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Necessary conditions of optimality in a problem of optimal control of moving sources for singular heat equation

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Abstract. A problem of optimal control of processes described by a singular heat equation and systems of ordinary differential equations with moving sources is investigated in this paper. In spite of applied importance of problems with moving sources controls, they have not been studied enough so far [1-3],[7-8]. Sufficients conditions of Frechet differentiability of quality test and an expression for its gradient are obtained, necessary conditions of optimality in the form of point wise and integral maximum principles are established for an optimal control problem considered below.

Key Words and Phrases: moving sources, maximum principles, integral identity, reduced problem, necessary conditions of optimality

2000 Mathematics Subject Classifications: 35K20, 49K20, 49J20

1. Introduction

Practical examples of moving sources of influence are electronic, laser and ionic beams, an electric arch, the induction current raised by the moving inductor. The most widespread processes in which these sources are applied, processes of melting and metal refinement in metallurgy are; processes of heat treatment, welding and microprocessing in mechanical engineering and instrumentation; processes of manufacturing of semi-conductor and resistor elements in microelectronics; processes of activation, radiation and drying in biology, medicine, agriculture, etc. For the first time theoretical statement of problems optimal control of moving sources for systems with the distributed parameters was given in A.G.Butkovsky and L.M.Pustylnikovs works [2]. One of the main features of this systems is their nonlinearity concerning the control defining the law of movement of a sources. The problem of the moments becomes nonlinear. Thus, the method of the moments which is widely used for search of optimal control in linear systems with the distributed and concentrated parameters, becomes unsuitable for systems with moving sources. In this work the variation method to solve a problem of optimum control of moving sources for the heat conductivity processes described by totality of a parabolic type equation and ordinary differential equation with moving sources is considered. Considering that the received problem of optimum control wasnt studied earlier, for it questions of a correctness of the decision are investigated, uniqueness and existence theorems are proved, sufficient conditions of Frechet differentiability of criterion of quality are found and expression for its gradient is received. Necessary conditions of optimality in the form of point wise and integral maximum principles are established for the optimal control problems.

2. Problem statement

Let‘s consider a problem on minimization of the functional

$$J(\overline{\vartheta}) = \int_0^l [u(x,T) - y(x)]^2 dx + \alpha_1 \sum_{k=1}^{n} \int_0^T [p_k(t) - \tilde{p}_k(t)]^2 dt +$$

$$+\alpha_2 \sum_{m=1}^{r} \int_0^T [\vartheta_m(t) - \tilde{\vartheta}_m(t)]^2 dt, \tag{1}$$

on the set

$$V = \{\overline{\vartheta} = (p, \vartheta) : p = (p_1(t), ..., p_n(t)) \in L_2^n(0,T), \vartheta = (\vartheta_1(t), ..., \vartheta_r(t)) \in L_2^n(0,T),$$

$$0 \le p_i(t) \le A_i, 0 \le \vartheta_j(t) \le B_j, i = \overline{1,n}\ , j = \overline{1,r}\},$$

under conditions

$$u_t = a^2 u_{xx} + \sum_{k=1}^{n} p_k(t)\delta(x - s_k(t)), (x,t) \in \Omega = \{0 < x < l, 0 < t \le T\}, \tag{2}$$

$$u_x\,|_{x=0} = g_1(t),\ u_x\,|_{x=l} = g_2(t),\ 0 < t \le T, \tag{3}$$

$$u(x,o) = \varphi(x), 0 \le x \le l, \tag{4}$$

$$\dot{s}_k(t) = f_k(s(t), \vartheta(t), t),\ 0 < t \le T,\ s_k(o) = s_{k0}, k = \overline{1,n}, \tag{5}$$

