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Markov type integral inequality for Pseudo-integrals

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Abstract. In this paper, generalizations of the Markov type integral inequalities for pseudo-integrals are proved. There are considered two cases of the real semiring with pseudo-operations: One, when pseudo-operations are defined by monotone and continuous function g (then the pseudo-integrals reduces g-integral), and the second with a semiring $([a,b], max, \odot)$, where the pseudo-multiplication $\odot$ is generated.

Key Words and Phrases: Non-additive measure, Chebyshev type inequality, Pseudo-addition, Pseudo-multiplication, Pseudo-integral

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1. Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a,b] \subset [-\infty, \infty]$ endowed with pseudo-addition $\bigoplus$ and with pseudo-multiplication $\odot$ (see [13, 17, 20]). Based on this structure there where developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. Pseudo-analysis would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [1, 2, 4, 5, 6, 8, 15, 18, 19, 20].

The well-known Markov inequality is a part of the classical mathematical analysis. The following inequality is a classical Markov type inequality [9]:

$$\mu\{x \in A : f(x) \geq c\} \leq \frac{1}{c}\int_A f d\mu,$$

where f is a non-negative integrable function and $c > 0$. A. Flores-Franulič et al. have proved Markov type inequalities for fuzzy integrals in [7].

In this paper, we generalize their works for pseudo-integrals. In special case, if in Markov type inequalities for pseudo-integrals we put $\oplus = max$ and $\odot = min$, then we get the Markov type inequality for Sugeno integrals [3].

The paper is organized as follows: Section 2 and 3 contain some of preliminaries, such as pseudo-operations and pseudo-analysis as well as integrals. In Section 4, We have proved generalizations of the Markov type inequality for pseudo-integrals. Finally, a conclusion is given in Section 5.

2. Preliminaries

In this section, we are going to review some well-known definitions of pseudo-operations. We refer to [10, 11, 12, 13, 14, 17].

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring (see [10, 14]). In this paper we consider semirings with the continuous operations those that are discussed in [2, 13, 16].In this paper we consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) Suppose that $x \oplus y = sup(x, y), \odot$ is arbitrary and is not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $sup(x, y) = y$.

(b) Suppose that $x \oplus y = inf(x, y), \odot$ is arbitrary and is not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x) + g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$.
If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = 1$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = 1$. If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have

(a) Suppose that $x \oplus y = sup(x, y), x \odot y = inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $sup(x, y) = y$).

(b) Suppose that $x \oplus y = inf(x, y), x \odot y = sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order ($x \preceq y$ if and only if $inf(x, y) = y$).

Let X be a non-empty set. Let $\mathbb{A}$ be a σ-algebra of subsets of a set X.

Definition 2.3. A set function $m : \mathbb{A} \to [a, b]_+$ (or semiclosed interval) is a $\oplus$-measure if there hold:

(i) $m(\phi) = \mathbf{0}$ (if $\oplus$ is not idempotent);

(ii) m is $\sigma - \oplus -$(decomposable) measure, i.e.

$$m\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $A_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathbb{A}$.

We suppose that $([a,b],\oplus)$ and $([a,b],\odot)$ are complete lattice ordered semigroups. Further, suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf, i.e. $lim_{n\to\infty} sup x_n = x$ and $lim_{n\to\infty} inf x_n = x$, imply $lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y, x' \oplus y') \leq d(x,x') + d(y,y')$,

(b) $d(x \oplus y, x' \oplus y') \leq max\{d(x,x'), d(y,y')\}$.

Both conditions (a) and (b) imply:

$d(x_n, y_n) \to 0 \Rightarrow d(x_n \oplus z, y_n \oplus z) \to 0$.

Metric d is also monotonic, i.e.,

$x \leq z \leq y \Rightarrow d(x,y) \geq sup\{d(y,z), d(x,z)\}$.

Let f and g be two functions defined on X and with values in a semiring $([a,b],\oplus,\odot)$. Then for any $x \in X$ and for any $\lambda \in [a,b]$ we define $(f \oplus g)(x) = f(x) \oplus g(x)$, $(f \odot g)(x) = f(x) \odot g(x)$ and $(\lambda \odot f)(x) = \lambda \odot f(x)$.

