *be a bounded operator. Then the following statements are equivalent:*

(i) $(\{g_k\}_{k\in\mathbb{N}},S)$ *is a Banach frame for* X *with respect to* $\mathcal{K}$.

(ii) $(\{g_k\}_{k\in\mathbb{N}},\{S(\delta_k)\}_{k\in\mathbb{N}})$ *is an atomic decomposition of* X *with respect to* $\mathcal{K}$.

From these statements we directly obtain that if any element in X can not be expanded with respect to the system $\{x_n\}_{n\in\mathbb{N}} \subset X$, then it doesn't form a frame for X.

More details about these and related facts can be found in [16-18].

We will also use the symbol " $\sim$ ". The expression $f \sim g$, $t \to a$, means that in sufficiently small neighborhood of the point $t = a$ there holds the inequality $0 < \delta \leq \left|\frac{f(t)}{g(t)}\right| \leq \delta^{-1} < +\infty$.

Let us recall the definition of the Hardy classes. By H_p^+ we denote the usual Hardy class of analytical functions inside the unit circle furnished with the norm

$$\|f\|_{H_p^+} = \sup_{0<r<1} \left(\int_{-\pi}^{\pi} \left|f\left(re^{it}\right)\right|^p dt\right)^{1/p}, \ p \geq 1.$$

Restriction of class H_p^+ to the unit circumference $\partial\omega$ will be denoted by L_p^+ . Spaces H_p^+ and L_p^+ are isomorphic and isometric.

3. Basicity

Consider the case when the system $E_+^0(\rho))$ forms a basis for H_p^+ with $E_+^{(k)}(\rho) \equiv \left\{\rho(t)\, e^{int}\right\}_{n\geq k}$. We will assume that the degenerate coefficient ρ is given in the form of power function

$$\rho(t) = \left(e^{it} - 1\right)^{\alpha_0} \prod_{k=1}^{r} \left(e^{it} - e^{it_k}\right)^{\alpha_k},$$

where $\{t_k\}_1^r \subset (-\pi, \pi] \setminus \{0\}$ are different points and $\{\alpha_k\}_0^r \subset \mathbb{R}$. By $\mathcal{M}_p$ we denote the class of weights $\nu(t)$ satisfying the Muckenhoupt condition (see e.g. [5])

$$\sup_{I\subset[-\pi,\pi]} \left(\frac{1}{|I|}\int_I \nu(t)\, dt\right) \left(\frac{1}{|I|}\int_I [\nu(t)]^{-\frac{1}{p-1}}\, dt\right)^{p-1} < +\infty,$$

where *sup* is taken over all intervals $I \subset [-\pi, \pi]$ and $|I|$ is the Lebesgue measure I. It is easy to see that $|\rho|^{\frac{1}{p}} \in \mathcal{M}_p$ if and only if the following inequalities are true

$$-\frac{1}{p} < \alpha_k < 1 - \frac{1}{p}, \ \ k = \overline{0, r}. \tag{1}$$

Consider the system $E_+^{(k)}(\rho) \equiv \left\{\rho(t)\, e^{int}\right\}_{n\geq k}$.

The following theorem is true.

Theorem 1. *Let the inequalities (1) be fulfilled. Then the system* $E_+^{(0)}(\rho)$ *forms a basis for* L_p^+, $1 < p < +\infty$.

4. Defect case

We will consider the case when $|\rho|^{\frac{1}{p}} \notin \mathcal{M}_p$. Let the following inequalities hold

$$1-\frac{1}{p} \leq \alpha_0 < 2-\frac{1}{p}\ ,\ \ -\frac{1}{p} < \alpha_k < 1-\frac{1}{p}\ ,\ \ k=\overline{1,r}. \tag{2}$$

Assume

$$H_p^+(0) = \{f \in H_p^+ : f(0) = 0\}$$

Every functional $l \in \left(H_p^+\right)^*$ can be determined by $g \in L_q$ through the expression

$$l(f) = l_g(f) = \int_{-\pi}^{\pi} \left(g\left(e^{it}\right) + F\left(e^{it}\right)\right) f\left(e^{it}\right) dt\ ,\ \ \forall f \in H_p^+,$$

where $F \in H_q^+(0)$ is an arbitrary function. Consequently, zero functional is generated by zero function. Let us assume that the functional $l_g \in \left(L_p^+\right)^* = \left(H_p^+\right)^*$ cancels the system $E_+^{(0)}(\rho)$ out, i.e.

