Caspian Journal of Applied Mathematics, Ecology and Economics
V. 2, No 1, 2014, July
ISSN 1560-4055

Identities Involving the Terms of a Balancing-Like Sequence Via Matrices

P. K. Ray

Abstract. The goal of this paper is to establish some identities involving the terms of a newly introduced sequence $\{u_n\}_{n=0}^{\infty}$ called as the balancing-like sequence defined recursively by $u_n = 6au_{n-1} - u_{n-2}$ with initials $u_0 = 0, u_1 = 1$ via certain matrices.

Key Words and Phrases: Balancing numbers, Recurrence relation, Balancing matrix.
2000 Mathematics Subject Classifications: 11B37, 11B39

1. Introduction

It is well known that, the sequence of balancing numbers $\{B_n\}$ is defined recursively by the equation

$$B_n = 6B_{n-1} - B_{n-2},\ n \geq 2,$$

with initial conditions $B_0 = 0$ and $B_1 = 1$ [1]. Whereas the companion to these numbers is the sequence of Lucas-balancing numbers $\{C_n\}$ which is defined recursively by

$$C_n = 6C_{n-1} - C_{n-2},\ n \geq 2$$

with $C_0 = 1$ and $C_1 = 3$ [7, 8]. Both these numbers can also be extended negatively. The following results were established in [1].

$$B_{-n} = -B_n,\ \ C_{-n} = C_n.$$

The Binet's formulas for both balancing and Lucas-balancing numbers are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_1^n}{\lambda_1 - \lambda_2},\ \ C_n = \frac{\lambda_1^n + \lambda_1^n}{2}.$$

The generalizations for balancing numbers were done in different ways. To know in details about balancing numbers and their generalization, one can go through [2-5]. There is another way to generate balancing numbers through matrices. In [9], Ray introduced balancing matrix

$$Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix},$$

which is a second order matrix and whose entries are the first three balancing numbers 0, 1 and 6. He has also shown that

$$Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix},$$

for every integer n [9]. Later, he has deduced nice product formulas for both negatively and positively subscripted balancing and Lucas-balancing numbers [10, 12]. Ray has established many interesting identities for both balancing and Lucas-balancing numbers through matrices [9-15].

In this study, we will first introduce a new sequence which we call balancing-like sequence $\{u_n\}_{n=0}^{\infty}$ defined recursively by

$$u_n = 6au_{n-1} - u_{n-2}, \tag{1.1}$$

with initials $u_0 = 0, u_1 = 1$, where $n \geq 2$. Then we will define a second order matrix which we call balancing Q_B-like matrix whose entries are the first three balancing-like numbers $0, 1$, and $6a$. Later, we will show that the higher powers of this matrix also contain the balancing-like numbers. These matrices will be used to obtain identities involving the terms of a balancing-like sequence.$\{u_n\}_{n=0}^{\infty}$. From (1.1) we notice that, the first few terms of the balancing-like sequence are

$$0,\ 1,\ 6a,\ 36a^2-1,\ 216a^3-12a,\ 1296a^4-108a^2+1,$$

$$7776a^5-864a^3+18a,\ 46656a^6-6480a^4+216a^2-1.$$

Also observe that, for $a = 1$ the balancing-like numbers $\{u_n\}$ reduce to the balancing numbers.

2. Some identities involving balancing-like numbers

2.1. Binet's formula

In this section, we will establish Binet' formula for balancing-like numbers and the identity involving negatively subscripted balancing-like numbers.

Solving the homogenous recurrence relation (1.1), its characteristic equation $\lambda^2-6a\lambda+1=0$ has the roots

$$v = 3a + \sqrt{9a^2-1}, \quad w = 3a - \sqrt{9a^2-1}.$$

The general solution of (1.1) is given by

$$u_n = c_1v^n + c_2w^n, \tag{2.1}$$

where c_1 and c_2 are arbitrary constants. Using the initial conditions given in (1.1), we obtain the following system of equations

$$u_0 = c_1 + c_2 = 0, \quad u_1 = c_1v + c_2w = 1.$$

Solving these two equations, we get

$$c_1 = \frac{1}{2\sqrt{9a^2-1}}, \quad c_2 = \frac{-1}{2\sqrt{9a^2-1}},$$

for $3a \neq \pm 1$. Therefore, equation (2.1) reduces to the following identity

$$u_n = \frac{v^n - w^n}{2\sqrt{9a^2-1}}, \tag{2.2}$$

which we call the Binet's formula of balancing-like numbers.

