Markov type integral inequality for Pseudo-integrals
Daraby B.

Abstract. In this paper, generalizations of the Markov type integral inequalities for pseudo-integrals are proved. There are considered two cases of the real semiring with pseudo-operations: One, when pseudo-operations are defined by monotone and continuous function $g$ (then the pseudo-integrals reduces $g$-integral), and the second with a semiring $([a,b], max, ⊙)$, where the pseudo-multiplication $⊙$ is generated.

Key Words and Phrases: Non-additive measure, Chebyshev type inequality, Pseudo-addition, Pseudo-multiplication, Pseudo-integral

2000 Mathematics Subject Classifications: 03E72, 28E10, 26E50

1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a,b] \subset [-\infty, \infty]$ endowed with pseudo-addition $⊕$ and with pseudo-multiplication $⊙$ (see [13, 17, 20]). Based on this structure there where developed the concepts of $⊕$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. Pseudo-analysis would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [1, 2, 4, 5, 6, 8, 15, 18, 19, 20].

The well-known Markov inequality is a part of the classical mathematical analysis. The following inequality is a classical Markov type inequality [9]:

$$\mu\{x \in A : f(x) \geq c\} \leq \frac{1}{c} \int_A f d\mu,$$
where $f$ is a non-negative integrable function and $c > 0$. A. Flores-Franulíč et al. have proved Markov type inequalities for fuzzy integrals in [7].

In this paper, we generalize their works for pseudo-integrals. In special case, if in Markov type inequalities for pseudo-integrals we put $\oplus = \text{max}$ and $\odot = \text{min}$, then we get the Markov type inequality for Sugeno integrals [3].

The paper is organized as follows: Section 2 and 3 contain some of preliminaries, such as pseudo-operations and pseudo-analysis as well as integrals. In Section 4, We have proved generalizations of the Markov type inequality for pseudo-integrals. Finally, a conclusion is given in Section 5.

2. Preliminaries

In this section, we are going to review some well-known definitions of pseudo-operations. We refer to [10, 11, 12, 13, 14, 17].

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

**Definition 2.1.** The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $0$, i.e., for each $x \in [a, b], 0 \oplus x = x$ holds (usually $0$ is either $a$ or $b$).

Let $[a, b]_+ = \{x|x \in [a, b], 0 \preceq x\}$.

**Definition 2.2.** The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $1 \in [a, b]$, i.e., for each $x \in [a, b], 1 \odot x = x$.

We assume also $0 \odot x = 0$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring (see [10, 14]). In this paper we consider semirings with the continuous operations those that are discussed in [2, 13, 16]. In this paper we consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.
(a) Suppose that \( x \oplus y = \text{sup}(x, y), \odot \) is arbitrary and is not idempotent pseudo-multiplication on the interval \([a, b]\). We have \( 0 = a \) and the idempotent operation \( \text{sup} \) induces a full order in the following way: \( x \preceq y \) if and only if \( \text{sup}(x, y) = y \).

(b) Suppose that \( x \oplus y = \text{inf}(x, y), \odot \) is arbitrary and is not idempotent pseudo-multiplication on the interval \([a, b]\). We have \( 0 = b \) and the idempotent operation \( \text{inf} \) induces a full order in the following way: \( x \preceq y \) if and only if \( \text{inf}(x, y) = y \).

Case II: The pseudo-operations are defined by a monotone and continuous function 
\[ g : [a, b] \to [0, \infty], \] i.e., pseudo operations are given with \( x \oplus y = g^{-1}(g(x) + g(y)) \) and \( x \odot y = g^{-1}(g(x)g(y)) \).

If the zero element for the pseudo-addition is \( a \), we will consider increasing generators. Then \( g(a) = 0 \) and \( g(b) = 1 \). If the zero element for the pseudo-addition is \( b \), we will consider decreasing generators. Then \( g(b) = 0 \) and \( g(a) = 1 \). If the generator \( g \) is increasing (respectively decreasing), then the operation \( \oplus \) induces the usual order (respectively opposite to the usual order) on the interval \([a, b]\) in the following way: \( x \preceq y \) if and only if \( g(x) \leq g(y) \).

Case III: Both operations are idempotent. We have

(a) Suppose that \( x \oplus y = \text{sup}(x, y), x \odot y = \text{inf}(x, y) \), on the interval \([a, b]\). We have \( 0 = a \) and \( 1 = b \). The idempotent operation \( \text{sup} \) induces the usual order \( (x \preceq y \) if and only if \( \text{sup}(x, y) = y \).