where $s_{k0} \in [0,l]$,α_1,α_2 $\ge$0, $\alpha_1 + \alpha_2 > 0$, a, l, T, $A_i > 0$, $i = \overline{1,n}$,$B_j > 0, j = \overline{1,r}$ are the given numbers; $s_k(t) = s_k(t;\vartheta) \in C(0,T)$, $0 \le s_k(t) \le l, k = \overline{1,n}$ is a solution of problem (5) corresponding to the control $\vartheta = \vartheta(t) = (\vartheta_1(t), \vartheta_2(t), ..., \vartheta_r(t)) \in L_2^r(0,T)$; the functions $f_k\,(s, \vartheta, t)\,,\ k = \overline{1,n}$, are continuous and have continuous derivatives with respect to s and ϑ for $(s, \vartheta, t) \in E^n \times E^r \times [0,T]$; $g_1(t), g_2(t) \in L_2(0,T)$, $\varphi(x) \in L_2(0,l)$, $y(x) \in L_2(0,l)$, $\delta\,(\cdot)$ is a Dirac function; $\omega = (\tilde{p}(t), \tilde{\vartheta}(t))$, $\tilde{p}(t) = (\tilde{p}_1(t), \tilde{p}_2(t), ..., \tilde{p}_n(t)) \in L_2^n(0,T)$, $\tilde{\vartheta}(t) = (\tilde{\vartheta}_1(t), \tilde{\vartheta}_2(t), ..., \tilde{\vartheta}_r(t)) \in L_2^r(0,T)$ are the given functions.

For the sake of brevity, we denote by $H = L_2^n(0,T) \times L_2^r(0,T)$ a Hilbert space of pairs $\bar{\vartheta} = (p(t), \vartheta(t))$ with scalar product $< \bar{\vartheta}^1, \bar{\vartheta}^2 >_H = \int_0^T [p^1(t)p^2(t) + \vartheta^1(t)\vartheta^2(t)]dt$, and the norm $\|\bar{\vartheta}\|_H = \sqrt{(< \bar{\vartheta}, \bar{\vartheta} >_H)} = \sqrt{(\|p\|_{L_2}^2 + \|\vartheta\|_{L_2}^2)}$.

3. Correctness of problem statement

Definition. A problem of finding the function $(u(x,t),s(t)) = (u(x,t;\overline{\vartheta}),s(t;\vartheta))$ satisfying the conditions (2) – (5) for the given control $\overline{\vartheta} \in V$ is said to be a reduced problem. Under the solution of reduced problem (2) – (5) corresponding to the control $\overline{\vartheta} = (p(t),\vartheta(t)) \in V$ we understand the function $(u(x,t),s(t))$ from $(V_2^{1,0}(\Omega),C[0,T])$, where the function $u = u(x,t)$ satisfies the integral identity

$$\int_0^l \int_0^T [-u\eta_t + a^2 u_x \eta_x] dxdt = a^2 \int_0^T [g_2(t)\eta(l,t) - g_1(t)\eta(0,t)]\, dt + \int_0^l \varphi(x)\eta(x,0)dx+$$

$$+\sum_{k=1}^n \int_0^T p_k(t)\eta(s_k(t),t)dt, \tag{6}$$

for $\forall\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ and $\eta(x,T) = 0$, and the function $s_k(t)$ satisfies the integral equation

$$s_k(t) = \int_0^t f_k(s(\tau),\vartheta(\tau),\tau)d\tau + s_{k0}, 0 \le t \le T, k = \overline{1,n}. \tag{7}$$

It follows from the results of the papers [5-6] that for each fixed $\overline{\vartheta} \in V$, the reduced problem (2) – (5) has a unique solution from $(V_2^{1,0}(\Omega),C[0,T])$. Let the conditions in the problem (1) – (5) be fulfilled. Then problem (1) – (5) has a unique solution [7] :

Theorem 1. *There exists a dense subset K of the space H, such that for each $\omega \in K$ and $\alpha_i > 0 (i = \overline{1,2})$ problem (1)-(5) has a unique solution.*

4. Differentiability of functional and necessary conditions of optimality

Let $\psi = \psi(x,t)$ be a solution from $V_2^{1,0}(\Omega)$ of the problem

$$\psi_t + a^2\psi_{xx} = 0, \;\; (x,t) \in \Omega, \tag{8}$$

$$\psi_x|_{x=0} = \psi_x|_{x=\ell} = 0, \;\; t \in [0,T), \tag{9}$$

$$\psi(x,T) = 2[u(x,T) - y(x)], \;\; x \in [0,\ell], \tag{10}$$

conjugated to (1) – (5), where $u(x,T)$ is a solution of reduced problem (1) – (5) for $t = T$,and $q = q(t)$ is a solution of the conjugated problem

$$\dot{q}_k(t) = -\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) + \psi_x(s_k(t),t)p_k(t), \;\; 0 \le t < T, \;\; q_k(T) = 0, k = \overline{1,n}. \tag{11}$$

from $C[0,T]$.

