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On an Employment Period of a Class of Service Systems

T.M. Aliev

Abstract. The condition of the property of the employment period a one class of service systems is found in work under enough general conditions on a stream of demands and character of service. It is established that for the considered service system the necessary and sufficient condition of the property of the employment period is equivalent to the ergodicity condition for the same system.

Key Words and Phrases: Markov process, service system, reliability, ergodicity condition.
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1. Introduction

Investigation of any service system makes necessary to analyze a random process related with transmission of this system from one state to another one. A lot of these systems with nonreliable devices are described by homogeneous Markov process with two components.

Investigations on the reliability of the service was founded by B.V. Gnedenko [1]. Then these investigations were continued by various authors. Statement of the problems and investigated in these works process mainly have different characters.

In the work by N.N. Yejov, T. Annaev [2] the period of employment of the service system with non-reliable devices, when we have non-homogeneous and Puasson input flow. G.P. Basharin [1] considered the systems with bounded turn of non-reliable devices, three possible service subjects (direct, inverse, random choice of the demand from the turn) and when input flow consists of ˆ the sum of finite number of simple Hows, each which corresponds to its own parameter of the exponential service low. Enough general one dimensional system with non-reliable devices was studied by G.P. Klimov [4], using the method of included Markov chain. In the work [5] using the methods of the functional analysis the process is investigated that describe a wide class of service systems considered in [2,3]. Considering the result of [2] and [5] in [6] the ergodicity condition is found for the service system with $n\,(n \geq 1)$ number of non-reliable devices and non-homogeneous input of the demands.

As is known independently from the character of the service the employment period is the main characteristics of the service system. Namely by these characteristics each service system works.

In the present work the property condition of the period of employment of a class of service systems is investigated.

The advantage of the obtained results consists of the fact that for the considered service system the necessary and sufficient condition of property of the employment period, related with some functional equation is equivalent to the ergodicity condition for the same system. This condition is found in the sense of mathematical expectation in the form of inequality.

Let's consider homogeneous Markov process $\{\xi_t, \eta_t\}$, $t \geq 0$ with phase process $\{0^+, 1^\pm, 2^\pm, ...\}\, R^+$ where $R^+ = \{x : x \geq 0\}$ and satisfies to the following transmission probabilities by $\Delta \downarrow 0$

$$(0^+, x) \left\{ \begin{array}{c} (0^+, x+\Delta) : 1 - \lambda(x)\Delta + o(\Delta) \\ (k^+, 0) : 1 - \lambda_k(x)\Delta + o(\Delta), k \geq 1 \end{array} \right.$$

$$\left(k^+, \Delta\right) \left\{ \begin{array}{c} (k^+, x+\Delta) : 1 - \left[\lambda^+(x) + \mu^-(x) + \nu(x)\right]\Delta + o(\Delta) \\ \left((k^+, r)^+, x+\Delta\right) : \lambda_r^+(x)\Delta + o(\Delta), r \geq 1 \\ \left((k-r)^+, 0\right) : \nu(x)\Delta + o(\Delta) \\ (k^-, 0) : \mu^-(x)\Delta + o(\Delta), \end{array} \right. \tag{1}$$

$$\left(k^-, \Delta\right) \left\{ \begin{array}{c} (k^-, x+\Delta) : 1 - \left[\lambda^-(x) + \mu^+(x) + \nu(x)\right]\Delta + o(\Delta) \\ \left((k+r)^-, x+\Delta\right) : \lambda_r^-(x)\Delta + o(\Delta), r \geq 1 \\ \left((k-1)^+, 0\right) : \nu(x)\Delta + o(\Delta) \\ (k^+, 0) : \mu^+(x)\Delta + o(\Delta), \end{array} \right.$$

where $\nu(x), \mu^\pm(x), \lambda_2^\pm(x), r \geq 1$- non negative functions and

$$\lambda^\pm(x) = \sum_{k=1}^{\infty} \lambda_k^\pm(x), \qquad \lambda(x) = \sum_{k=1}^{\infty} \lambda_k(x).$$

Such kind of process are met in the investigation of the service systems with non-reliable devices and intensities depending on some parameter $x \in R^+ = [x, +\infty\}$. In this work by means of mathematical expectation the property condition and Laplace transformation is found for the employment period of the considered service system.