(b) Suppose that \( x \oplus y = \text{inf}(x, y), x \odot y = \text{sup}(x, y) \), on the interval \([a, b]\). We have \( 0 = b \) and \( 1 = a \). The idempotent operation \( \text{inf} \) induces an order opposite to the usual order \( (x \preceq y \) if and only if \( \text{inf}(x, y) = y \).

Let \( X \) be a non-empty set. Let \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets of a set \( X \).

**Definition 2.3.** A set function \( m : \mathcal{A} \to [a, b]_+ \) (or semiclosed interval) is a \( \oplus \)-measure if there hold:

(i) \( m(\emptyset) = 0 \) (if \( \oplus \) is not idempotent);
(ii) \( m \) is \( \sigma - \oplus - \) (decomposable) measure, i.e.

\[
m\left( \bigcup_{i=1}^{\infty} A_i \right) = \bigoplus_{i=1}^{\infty} m(A_i)
\]

holds for any sequence \( A_i \in \mathbb{N} \) of pairwise disjoint sets from \( A \).

We suppose that \( ([a,b], \oplus) \) and \( ([a,b], \odot) \) are complete lattice ordered semigroups. Further, suppose that \([a,b]\) is endowed with a metric \( d \) compatible with \( \sup \) and \( \inf \), i.e. \( \lim_{n \to \infty} \sup x_n = x \) and \( \lim_{n \to \infty} \inf x_n = x \), imply \( \lim_{n \to \infty} d(x_n, x) = 0 \), and which satisfies at least one of the following conditions:

(a) \( d(x \oplus y, x' \oplus y') \leq d(x, x') + d(y, y') \),

(b) \( d(x \oplus y, x' \oplus y') \leq \max\{d(x, x'), d(y, y')\} \).

Both conditions (a) and (b) imply:

\( d(x_n, y_n) \to 0 \Rightarrow d(x_n \oplus z, y_n \oplus z) \to 0 \).

Metric \( d \) is also monotonic, i.e.,

\( x \leq z \leq y \Rightarrow d(x, y) \geq \sup\{d(y, z), d(x, z)\} \).

Let \( f \) and \( g \) be two functions defined on \( X \) and with values in a semiring \( ([a,b], \oplus, \odot) \). Then for any \( x \in X \) and for any \( \lambda \in [a,b] \) we define \( (f \oplus g)(x) = f(x) \oplus g(x) \), \( (f \circ g)(x) = f(x) \odot g(x) \) and \( (\lambda \odot f)(x) = \lambda \odot f(x) \).

**Definition 2.4.** The characteristic function with values in a semiring \( ([a,b], \oplus, \odot) \) is defined by

\[
\chi_A(x) = \begin{cases} 
1, & \text{if } x \in A, \\
0, & \text{if } x \notin A.
\end{cases}
\]

Where \( 0 \) is zero element for \( \oplus \) and \( 1 \) is unit element for \( \odot \).

**Definition 2.5.** An elementary (measurable) function is a mapping \( e : X \to [a,b] \) that has the following representation:

\[
e = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i},
\]

where \( a_i \in [a,b] \) and sets \( A_i \in A \) are pairwise disjoint if \( \oplus \) is nonidempotent.
Definition 2.6 ([11]). Let $\epsilon$ be a positive real number and $B \subset [a, b]$. A subset $\{l_i^\epsilon\}$ of the set $B$ is a $\epsilon$-net on $B$ if for each $x \in B$ there exists $l_i^\epsilon$ such that $d(l_i^\epsilon, x) \leq \epsilon$. If we, also, have $l_i^\epsilon \preceq x$, then we call $\{l_i^\epsilon\}$ a lower $\epsilon$-net. If $l_i^\epsilon \preceq l_{i-1}^\epsilon$ holds, then $\{l_i^\epsilon\}$ is monotone.

Definition 2.7. Let $m : A \to [a, b]$ be a $\oplus$-measure.

(i) The pseudo-integral of an elementary function $e : X \to [a, b]$ with respect to $m$ is defined by

\[ \int_X e \odot dm = \bigoplus_{i=1}^n a_i \odot m(A_i). \]

(ii) The pseudo-integral of a bounded measurable function $f : X \to [a, b]$, (if $\oplus$ is not idempotent we suppose that for each $\epsilon > 0$ there exists a monotone $\epsilon$-net in $f(X)$) is defined by

\[ \int_X f(x) \odot dm = \lim_{n \to \infty} \int_X e_n(x) \odot dm, \]

where $(e_n)_{n \in \mathbb{N}}$ is a sequence of elementary functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$. For more details see [14, 16].

3. Two important cases: generated and max-plus semirings

We shall consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$. Then the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the $g$-integral [12, 13],

\[ \int_{[c,d]} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right). \]

Now easily we can obtain the properties listed in the following proposition.