Definition 2.4. The characteristic function with values in a semiring $([a,b],\oplus,\odot)$ is defined by

$$\chi_A(x) = \begin{cases} \mathbf{1}, & \text{if } x \in A, \\ \mathbf{0}, & \text{if } x \notin A. \end{cases}$$

Where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 2.5. An elementary (measurable) function is a mapping
$e : X \to [a,b]$ that has the following representation:

$$e = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i} \qquad ,$$

where $a_i \in [a,b]$ and sets $A_i \in \mathbb{A}$ are pairwise disjoint if $\oplus$ is nonidempotent.

Definition 2.6 ([11]). Let ϵ be a positive real number and $B \subset [a,b]$. A subset $\{l_i^\epsilon\}$ of the set B is a ϵ-net on B if for each $x \in B$ there exists l_i^ϵ such that $d(l_i^\epsilon, x) \leq \epsilon$. If we, also, have $l_i^\epsilon \preceq x$, then we call $\{l_i^\epsilon\}$ a lower ϵ-net. If $l_i^\epsilon \preceq l_{i-1}^\epsilon$ holds, then $\{l_i^\epsilon\}$ is monotone.

Definition 2.7. Let $m : \mathbb{A} \to [a,b]$ be a $\oplus$-measure.

(i) The pseudo-integral of an elementary function $e : X \to [a,b]$ with respect to m is defined by

$$\int_X^{\oplus} e \odot dm = \bigoplus_{i=1}^{n} a_i \odot m(A_i).$$

(ii) The pseudo-integral of a bounded measurable function $f : X \to [a,b]$, (if $\oplus$ is not idempotent we suppose that for each $\epsilon > 0$ there exists a monotone ϵ-net in $f(X)$) is defined by

$$\int_X^{\oplus} f(x) \odot dm = lim_{n\to\infty} \int_X^{\oplus} e_n(x) \odot dm,$$

where $(e_n)_{n\in\mathbb{N}}$ is a sequence of elementary functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$. For more details see [14, 16].

3. Two important cases: generated and max-plus semirings

We shall consider the semiring $([a,b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0,\infty]$. Then the pseudo-integral for a function $f : [c,d] \to [a,b]$ reduces on the g-integral [12, 13],

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\Big(\int_c^d g(f(x))dx\Big).$$

Now easily we can obtain the properties listed in the following proposition.

Proposition 3.1 ([16]). Let $(X, \mathcal{F}, \mu, \mathbb{R}_+^-, \oplus, \odot)$ is a pseudo-space and $f, g \in \mathcal{F}$, then:
(1) If $f = 0$ on A a.e., then $\int_A^{\oplus} f d\mu = 0$.
(2) If $\mu(A) = 0$, then $\int_A^{\oplus} f d\mu = 0$.
(3) $\int_A^{\oplus} a d\mu \geq a \odot \mu(A)$.
(4) If $f \leq g$ on A, then $\int_A^{\oplus} f d\mu \leq \int_A^{\oplus} g d\mu$.
(5) If $A \subset B$, then $\int_A^{\oplus} f d\mu \leq \int_B^{\oplus} f d\mu$.

Second case is when the semiring is of the form $([a,b], max, \odot)$. Then the pseudo-integral for a function $f : \mathbb{R} \to [a,b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = sup_{x\in\mathbb{R}}\Big(f(x) \odot \psi(x)\Big),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [11]. We shall denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = ess\ sup_{\mu}(x \mid x \in A) = sup\{a \mid \mu(\{x \mid x \in A, x > a\}) > 0\}.$$

We have by [11]:

Theorem 3.1. *Let m be a sup-measure on $([0,\infty], \mathbb{B}([0,\infty]))$, where $\mathbb{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A) = ess\ sup_{\mu}(\psi(x)|x \in A)$, and $\psi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$–measure on $([0,\infty), \mathbb{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0,\infty)$, such that $lim_{\lambda\to\infty} m_\lambda = m$.*

For any continuous function $f : [0,\infty] \to [0,\infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [11].

Theorem 3.2. *Let $([0,\infty]$, sup, $\odot)$ be a semiring with $\odot$ generated by some increasing generator g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a,b]$. Let m be the same as in Theorem 3.1. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by $g^\lambda, \lambda \in (0,\infty)$, such that for every continuous function $f : [0,\infty] \to [0,\infty]$*

$$\int^{sup} f \odot dm = lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int g^\lambda(f(x))dx\Big).$$

4. Main results

Lemma 4.1. *Let $g : [a,b] \to [0,\infty]$ be a continuous and increasing function, then for any non-negative integrable function $f : [c,d] \to [a,b]$ the inequality*

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A^{\oplus} f^2 d\mu, \tag{4.1}$$

holds where $A = [c, d]$ *and* $e \in [a, b]$.