2.2. Negatively subscripted balancing-like numbers

We can also extend the sequence $\{u_n\}$ to the negative values of n. Since the product of v and w is one, therefore for $n \geq 0$, the Binet's formula (2.2) reduces to

$$u_{-n} = \frac{v^{-n} - w^{-n}}{2\sqrt{9a^2-1}} = \frac{w^n - v^n}{2\sqrt{9a^2-1}} = -u_n. \tag{2.3}$$

It is easy to show that, (2.3) holds for all negative integers n. Let $n < 0$, say $n = -m$, where $m > 0$. Using (2.3), we observe that

$$\begin{aligned} u_n - 6au_{n-1} + u_{n-2} &= u_{-m} - 6au_{-m-1} + u_{-m-2} \\ &= -u_m + 6au_{m+1} - u_{m+2} = u_{m+2} - u_{m+2} = 0. \end{aligned}$$

3. Balancing-like numbers via matrices

As stated before, the balancing Q_B matrix has introduced by Ray in [9]. In this section, we introduce balancing-like Q_B matrix, denoted by Q_u and defined by

$$Q_u = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

whose entries are the first three balancing-like numbers $0, 1$ and $6a$, where a is arbitrary.

The following theorem shows that the sequence $\{u_n\}_{n=0}^{\infty}$ can also be generated by matrix multiplication.

Theorem 3.1. *If* $Q_u = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$ *as defined in (3.1), then for all positive integers n,* $Q_u^n = \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix}$.

Proof. This result can be proved by method of induction. The basis step is clear. In the inductive step, suppose the result holds for all integers $\leq n$. Using (1.1) and the hypothesis, we observe that

$$Q_u^{n+1} = Q_u^n \cdot Q_u = \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix} \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6au_{n+1} - u_n & -u_{n+1} \\ u_{n+1} & u_n \end{pmatrix} = \begin{pmatrix} u_{n+2} & -u_{n+1} \\ u_{n+1} & -u_n \end{pmatrix},$$

which completes the proof of the theorem.

We notice that, the balancing-like matrices also satisfy the same recurrence relation as that of balancing-like numbers, that is for $n \geq 1$ we have

$$Q_u^{n+1} = 6aQ_u^n - Q_u^{n-1}, \tag{3.2}$$

with initial conditions $Q_u^0 = I$ and $Q_u^1 = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$, where I is the identity matrix.

Since $Q_u^{m+n} = Q_u^m \cdot Q_u^n$, comparing the corresponding entries of both the matrices, we have the following identities:

$$\begin{aligned} &i)\ \ u_{m+n+1} = u_{m+1}u_{n+1} - u_m u_n \\ &ii)\ \ u_{m+n} = u_{m+1}u_n - u_m u_{n-1} = u_{n+1}u_m - u_n u_{m-1} \\ &iii)\ u_{m+n-1} = u_{m-1}u_{n-1} - u_m u_n. \end{aligned}$$

For $a = 1$, the similar results for balancing numbers are found in [7-9].

Theorem 3.1 can be used to obtain the following identities involving terms of the sequence $\{u_n\}$.

Corollary 3.1. *For $n \geq 1$, the following identities are valid:*

$$u_{2n+1} = u_{n+1}^2 - u_n^2 \ \text{ and } \ u_{2n} = u_n(u_{n+1} - u_{n-1}).$$

Proof. Since $Q_u^{2n} = (Q_u^n)^2$, we have

$$\begin{aligned} \begin{pmatrix} u_{2n+1} & -u_{2n} \\ u_{2n} & -u_{2n-1} \end{pmatrix} &= \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix} \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} u_{n+1}^2 - u_n^2 & u_n u_{n-1} - u_n u_{n+1} \\ u_n u_{n+1} - u_n u_{n-1} & -(u_n^2 - u_{n-1}^2) \end{pmatrix}. \end{aligned}$$

The results follow by equating corresponding rows and columns from both sides as the above matrix.

We notice that for $a = 1$, formulas for balancing numbers analogous to the last two identities are given in [7, 8]. In general, if k is a positive integer, then $Q_u^{kn} = (Q_u^n)^k$, that is

$$\begin{pmatrix} u_{kn+1} & -u_{kn} \\ u_{kn} & -u_{kn-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} & -u_n \\ u_n & -u_{n-1} \end{pmatrix}^k.$$

The determinants of the above matrices give the identity

$$u_{kn}^2 - u_{kn+1}u_{kn-1} = (u_n^2 - u_{n+1}u_{n-1})^k.$$

In fact, both sides of the above equation are equal to 1 as $\det Q_u = 1$. Therefore we have the following important identity for balancing-like numbers:

$$u_{kn}^2 - u_{kn+1}u_{kn-1} = 1, \tag{3.3}$$

which we call Cassini formula for balancing-like numbers. Recall that [8], the Cassini formula for balancing numbers is given by the identity

$$B_n^2 - B_{n+1}B_{n-1}.$$

This formula play a vital role to find many important identities for balancing numbers and their related sequences and can be obtained by setting $a = 1$ in (3.3). Since $det(Q_u) \neq 0$, for $n \geq 0$ the following identity is valid:

$$Q_u^{-n} = (Q_u^{-1})^n. \tag{3.4}$$

The following result can be easily shown by induction.