The function $\psi = \psi(x,t)$ satisfies the integral identity

$$\int_0^l \int_0^T [\psi \eta_{1t} + a^2 \psi_x \eta_{1x}] dx dt = 2 \int_0^l [u(x,T) - Y(x)] \eta_1(x,T) dx, \tag{12}$$

for $\forall \eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ and $\eta_1(x,0) = 0$, and the function $q_k(t)$ satisfies the integral identity

$$q_k(t) = \int_t^T \left[\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(\tau) - p_k(\tau) \psi_x(s_k(\tau), \tau) \right] d\tau, 0 \le t \le T, k = \overline{1,n}. \tag{13}$$

The conjugated problem (8) – (11) is a mixed problem for a linear parabolic equation. If in relations (8) – (11), instead of the variable t we take a new independent variable $\tau = T - t$, we get a boundary value problem of the same types as (2) – (5). Therefore, it follows from the facts established for problem (2) – (5) that for each given $\overline{\vartheta} = (p(t), \vartheta(t)) \in V$ problem (8) – (11) has a unique solution from $(V_2^{1,0}(\Omega), C[0,T])$.

Let $\Delta\overline{\vartheta} = (\Delta p, \Delta\vartheta) \in V$ be an increment of the control on the element $\overline{\vartheta} = (p, \vartheta) \in V$ such that $\overline{\vartheta} + \Delta\overline{\vartheta} \in V$. Denote $u \equiv u(x,t;\bar{\vartheta})$, $s_k \equiv s_k(t;\bar{\vartheta})$, $\Delta u(x,t) \equiv \equiv u(x,t;\bar{\vartheta} + \Delta\bar{\vartheta}) - u(x,t,\bar{\vartheta})$, $\Delta s_k \equiv \Delta s_k(t) = s_k(t;\bar{\vartheta} + \Delta\overline{\vartheta}) - s_k(t;\overline{\vartheta})$, $p_k = p_k(t)$, $\Delta p_k = \Delta p_k(t)$.

It follows from (2)-(5) that $\Delta u(x,t)$ is a generalized solution of the boundary value problem

$$\Delta u_t = a^2 \Delta u_{xx} + \sum_{k=1}^n [(p_k + \Delta p_k)\delta(x - (s_k + \Delta s_k)) - p_k \delta(x - s_k)], \ \ (x,t) \in \Omega, \tag{14}$$

$$\Delta u_x|_{x=0} = \Delta u_x|_{x=l} = 0, \ \ t \in [0,T], \tag{15}$$

$$\Delta u|_{t=0} = 0, \ \ x \in [0,l], \tag{16}$$

and functions $\Delta s_k(t),\ k = \overline{1,n}$, are the solutions of the Cauchy problem

$$\Delta \dot{s}_k(t) = f_k(s + \Delta s, \vartheta + \Delta\vartheta, t) - f_k(s, \vartheta, t), \quad \Delta s_k(0) = 0, \ k = \overline{1,n}. \tag{17}$$

It follows from (6) that the function $\Delta u(x,t)$ satisfies the integral identity

$$\int_0^l \int_0^T [-\Delta u \eta_t + a^2 \Delta u_x \eta_x] dx dt = \sum_{k=1}^n \int_0^T [(p_k(t) + \Delta p_k)\eta(s_k(t) + \Delta s_k, t) -$$

$$-p_k(t)\eta(s_k(t), t)] dt, \tag{18}$$

for $\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega),\ \eta(x,T) = 0$.

The function

$$H(t,s,\psi,q,\overline{\vartheta}) = -\left\{\sum_{k=1}^{n}\left[-f_k(s(t),\vartheta(t),t)q_k(t) + \psi(s_k(t),t)p_k(t) + \right.\right.$$

$$\left.\left. +\ \alpha_1\left(p_k(t)-\tilde{p}_k(t)\right)^2\right] + \alpha_2\sum_{m=1}^{r}\left(\vartheta_m(t)-\tilde{\vartheta}_m(t)\right)^2\right\}, \qquad (19)$$

is said to be Hamilton-Pontryagin function of problem (1)-(5). Now, we state sufficient conditions of Frechet differentiability of functional (1) and find an expression for its gradient.