Proposition 3.1 ([16]). Let $(X, F, \mu, \mathbb{R}_+, \oplus, \odot)$ is a pseudo-space and $f, g \in F$, then:

1. If $f = 0$ on $A$ a.e., then $\int_A^\oplus f d\mu = 0$.
2. If $\mu(A) = 0$, then $\int_A^\oplus f d\mu = 0$.
3. $\int_A^\oplus a d\mu \geq a \odot \mu(A)$.
4. If $f \preceq g$ on $A$, then $\int_A^\oplus f d\mu \leq \int_A^\oplus g d\mu$.
5. If $A \subset B$, then $\int_A^\oplus f d\mu \leq \int_B^\oplus f d\mu$. 
Second case is when the semiring is of the form \([a, b, max, \odot]\). Then the pseudo-integral for a function \(f : \mathbb{R} \rightarrow [a, b]\) is given by
\[
\int_{\mathbb{R}} f \odot dm = \sup_{x \in \mathbb{R}} \left( f(x) \odot \psi(x) \right),
\]
where function \(\psi\) defines sup-measure \(m\). Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [11]. We shall denote by \(\mu\) the usual Lebesgue measure on \(\mathbb{R}\). We have
\[
m(A) = \text{ess sup}_\mu(x \mid x \in A) = \sup \{ a \mid \mu(\{ x \mid x \in A, x > a \}) > 0 \}.
\]

We have by [11]:

**Theorem 3.1.** Let \(m\) be a sup-measure on \(([0, \infty], \mathbb{B}([0, \infty]))\), where \(\mathbb{B}([0, \infty])\) is the Borel \(\sigma\)-algebra on \([0, \infty]\), \(m(A) = \text{ess sup}_\mu(\psi(x) \mid x \in A)\), and \(\psi : [0, \infty] \rightarrow [0, \infty]\) is a continuous density. Then for any pseudo-addition \(\oplus\) with a generator \(g\) there exists a family \(\{m_\lambda\}\) of \(\oplus_\lambda\)-measures on \(([0, \infty], \mathbb{B})\), where \(\oplus_\lambda\) is generated by \(g^\lambda\) (the function \(g\) of the power \(\lambda\), \(\lambda \in (0, \infty)\), such that \(\lim_{\lambda \rightarrow \infty} m_\lambda = m\).

For any continuous function \(f : [0, \infty] \rightarrow [0, \infty]\) the integral \(\int \oplus f \odot dm\) can be obtained as a limit of \(g\)-integrals, [11].

**Theorem 3.2.** Let \(([0, \infty], \sup, \odot)\) be a semiring with \(\odot\) generated by some increasing generator \(g\), i.e., we have \(x \odot y = g^{-1}(g(x)g(y))\) for every \(x, y \in [a, b]\). Let \(m\) be the same as in Theorem 3.1. Then there exists a family \(\{m_\lambda\}\) of \(\oplus_\lambda\)-measures, where \(\oplus_\lambda\) is generated by \(g^\lambda\), \(\lambda \in (0, \infty)\), such that for every continuous function \(f : [0, \infty] \rightarrow [0, \infty]\)
\[
\int \sup f \odot dm = \lim_{\lambda \rightarrow \infty} \int \oplus_\lambda f \odot dm_\lambda = \lim_{\lambda \rightarrow \infty} \left( g^\lambda(\int g^\lambda(f(x))dx) \right).
\]

4. Main results

**Lemma 4.1.** Let \(g : [a, b] \rightarrow [0, \infty]\) be a continuous and increasing function, then for any non-negative integrable function \(f : [c, d] \rightarrow [a, b]\) the inequality
\[ \mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A f^2 \, d\mu, \quad (4.1) \]

holds where \( A = [c, d] \) and \( e \in [a, b] \).

**Proof.** Let us consider \( A^* = \{x \in A : f(x) \geq e\} \). We must show that:

\[ \int_A f^2 \, d\mu \geq e^2.\mu(A^*). \quad (4.2) \]

As \( A^* \subseteq A \), then by (5) of Proposition 3.1 we have

\[ \int_A f^2 \, d\mu \geq \int_{A^*} f^2 \, d\mu. \]

Since \( f(x) \geq e \) for all \( x \in A^* \), we have

\[ (f)^2 \geq (e)^2. \]

Since \( g \) is an increasing function, then \( g(f^2) \geq g(e^2) \). Therefore by (4) of Proposition 3.1 we have

\[ \int_{A^*} g(f^2) \, d\mu \geq \int_{A^*} g(e^2) \, d\mu. \]