Proof. Let us consider $A^* = \{x \in A : f(x) \geq e\}$. We must show that:

$$\int_A^{\oplus} f^2 d\mu \geq e^2.\mu(A^*).$$

As $A^* \subseteq A$, then by (5) of Proposition 3.1 we have

$$\int_A^{\oplus} f^2 d\mu \geq \int_{A*}^{\oplus} f^2 d\mu. \tag{4.2}$$

Since $f(x) \geq e$ for all $x \in A^*$, we have

$$(f)^2 \geq (e)^2.$$

Since g is an increasing function, then $g(f^2) \geq g(e^2)$. Therefore by (4) of Proposition 3.1 we have

$$\int_{A^*} g(f^2) d\mu \geq \int_{A^*} g(e^2) d\mu.$$

Since inverse of increasing function is increasing, so g^{-1} is also increasing. It follows that

$$\begin{aligned} g^{-1}\Big(\int_{A^*} g(f^2) d\mu\Big) &\geq g^{-1}\Big(\int_{A^*} g(e^2) d\mu\Big) \\ &= g^{-1} g(e^2).\mu(A^*) \\ &= e^2.\mu(A^*) \end{aligned}$$

i.e.

$$\begin{aligned} \int_{A^*}^{\oplus} f^2 d\mu &= g^{-1}\Big(\int_{A^*} g(f^2) d\mu\Big) \\ &\geq e^2.\mu(A^*). \end{aligned}$$

From (4.2) we have

$$\begin{aligned} \int_A^{\oplus} f^2 d\mu &\geq \int_{A^*}^{\oplus} f^2 d\mu \\ &\geq e^2.\mu(A^*). \end{aligned}$$

Consequently

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e^2} \int_A^{\oplus} f^2 d\mu,$$

which completes the proof.

The following result is generalization of the Markov type inequality for pseudo-integrals.

Theorem 4.2. *If $g : [a, b] \to [0, \infty]$ is a continuous and increasing function, then for every non-negative integrable function $f : [c, d] \to [a, b]$, the inequality*

$$\mu(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_A^{\oplus} f d\mu$$

holds, where $e \in [a, b]$ and $A = [c, d]$.

Proof. As $f \geq 0$ and $\{x \in A^* : f(x) \geq e\} = \{x \in A : f(x) \geq e\}$, by Lemma 4.1 we have

$$\begin{aligned} \mu(\{x \in A^* : f(x) \geq e\}) &= \mu(\{x \in A : f(x) \geq e\}) \\ &\leq \frac{1}{(e)^2} \int_A^{\oplus} (f(x))^2 d\mu \\ &= \frac{1}{e} \int_A^{\oplus} f d\mu, \end{aligned}$$

which implies that the Theorem 4.2 holds.

Example 4.3. Let $f(x) = x$, for all $x \in [1, 2]$ and $g : [1, 2] \to [0, \infty]$ be defined as $g(x) = e^x$. Taking $A = [1, 2]$ and $e = \frac{3}{2}$, we have

$$\begin{aligned} \mu(\{x \in A : f(x) \geq e\}) &= \mu(\{x \in [1, 2] : x \geq \frac{3}{2}\}) \\ &= \mu([\frac{3}{2}, 2]) \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \int_A^{\oplus} f d\mu &= \int_A^{\oplus} x d\mu \\ &= g^{-1}(\int_1^2 g(x) dx) \\ &= g^{-1}(e^2 - e) \\ &= \ln(e^2 - e). \end{aligned}$$

Therefore

$$\mu(\{x \in A : f(x) \geq e\}) = frac12 \leq Ln(e^2 - e) = \frac{1}{e} \int_A^{\oplus} f d\mu.$$

In the sequel, we generalize the Markov inequality by the semiring $([a, b], max, \odot)$, where $\odot$ is generated.

Theorem 4.4. *Let $f : [c, d] \to [a, b]$ be a non-negative integrable function. If $\odot$ is represented by an increasing multiplicative generator g and m is the same as in Theorem 3.1, then the inequality*

$$m(\{x \in A : f(x) \geq e\}) \leq \frac{1}{e} \int_{A}^{\sup} f \odot dm$$

holds, where $A = [c, d]$ and $e \in [a, b]$.