$$\int_{-\pi}^{\pi} \left(g\left(e^{it}\right) + F\left(e^{it}\right)\right) \rho(t)\, e^{int} dt = 0\ ,\ \ \forall n \in \mathbb{Z}_+, \tag{3}$$

where $F \in H_q^+(0)$ is an arbitrary function. Take $\forall \beta \geq 0{:}(\alpha_0 - \beta) \in \left(-\frac{1}{p}, 1-\frac{1}{p}\right)$ and assume

$$\rho(t) \equiv \tilde{\rho}(t) \left(e^{it} - 1\right)^{\beta}.$$

Let $\tilde{g}\left(e^{it}\right) = g\left(e^{it}\right)\left(e^{it}-1\right)^{\beta}$ and $\tilde{F}(z) = F(z)(z-1)^{\beta}$. It is clear that $\tilde{g} \in L_q$ and $\tilde{F} \in H_q^+(0)$, as, $\beta \geq 0$ and $\tilde{F}(0) = 0$. The relation (3) can be rewritten as follows

$$\int_{-\pi}^{\pi} \left(\tilde{g}\left(e^{it}\right) + \tilde{F}\left(e^{it}\right)\right) \tilde{\rho}(t)\, e^{int} dt = 0\ ,\ \ \forall n \in \mathbb{Z}_+. \tag{4}$$

As $|\tilde{\rho}|^{\frac{1}{p}} \in \mathcal{M}_p$, it is clear that the system $E_+^{(0)}(\tilde{\rho})$ forms a basis for L_p^+, and, moreover, is complete in L_p^+. Then from (4) it follows $\tilde{g} = 0 \Rightarrow g = 0$. As a result, we obtain that the system $E_+^{(0)}(\rho)$ is complete in L_p^+. In a similar way we prove that, with the conditions

$$\alpha_k > -\frac{1}{p},\ \ k = \overline{0,r}, \tag{5}$$

fulfilled, the system $E_+^{(0)}(\rho)$ is complete in L_p^+.

Let us show that if the inequalities (2) hold, then the system $E_+^{(1)}(\rho)$ is minimal in L_p^+. Consider the system

$$\left\{\overline{\rho^{-1}(t)}\left(e^{int}-1\right)\right\}_{n\geq 1}. \tag{6}$$

We have

$$\int_{-\pi}^{\pi}\rho(t)\,e^{int}\rho^{-1}(t)\,\left(e^{-imt}-1\right)\,dt=\int_{-\pi}^{\pi}e^{i(n-m)t}dt-\int_{-\pi}^{\pi}e^{int}dt=$$
$$=2\pi\delta_{nm}\,,\ \forall n,m\in\mathbb{N}.$$

The relations

$$e^{int}-1\sim t\,,\ \ t\to 0\,,\ \ e^{it}-e^{it_k}\sim t-t_k\,,\ \ t\to t_k,$$

imply

$$\left|\overline{\rho^{-1}(t)}\left(e^{int}-1\right)\right|^{q}\sim|t|^{q(1-\alpha_0)}\prod_{k=1}^{r}|t-t_k|^{-\alpha_k q}\,,$$

on $(-\pi,\pi)$. Consequently

$$\left\|\overline{\rho^{-1}(t)}\left(e^{int}-1\right)\right\|_q^q=\int_{-\pi}^{\pi}\left|\overline{\rho^{-1}(t)}\left(e^{int}-1\right)\right|^{q}dt\sim$$
$$\int_{-\pi}^{\pi}|t|^{q(1-\alpha_0)}\prod_{k=1}^{r}|t-t_k|^{-\alpha_k q}\ dt.$$

Hence, from (2) we obtain that the system (6) belongs to L_q. As a result, the previous relations yield the minimality of system $E_+^{(1)}(\rho)$ in L_p^+. We can similarly prove that the system $E_+^{(1)}(\rho)$ is complete in L_p^+. So, under condition (2) the system $E_+^{(1)}(\rho)$ is complete and minimal in L_p^+. In this case the system $E_+^{(1)}(\rho)$ is not uniformly minimal in L_p^+, and, moreover, it doesn't form a basis for L_p^+. Consequently, the system $E_+^{(0)}(\rho)$ has a defect equal to 1. We can similarly prove that if the inequalities

$$k-\frac{1}{p}\leq\alpha_0<k+\frac{1}{q}\,,\ -\frac{1}{p}<\alpha_k<\frac{1}{q}\,,\ k=\overline{1,r}, \tag{7}$$

are fulfilled, then the system $E_+^{(k)}(\rho)$ is complete and minimal in L_p^+, but it doesn't form a basis for L_p^+. Consequently, in this case the system $E_+^{(0)}(\rho)$ has a defect equal to (k). As a result, we get the validity of

Theorem 2. *Let the inequalities (7) hold. Then the system $E_+^{(0)}(\rho)$ has a defect equal to (k) in L_p^+. In addition, the system $E_+^{(k)}(\rho)$ is complete and minimal in L_p^+, but is not uniformly minimal in it, and, consequently, it doesn't form a basis for L_p^+.*

We also proved that, in case when the Muckenhoupt condition does not hold, any function from closure of the linear span of the system $E_+^{(0)}(\rho)$ can not be expanded with respect to this system, i.e. the following theorem is true.

Theorem 3. *Let the inequalities $-\frac{1}{p}<\alpha_i<\frac{1}{q}$, $i=\overline{1,r}$ hold. Then the system $E_+^{0}(\rho)$ forms a frame for L_p^+ if and only if $-\frac{1}{p}<\alpha_0<\frac{1}{q}$.*

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