Theorem 2. *Let the function $f\left(s,\vartheta,t\right)$ be continuous in totality of all its arguments together with all its partial derivatives with respect to variables s,ϑ for $(s,\vartheta,t)\in E^n\times E^r\times[0,T]$ and the following conditions*

$$\begin{array}{l} |f_k(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_k(s,\vartheta,t)|\le L(|\Delta s|+|\Delta\vartheta|), \\ |f_{ks}(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_{ks}(s,\vartheta,t)|\le L(|\Delta s|+|\Delta\vartheta|), \\ |f_{k\vartheta}(s+\Delta s,\vartheta+\Delta\vartheta,t)-f_{k\vartheta}(s,\vartheta,t)|\le L(|\Delta s|+|\Delta\vartheta|),\ k=\overline{1,n}, \end{array}$$

be fulfilled for all $(s+\Delta s,\vartheta+\Delta\vartheta,t)$, $\quad(s,\vartheta,t)\in E^n\times E^r\times[0,T]$, where $L=const\ge0$.

Then the functional (1) *is Frechet differentiable and the expression*

$$J'(\bar{\vartheta}) = -\frac{\partial H}{\partial\overline{\vartheta}} \equiv \left(-\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial\vartheta}\right), \qquad (20)$$

where

$$\begin{array}{l} \frac{\partial H}{\partial p} = \left(\frac{\partial H}{\partial p_1},\frac{\partial H}{\partial p_2},...,\frac{\partial H}{\partial p_n}\right), \quad \frac{\partial H}{\partial\vartheta} = \left(\frac{\partial H}{\partial\vartheta_1},\frac{\partial H}{\partial\vartheta_2},...,\frac{\partial H}{\partial\vartheta_r}\right), \\ \frac{\partial H}{\partial p_k} = -\psi(s_k(t),t) - 2\alpha_1\left(p_k(t)-\tilde{p}_k(t)\right), k=\overline{1,n}, \\ \frac{\partial H}{\partial\vartheta_m} = \sum_{k=1}^{n}\frac{\partial f_k(s(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t) - 2\alpha_2\left(\vartheta_m(t)-\tilde{\vartheta}_m(t)\right), m=\overline{1,r}, \end{array}$$

is valid for its gradient.

Proof. Consider the increment of the functional

$$\begin{array}{l} \Delta J \equiv J(\bar{\vartheta}+\Delta\bar{\vartheta})-J(\bar{\vartheta}) = 2\int_0^l[u(x,T)-y(x)]\Delta u(x,T)dx + \int_0^l\left|\Delta u(x,T)\right|^2dx + \\ +\sum_{k=1}^{n}\left\{2\alpha_1\int_0^T\left[p_k(t)-\tilde{p}_k(t)\right]\Delta p_k(t)dt + \alpha_1\int_0^T\left|\Delta p_k\right|^2dt\right\} + \\ +\sum_{m=1}^{r}\left\{2\alpha_2\int_0^T\left[\vartheta_m(t)-\tilde{\vartheta}_m(t)\right]\cdot\Delta\vartheta_m(t)dt + \alpha_2\int_0^T\left|\Delta\vartheta_m\right|^2dt\right\} \end{array} \qquad (21)$$

where $\bar{\vartheta}=(p,\vartheta)\in V$, $\bar{\vartheta}+\Delta\bar{\vartheta}\in V$, $\Delta u(x,T)\equiv u(x,T;\bar{\vartheta}+\Delta\bar{\vartheta})-u(x,T;\bar{\vartheta})$, $u\equiv u(x,T;\bar{\vartheta})$. Prove that

$$\begin{array}{l} 2\int_0^l[u(x,T)-y(x)]\Delta u(x,T)dx = \sum_{k=1}^{n}\left\{\int_0^T\psi(s_k(t),t)\ \Delta p_k(t)dt + \right. \\ \left. +\sum_{m=1}^{r}\int_0^T\frac{\partial f_k(s(t),\vartheta(t),t)}{\partial\vartheta_m}q_k(t)\Delta\vartheta_m(t)dt\right\} + R_1, \end{array} \qquad (22)$$

where $R_1 = \sum_{k=1}^{n} \int_0^T \psi_x(s_k(t),t)\Delta p_k(t)\Delta s_k(t)dt$.