Let to the one-line service system with expectation non-homogeneous Puasson flow input with intensity $\lambda^\pm(x), \lambda^\pm(x) = \sum\limits_{i=1}^{\infty} \lambda_k^\pm(x)$. The service period has intensity $\nu(x)$. The device may be broken during the service then repaired. The intensity of breakage and repairing are $\mu^-(x), \mu^+(x)$ correspondingly.

The state $(0^+, x)$ means that at the considered time moment the device is free, capable, and the stopping period is equal to x. The state $(k^\pm, x), x \geq 1$ means that at the considered time moment there exist k number of demands to the system, the device is capable (non-capable) x units of time (the time expended to repairing is also equal to x).

It is clear that this system will be described by the process $\{\xi_t, \eta_t\}, t \geq 0$ given in (1).

If the initial state of the system is $(k^+, 0), k \geq 1$ then ξ^+ defines the time till the end of the service, or breakage of the device. If the initial state is $(k^-, 0), k \geq 1$ then ξ^- is the period expended for the repairing of the device.

2. Main result

Let's investigate employment period of transmissions probabilities (1) .

Let τ_k^+ be the period of transmissions of the process $\{\xi_t, \eta_t\}$ from the state $(k^+, 0)\,, k \geq 1$ to the state $(0,0)$.

The quantity τ_k^+ is called the employment period of the service system following to Takara [7].

Let's define by $\pi^{\pm}(x)$ the Puasson process with local characteristics $\lambda^{\pm}(u)\,, k \geq 1$ by $\xi^{\pm} = x$. Then

$$\begin{array}{lll} \tau_k^+ = \xi^+_{k+S^+(\xi^+)-1} & \text{with probability} & \frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}, \\ \tau_k^+ = \xi^+_{k+S^+(\xi^+)} & \text{with probability} & \frac{\mu^-(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}, \end{array} \tag{2}$$

It is easy to check that

$$\tau_k^- = \xi^- + \tau^+_{k+\pi^-(\xi^-)}. \tag{3}$$

Introduce

$$Me^{-s\tau_k^{\pm}} = \varphi_k^{\pm}(s)\,, \quad (k \geq 1)$$

Thus

$$\tau_k^+ = \tau_k^{+k-1} + \tau_{k-1}^{+k-2} + ... + \tau_1^{+0} \tag{4}$$

where all terms are independent and have such distribution that $\tau_1^+ \equiv \tau_1^{+0}$ and considering (4)

$$\varphi_k^+(s) = [\omega(s)]^k\,,$$

where $\omega(s) = \varphi_1^+(s)$. Then from (3) we obtain

$$\varphi_k^-(s) = Me^{-s\xi-}\left[\omega(s)\right]^{k+\pi^-(\xi^-)}.$$

Considering this from (2) we can get

$$\omega^k = Me^{-s\xi^+}\left[\frac{\nu(\xi^+)}{\nu(\xi^+) + \mu^-(\xi^+)}\omega^{k+\pi^+(\xi^+)-1} + \right.$$

$$\left. + \frac{\mu^-(\xi^+)}{\nu(\xi^+) + \mu^-(\xi^+)} Me^{-s\xi^-}\omega^{k+\pi^+(\xi^+)+\pi^-(\xi^-)}\right]$$

or

$$Me^{-s\xi^+}\frac{\nu(\xi^+)}{\nu(\xi^+) + \mu^-(\xi^+)}\omega^{\pi^+(\xi^+)-1} +$$

$$\left. + Me^{-s\xi^+}\frac{\mu^-(\xi^+)}{\nu(\xi^+) + \mu^-(\xi^+)}\omega^{\pi^+(\xi^+)} Me^{-s\xi^-}\omega^{\pi^-(\xi^-)} = 1\right] \tag{5}$$