Theorem 3.2. *If* $Q_u = \begin{pmatrix} 6a & -1 \\ 1 & 0 \end{pmatrix}$ *as defined in (3.1), then for* $n \geq 0$, $Q_u^{-n} =$
$= \begin{pmatrix} u_{-n+1} & -u_{-n} \\ u_{-n} & -u_{-n-1} \end{pmatrix}$.

Since Theorem 3.2 is a special case of Theorem 3.1 with n negative, the identities proved in Theorem 3.1 also holds for $n < 0$.

At the end of the section, we now establish the following two important results.

Theorem 3.3. *If* $\binom{n}{k}$ *denote the usual notation for combination, then for* $n \geq 0$

$$u_{2n+1} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} u_{k+1} \quad \text{and} \quad u_{2n} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} u_k.$$

Proof. By (3.2), $Q_u^2 = 6aQ_u - I$. Both sides of this relation raised to a power n yields the following expression

$$Q_u^{2n} = (-I + 6a)^n.$$

For $n \geq 0$, the binomial expansion of the right hand side expression gives the following:

$$Q_u^{2n} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} u_k.$$

Therefore by virtue of Theorem 3.1, we get

$$\begin{pmatrix} u_{2n+1} & -u_{2n} \\ u_{2n} & -u_{2n-1} \end{pmatrix} = \sum_{k=0}^{n} (-1)^{k+1}(6a)^k \binom{n}{k} \begin{pmatrix} u_{k+1} & -u_k \\ u_k & -u_{k-1} \end{pmatrix}.$$

Equating the corresponding entries, we get the desired result.

By (3.4) we observe that, $Q_u^{-2} = -I + 6aQ_u^{-1}$ and present following result.

Theorem 3.4. *For $n \geq 0$, we have*

$$u_{-2n+1} = \sum_{k=0}^{n} (-1)^{k+1}(-6a)^k \binom{n}{k} u_{-k+1} \quad \text{and} \quad u_{-2n} = \sum_{k=0}^{n} (-1)^{k+1}(-6a)^k \binom{n}{k} u_{-k}.$$

Proof. The proof is analogous to that of Theorem 3.3.

Remark 3.1. *All these identities we have proven so far in this paper, can be obtained directly by using the Binet's formula for balancing-like numbers given in (2.2). However, the matrix method is noticeably simpler.*

References

[1] A. Behera and G.K. Panda, On the square roots of triangular numbers, Fibonacci Quarterly, 37(2), 1999, 98-105.

[2] A. Berczes, K. Liptai, I. Pink, On generalized balancing numbers, Fibonacci Quarterly, 48(2), 2010, 121-128.

[3] R. Keskin and O. Karaatly, Some new properties of balancing numbers and square triangular numbers, Journal of Integer Sequences, 15(1), 2012.

[4] K. Liptai, F. Luca, A. Pinter and L. Szalay, Generalized balancing numbers, Indagationes Math. (N.S.), 20, 2009, 87-100.

[5] P. Olajos, Properties of balancing, cobalancing and generalized balancing numbers, Annales Mathematicae et Informaticae, 37, 2010, 125-138.

[6] G.K. Panda and P.K. Ray, Cobalancing numbers and cobalancers, International Journal of Mathematics and Mathematical Sciences, 2005(8), 2005, 1189-1200.

[7] G.K. Panda and P.K. Ray, Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, Bulletin of the Institute of Mathematics, Academia Sinica (N.S.), 6(1), 2011, 41-72.

[8] G.K. Panda, Some fascinating properties of balancing numbers, Proceeding Eleventh International Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium,194, 2009, 185-189.

[9] P. K. Ray, Certain matrices associated with balancing and Lucas-balancing numbers, Matematika, 28 (1), 2012, 15-22.

[10] P.K. Ray, Application of Chybeshev polynomials in factorization of balancing and Lucas-balancing numbers, Boletim da Sociedade Paranaense de Matemtica 30 (2), 2012, 49-56.

[11] P.K. Ray, Curious congruences for balancing numbers, International Journal of Contemporary Mathematical Sciences, 7 (18), 2012, 881-889.

[12] P.K. Ray, Factorization of negatively subscripted balancing and Lucas-balancing numbers, Boletim da Sociedade Paranaense de Matemtica, 31 (2), 2013, 161-173.

[13] P.K. Ray, Some congruences for balancing and Lucas-balancing numbers and their applications, Integers, 14, 2014, #A8.

[14] P.K. Ray, On the properties of Lucas-balancing numbers by matrix method, Sigmae, Alfenas, 3(1), 2014, 1-6.

[15] P.K. Ray, On the properties of Lucas-balancing numbers by matrix method, Notes on Number Theory and Discrete Mathematics, 20(1), 2014, 49-58.

Prasanta Kumar Ray
International Institute of Information Technology Bhubaneswar
Bhubaneswar-751003, India
E-mail: praanta@iiit-bh.ac.in

Received 19 March 2014
Accepted 15 June 2014