Since inverse of increasing function is increasing, so \( g^{-1} \) is also increasing. It follows that

\[ g^{-1}(\int_{A^*} g(f^2) \, d\mu) \geq g^{-1}(\int_{A^*} g(e^2) \, d\mu) \]

\[ = g^{-1}g(e^2).\mu(A^*) \]

\[ = e^2.\mu(A^*) \]

i.e.

\[ \int_{A^*} f^2 \, d\mu = g^{-1}(\int_{A^*} g(f^2) \, d\mu) \]

\[ \geq e^2.\mu(A^*). \]

From (4.2) we have

\[ \int_A f^2 \, d\mu \geq \int_{A^*} f^2 \, d\mu \]

\[ \geq e^2.\mu(A^*). \]

Consequently

\[ \mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A f^2 \, d\mu, \]

which completes the proof.

The following result is generalization of the Markov type inequality for pseudo-integrals.
Theorem 4.2. If \( g : [a,b] \to [0,\infty] \) is a continuous and increasing function, then for every non-negative integrable function \( f : [c,d] \to [a,b] \), the inequality

\[
\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A f d\mu
\]

holds, where \( e \in [a,b] \) and \( A = [c,d] \).

Proof. As \( f \geq 0 \) and \( \{x \in A^* : f(x) \geq e\} = \{x \in A : f(x) \geq e\} \), by Lemma 4.1 we have

\[
\mu(\{x \in A^* : f(x) \geq e\}) = \mu(\{x \in A : f(x) \geq e\}) \\
\leq \frac{1}{(e)^2} \int_A (f(x))^2 d\mu \\
= \frac{1}{e} \int_A f d\mu,
\]

which implies that the Theorem 4.2 holds.

Example 4.3. Let \( f(x) = x \), for all \( x \in [1,2] \) and \( g : [1,2] \to [0,\infty] \) be defined as \( g(x) = e^x \). Taking \( A = [1,2] \) and \( e = \frac{3}{2} \), we have

\[
\mu(\{x \in A : f(x) \geq e\}) = \mu(\{x \in [1,2] : x \geq \frac{3}{2}\}) \\
= \mu(\left[\frac{3}{2},2\right]) \\
= \frac{1}{2}
\]

and

\[
\int_A f d\mu = \int_A x d\mu \\
= g^{-1}(\int_1^2 g(x) dx) \\
= g^{-1}(e^2 - e) \\
= \ln(e^2 - e).
\]

Therefore

\[
\mu(\{x \in A : f(x) \geq e\}) = \frac{1}{e} \int_A f d\mu \\
= \frac{1}{e} \int_A f d\mu.
\]

In the sequel, we generalize the Markov inequality by the semiring \(([a,b],\max,\ominus)\), where \( \ominus \) is generated.
Theorem 4.4. Let $f : [c, d] \to [a, b]$ be a non-negative integrable function. If $\odot$ is represented by an increasing multiplicative generator $g$ and $m$ is the same as in Theorem 3.1, then the inequality

$$m(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A f \odot dm$$

holds, where $A = [c, d]$ and $e \in [a, b]$.

Proof. Suppose that $A^* = \{x \in A : f(x) \geq e\}$. Theorem 3.2 implies that

$$\int_{[c, d]}^\sup f \odot dm = \lim_{\lambda \to \infty} \int_{[c, d]}^{\oplus\lambda} f \odot dm_{\lambda}$$

$$= \lim_{\lambda \to \infty} (g^\lambda)^{-1} \left( \int_c^d g^\lambda(f(x)) dx \right)$$

$$\geq \lim_{\lambda \to \infty} (g^\lambda)^{-1} \left( \int_{A^*} g^\lambda(e) dx \right)$$

$$= \lim_{\lambda \to \infty} (g^\lambda)^{-1} g^\lambda(e) m(A^*)$$

$$= e \cdot m(A^*),$$

therefore

$$m(A^*) \leq \frac{1}{e} \int_{[c, d]}^\sup f \odot dm.$$

This completes the proof.

Note that the third important case $\oplus = \max$ and $\odot = \min$ for Theorem 4.2 has been studied in [3] and the pseudo-integral in such a case yields the Sugeno integral.

5. Conclusion

We have proved the Markov type inequalities for pseudo-integrals. There are two classes of pseudo-integrals. One of them concerning the pseudo-integrals based on a function reduces to the g-integral, where pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function $g$. The other one concerns the pseudo-integrals based on a semiring $([a, b], \max, \odot)$, where $x \odot y$ is generated by $g^{-1}(g(x)g(y))$. 

References


Bayaz Daraby
Department of Mathematics, University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran
E-mail: bayazdaraby@yahoo.com, bdaraby@maragheh.ac.ir