If we set $\eta_1 = \Delta u\,(x,t)$, in (12), $\eta = \upsilon\,(x,t)$ in (18), and then subtract the obtained relations, we have

$$\int_0^l \int_0^T [\psi\Delta u_t + a^2\psi_x\Delta u_x]dxdt = 2\int_0^l [u(x,T) - y(x)]\Delta u(x,T)dx,$$

$$\int_0^l \int_0^T [-\Delta u\psi_t + a^2\psi_x\Delta u_x]dxdt = \sum_{k=1}^{n}\int_0^T [(p_k + \Delta p_k)\psi(s_k + \Delta s_k, t) - p_k\psi(s_k,t)]dt,$$

$$\int_0^l 2[u(x,T) - y(x)]\Delta u(x,T)dx = \sum_{k=1}^{n}\int_0^T [(p_k + \Delta p_k)\psi(s_k + \Delta s_k, t) - p_k\psi(s_k,t)]dt. \quad (23)$$

It follows from (17) that the function $\Delta s_k(t)$ satisfies the integral identity

$$\int_0^T \left[\Delta s_k(t)\dot{\theta}_k(t) + \Delta f_k(s(t),\vartheta(t),t)\theta_k(t)\right] dt = 0, \quad (24)$$

for $\forall\theta_k(t) \in C[0,T]$,$\theta_k(T) = 0, k = \overline{1,n}$.

It follows from (11) that the function $q_k(t)$ satisfies the integral identity

$$\int_0^T \left[\dot{\theta}_{1k}(t)q_k(t) - \left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial s_k}q_i(t) - \psi_x(s_k(t),t)p_k(t)\right)\theta_{1k}(t)\right] dt = 0, \quad (25)$$

for $\forall\theta_{1k}(t) \in C[0,T]$,$\theta_{1k}(0) = 0, k = \overline{1,n}$.

In the same way, if we set $\theta_{1k} = \Delta s_k$ in (25), $\theta_k = q_k$ in (24) and then sum the obtained relations, we have

$$\int_0^T \left[\Delta\dot{s}_k(t)q_k(t) - \left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial s_k}q_i(t) - \psi_x(s_k(t),t)p_k(t)\right)\Delta s_k(t)\right] dt = 0,$$

$$\int_0^T [\dot{q}_k(t)\Delta s_k(t) + \Delta f_k(s(t),\vartheta(t),t)q_k(t)]\, dt = 0,$$

$$[\Delta s_k(t)q_k(t)]\,\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial s_k}q_i(t) - \psi_x(s_k(t),t)p_k(t)\right)\Delta s_k(t) - \Delta f_k q_k(t)\right] dt.$$

Considering conditions of the theorem, we can represent the function $\Delta f_k = \Delta f_k(s(t),\vartheta(t),t)$ in the form

$$\Delta f_k = \sum_{i=1}^{n}\frac{\partial f_k}{\partial s_i}\Delta s_i + \sum_{m=1}^{r}\frac{\partial f_k}{\partial\vartheta_m}\Delta\vartheta_m + R_2,$$

where $R_2 = o\left(\sqrt{\|\Delta s\|^2_{L_2(0,T)} + \|\Delta\vartheta\|^2_{L_2(0,T)}}\right)$ as $\|\Delta s\|_{L_2(0,T)} \to 0$, and $\|\Delta\vartheta\|_{L_2(0,T)} \to 0$.

Then, from the last equality we have:

$$[\Delta s_k(t) q_k(t)]\,\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) - \psi_x(s_k(t),t) p_k(t)\right)\Delta s_k(t) - \right.$$

$$\left. - \sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m} \Delta\vartheta_m(t) q_k(t) - \sum_{i=1}^n \frac{\partial f_k}{\partial s_i} \Delta s_i(t) q_k(t)\right] dt + R_2.$$

From(17) and (11) we get

$$\int_0^T \psi_x(s_k(t),t) p_k(t) \Delta s_k(t) dt = -\sum_{m=1}^r \int_0^T \frac{\partial f_k}{\partial \vartheta_m} \Delta\vartheta_m(t) q_k(t) dt -$$

$$-\sum_{i=1}^n \int_0^t \left[\frac{\partial f_k}{\partial s_i} q_k(t) \Delta s_i(t) - \frac{\partial f_i}{\partial s_k} q_i(t) \Delta s_k(t)\right] dt + R_2. \tag{26}$$

It is clear that under the assumptions made above, the expansion

$\psi(s_k + \Delta s_k, t) = \psi(s_k, t) + \psi_x(s_k(t), t)\Delta s_k + o\left(\|\Delta s\|_{C[0,T]}\right)$ as $\|\Delta s\|_{C[0,T]} \to 0$,

is valid.