Define

$$L(s,\omega) = Me^{-s\xi^+}\frac{\nu(\xi^+)}{\nu(\xi^+) + \mu^-(\xi^+)}\omega^{\pi^+(\xi^+)} +$$

$$+\omega M e^{-s\xi^+}\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}\omega^{\pi^+\left(\xi^+\right)}Me^{-s\xi^-}\omega^{\pi^-\left(\xi^-\right)}.$$

Then (5) may be rewritten in the form of equation

$$\omega = L\left(s,\omega\right). \tag{6}$$

Now let's consider the equation (6) in unit interval $0 \leq \omega \leq 1$.

Let $c > 0$. Then $L\left(s,\omega\right)$ is down convex in $\omega \in [0,1]$ function and the curve, describing this function has unique joint point ω^* with straight line $y = \omega$. Consequently in this case the equation (6) has unique solution ω^*.

Let's find necessary and sufficient condition for the property of the random quantity τ_1^+. For this purpose it is enough to check that the inequality

$$\left.\frac{dL\left(0,\omega\right)}{d\omega}\right|_{\omega=1} < 1 \tag{7}$$

is valid. From this that $\omega\left(0\right) = 1$, i.e. τ_1^+ is eigenrandom quantity. For $L\left(0,\omega\right)$ we have

$$L\left(0,\omega\right) = M\frac{\nu\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}\omega^{\pi^+\left(\xi^+\right)} + \omega M\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}\omega^{\pi^+\left(\xi^+\right)}M\omega^{\pi^-\left(\xi^-\right)},$$

$$\left.\frac{dL\left(0,\omega\right)}{d\omega}\right|_{\omega=1} = M\pi^+\left(\xi^+\right)\frac{\nu\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}\omega^{\pi^+\left(\xi^+\right)}+$$

$$+\omega M\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}\omega^{\pi^+\left(\xi^+\right)}M\omega^{\pi^-\left(\xi^-\right)},$$

$$\left.\frac{dL\left(0,\omega\right)}{d\omega}\right|_{\omega=1} = M\pi^+\left(\xi^+\right)\frac{\nu\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)} + M\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}+$$

$$+M\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}M\pi^-\left(\xi^-\right) + M\pi^+\left(\xi^+\right)\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}.$$

From the condition (7) one can get

$$M\pi^+\left(\xi^+\right) + M\frac{\mu^-\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)}M\pi^-\left(\xi^-\right) < M\frac{\nu\left(\xi^+\right)}{\nu\left(\xi^+\right)+\mu^-\left(\xi^+\right)} \tag{8}$$

As

$$M\theta^{\pi^\pm(x)} = \exp\left\{\int\limits_0^x \lambda^\pm\left(u,\theta\right)du\right\},$$

where

$$\lambda^\pm\left(u,\theta\right) = \sum_{i=1}^\infty \lambda_k^\pm\left(u\right)\theta^k$$

then

$$M\pi^{\pm(x)} = \int_0^x \dot{\lambda}^{\pm}(u,1)\,du,$$

(here $z(x,1)$ is a derivative of the function $z(x,\theta)$ by $x = 0$).

Considering last relations the inequality (8) takes a form

$$M\int_0^{\xi^+} \dot{\lambda}^{+}(u,1)\,du + M\frac{\mu^-(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}M\int_0^{\xi^-} \dot{\lambda}^{-}(u,1)\,du < M\frac{\nu(\xi^+)}{\nu(\xi^+)+\mu^-(\xi^+)}. \quad (9)$$

If $\varphi_1^+(s)$ is known, then it is possible to define $\varphi_k^+(s)$ for any $k \geq 2$.

In [8] is proved that the inequality (9) is ergodicity condition for the service system described above. We establish that this inequality is necessary and sufficient condition of property of the employment period of the service system.

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Tofig M. Aliev
Institute of Control Systems, Azerbaijan National Academy of Sciencs,
9 B. Vahabzade str. Az 1141, Baku, Azerbaijan
E-mail:tofiq_eliev44@rambler.ru

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