Proof. Suppose that $A^* = \{x \in A : f(x) \geq e\}$. Theorem 3.2 implies that

$$\begin{aligned}
\int_{[c,d]}^{\sup} f \odot dm &= \lim_{\lambda\to\infty} \int_{[c,d]}^{\oplus_\lambda} f \odot dm_\lambda \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int_c^d g^\lambda(f(x))dx\Big) \\
&\geq \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int_{A^*} g^\lambda(f(x))dx\Big) \\
&\geq \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int_{A^*} g^\lambda(e)dx\Big) \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} g^\lambda(e).m(A^*) \\
&= e.m(A^*),
\end{aligned}$$

therefore

$$m(A^*) \leq \frac{1}{e} \int_{[c,d]}^{\sup} f \odot dm.$$

This completes the proof.

Note that the third important case $\oplus = max$ and $\odot = min$ for Theorem 4.2 has been studied in [3] and the pseudo-integral in such a case yields the Sugeno integral.

5. Conclusion

We have proved the Markov type inequalities for pseudo-integrals. There are two classes of pseudo-integrals. One of them concerning the pseudo-integrals based on a function reduces to the g-integral, where pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function g. The other one concerns the pseudo-integrals based on a semiring $([a, b], max, \odot)$, where $x \odot y$ is generated by $g^{-1}\big(g(x)g(y)\big)$.

References

[1] H. Agahi, R. Mesiar and Y. Ouyang, *New general extensions of Chebyshev type inequalities for Sugeno integrals*, Int. J. Approx. Reason. **51** (2009), 135-140.

[2] H. Agahi, R. Mesiar and Y. Ouyang, *Chebyshev type inequalities for pseudo-integrals*, Nonlinear Analysis **72** (2010), 2737-2743.

[3] J. Caballero and K. Sadarangani, *Chebyshev inequality for Sugeno integrals*, Fuzzy Sets and Systems, **161** (2010), 1480-1487.

[4] B. Daraby, *Investigation of a Stolarsky type inequality for integrals in pseudo-analysis*, Fractional Calculus & Applied Analysis, **13 (5)** (2010), 467-473.

[5] B. Daraby, *Stolarsky type inequality for seminormed fuzzy integral.* In: Proc. 2nd Int. Conf. Mathematical Science (ICMS2) 2010, 30 November - 3 December 2010, Kuala Lumpur, Malaysia, 110-117.

[6] B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-interals*, Fuzzy Sets and Systems, **194** (2012), 90-96.

[7] A. Flores-Franulič, H. Román-Flores and Y. Chalco-Cano, *Markov type inequalities for fuzzy integrals*, Applied Mathematics and Computation, **207** (2009), 242-247.

[8] Y. Hong, X. Fang and B. Wang, *Intuitionistic fuzzy quasi-metric and pseudo-metric spaces*, Iranian Journal of Fuzzy Systems, **5 (3)** (2007), 81-88.

[9] A. Kolmogorov and S. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover, New York, 1999.

[10] W. Kuich, *Semirings, Automata, Languages*, Springer-Verlag, Berlin, 1986.

[11] R. Mesiar and E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets and Systems, **102** (1999), 385-392.

[12] E. Pap, *An integral generated by decomposable measure*, Univ. Novom Sadu Zb. Rad. Prirod. -Mat. Fak. Ser. Mat. **20 (1)** (1990), 135-144.

[13] E. Pap, *g-calculus*, Univ. Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **23 (1)** (1993), 145-156.

[14] E. Pap, *Null-additive Set Functions*, Kluwer, Dordrecht, 1995.

[15] E. Pap and N. Ralević, *Pseudo-Laplace transform*, Nonlinear Anal. **33** (1998), 553-560.

[16] E. Pap, *Pseudo-additive measures and their applications*, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002, pp. 1403-1465.

[17] E. Pap, *Generalized real analysis and its applications*, Int. J. Approx. Reason. **47** (2008), 368-386.

[18] E. Pap and M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Information Sciences, **180** (2010), 543-548.

[19] F. G. Shi, *Pointwise pseudo-metric on the L-real line*, Iranian Journal of Fuzzy Systems, **2 (2)** (2004), 15-20.

[20] M. Sugeno and T. Murofushi, *Pseudo-additive measures and integrals*, J. Math. Anal. Appl., **122** (1987), 197-222.

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