Considering this formula in (23), we get

$$2\int_0^l [u(x,T) - y(x)]\Delta u(x,T) dx = \sum_{k=1}^n \int_0^T [\psi_x(s_k(t),t) p_k(t) \Delta s_k(t) +$$
$$+ \psi(s_k(t),t)\Delta p_k(t) + \psi_x(s_k(t),t)\Delta p_k(t)\Delta s_k(t)]\, dt + o\left(\|\Delta s\|_{C[0,T]}\right).$$

In view of the fact that

$$\sum_{k=1}^n \sum_{i=1}^n \left[\frac{\partial f_k}{\partial s_i} q_k(t)\Delta s_i(t) - \frac{\partial f_i}{\partial s_k} q_i(t) \Delta s_k(t)\right] = 0,$$

from the last equality and the relation (26) we get

$$2\int_0^l [u(x,T) - y(x)]\Delta u(x,T) dx = \sum_{k=1}^n \int_0^T \left[-\sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m} q_k(t) \Delta\vartheta_m(t) + \right.$$
$$\left. + \psi(s_k,t)\Delta p_k\right] dt + R_3, \tag{27}$$

where

$$R_3 = \sum_{k=1}^n \int_0^T [\psi_x(s_k(t),t)\Delta p_k(t) \Delta s_k(t)]\, dt + R_2 + o(\|\Delta s\|_{C[0,T]}).$$

It is proved in (13) that the estimation

$$\|\Delta u(x,T)\|_{L_2(0,l)} \le c_1 \left\|\Delta\bar{\vartheta}\right\|_H \tag{28}$$

holds for the function $\Delta u(x,t)$ and in 6.3 of [6] it is established that the estimation

$$\|\Delta s\|_{L_2(0,T)} \le c_2 \|\Delta\vartheta\|_{L_2(0,T)}, \tag{29}$$

where $c_1 \ge 0, c_2 \ge 0$ are some constants, follows for the solution of problem (17).

Taking into account the estimation (29) in the expressions for R_1 and R_3, we get $R_3 = o\left(\left\|\Delta\overline{\vartheta}\right\|_H\right)$.

Considering these estimations in (21) and (22), we have:
$\Delta J(\bar{\vartheta}) = \sum_{k=1}^{n}\left(J_1(k) + \sum_{m=1}^{r} J_2(k,m)\right) + o\left(\left\|\Delta\bar{\vartheta}\right\|_H\right)$, as $\left\|\Delta\bar{\vartheta}\right\|_H \to 0$,
where

$$J_1(k) = \int_0^T \left[\psi(s_k(t),t) + 2\alpha_1\left(p_k(t) - \tilde{p}_k(t)\right)\right]\Delta p_k(t)dt,$$

$$J_2(k,m) = \int_0^T \left[-\frac{\partial f_k(s(t),\vartheta(t),t)}{\partial\vartheta_m} q_k(t) + 2\alpha_2\left(\vartheta_m(t) - \tilde{\vartheta}_m(t)\right)\right]\Delta q_m(t)dt.$$

Hence, allowing for expression of Hamilton-Pontryagin function, we get

$$\Delta J(\bar{\vartheta}) = \left(-\frac{\partial H}{\partial\bar{\vartheta}}, \Delta\bar{\vartheta}\right)_H + o\left(\left\|\Delta\bar{\vartheta}\right\|_H\right) \text{ as } \left\|\Delta\bar{\vartheta}\right\|_H \to 0,$$

that shows Frechet differentiability of functional (1) and validity of formula (20). The Theorem 2 is proved.

Now, let‘s get necessary conditions, i.e. control optimality conditions for problem (1)-(5).

Theorem 3. *Let all the conditions of theorem 1 be fulfilled and* $(u^*(x,t), s^*(t))$, $(\psi^*(x,t), q^*(t))$ *be solutions of problems* (2)-(5) *and* (8)-(11), *respectively, for* $\overline{\vartheta} = \overline{\vartheta}^* \in V$.*Then for optimality of the control* $\overline{\vartheta}^* = (p^*(t), \vartheta^*(t))$ *the condition*

$$H(t,s^*,\psi^*,q^*,\bar{\vartheta}^*) = \max_{\bar{\vartheta}\in V} H(t,s^*,\psi^*,q^*,\bar{\vartheta}), \tag{30}$$

should be fulfilled for $\forall(x,t)\in\Omega$.

Proof. Assume that $\bar{\vartheta}^* = (p^*,\vartheta^*)$ is an optimal control. Assume the contrary, i.e. assume there are a control $\tilde{\vartheta} = \bar{\vartheta}^* + h\cdot\Delta\bar{\vartheta} \in V$and the number $\beta > 0$ such that

$$H(t,s^*,\psi^*,q^*,\tilde{\vartheta}) - H(t,s^*,\psi^*,q^*,\bar{\vartheta}^*) \ge \beta > 0, \tag{31}$$

where $h > 0$ is some number, $\tilde{\vartheta} = (\tilde{p},\tilde{\vartheta}) \equiv (p^* + h\Delta p, \vartheta^* + h\Delta\vartheta)$.

If in (31) we take into account formula (20), we get

$$h\sum_{i=1}^{2}\left(\frac{\partial J(\breve{\vartheta}_i)}{\partial\bar{\vartheta}}, \Delta\bar{\vartheta}\right)_H \le -\beta < 0,$$

where $\breve{\vartheta}_1 = (h\theta_0 \Delta p, \tilde{\vartheta})$, $\breve{\vartheta}_2 = (p^*, h\theta_1 \Delta \vartheta)$, $\theta_i \in (0,1)$, $i = \overline{0,1}$ are some numbers. Hence, from the finite increment formula we have

$$J(\tilde{\vartheta}) - J(\bar{\vartheta}^*) = h \sum_{i=1}^{2} \left(\frac{\partial J(\widehat{\vartheta}_i)}{\partial \bar{\vartheta}}, \Delta \bar{\vartheta} \right)_H \le -\beta + h \cdot 0(\left\| \Delta \bar{\vartheta} \right\|_H), \tag{32}$$

where $\widehat{\vartheta}_1 = (h\gamma_0 \Delta p, \tilde{\vartheta})$, $\widehat{\vartheta}_2 = (p^*, h\gamma_1 \Delta \vartheta)$, $\gamma_i \in (0,1)$, $i = \overline{0,1}$ are some numbers.

Let $0<h1<h$ be such a number that $-\beta + h_1 o \left(\left\| \Delta \overline{\vartheta} \right\|_H \right) < 0$. Assume $\overline{\overline{\vartheta}} = (\overline{\overline{p}}, \overline{\overline{\vartheta}}) = (p^* + h_1 \Delta p, \vartheta^* + h_1 \Delta \vartheta)$. Reasoning as in the getting of inequality (32), we have

$$J(\overline{\overline{\vartheta}}) - J(\bar{\vartheta}^*) \le -\beta + h_1 o \left(\left\| \Delta \bar{\vartheta} \right\|_H \right) < 0.$$

This contradicts to the optimality of the control $\overline{\vartheta}^*$. Hence we get the validity of relation (30). The Theorem 3 is proved.

Using formula (20) and taking into account the expression of Hamilton-Pontryagin function, by the known theorem ([6], p.28) we get the validity of the following theorem:

Theorem 4. *Let the conditions of Theorem 1 be fulfilled. Then for the optimality of the control* $\overline{\vartheta}^* = (p^*(t), \vartheta^*(t)) \in V$ *, the condition*

$$\begin{aligned} &\int_0^T \sum_{k=1}^n [(\psi^*(s_k^*(t), t) + 2\alpha_1 (p_k^*(t) - \tilde{p}_k(t)), p_k(t) - p_k^*(t)) + \\ &+ \sum_{m=1}^r \left(-\tfrac{\partial f_k(s^*(t), \vartheta^*(t), t)}{\partial \vartheta_m} q_k^*(t) + 2\alpha_2 (\vartheta_m^*(t) - \tilde{\vartheta}_m(t)), \vartheta_m(t) - \vartheta_m^*(t) \right) \Big] \, dt \ge 0, \end{aligned}$$

should be fulfilled for $\forall \bar{\vartheta} = (p(t), \vartheta(t)) \in V$. *Here* $\psi^*(s_k^*(t), t), q_k^*(t)$ *are the solutions of problems* (8)-(10) *and* (11), *respectively, for* $\overline{\vartheta} = \overline{\vartheta}^* (p^*(t), \vartheta^*(t))$.

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