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A Variable Exponent Hardy's Inequality Approach for Some Nonlinear Eigenvalue Problem

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Abstract. Applying a new bounded ness and compactness result for Hardy's operator $(\int\limits_0^x f(t)\,dt)$ and its conjugate $(\int\limits_x^l f(t)\,dt)$ in variable exponent spaces $L^{p(\cdot)}(0,l)$ and applying the Mountain Pass Theorem approaches in this paper it has been proved an existence result for the eigenvalue problem

$$\begin{cases} -\left(|y'|^{p(x)-2}y'\right)' = \lambda\, y^{p(x)-1} + \left(\frac{y}{x^{\alpha}(l-x)^{\alpha}}\right)^{q(x)-1}\frac{a(x)}{x^{\alpha}(l-x)^{\alpha}}, \\ y(x)>0, \qquad 0<x<l, \\ y(0)=y(l)=0. \end{cases}$$

where the exponent function $p:(0,l)\to(1,\infty)$ is monotone near the origin and l also satisfying a log-regularity conditions in this points.

Key Words and Phrases: variable exponent spaces, inequality, eigenvalue problem, mountain pass theorem, functional.

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1. Introduction

In this paper, we shall study an existence result for the nonlinear eigenvalue problem

$$\begin{cases} -\left(\,|y'|^{p(x)-2}y'\right)' = \lambda\, y^{p(x)-1} + \left(\frac{y}{x^{\alpha}(l-x)^{\alpha}}\right)^{q(x)-1}\frac{a(x)}{x^{\alpha}(l-x)^{\alpha}}, \\ y(x)>0, \quad 0<x<l, \\ y(0)=y(l)=0. \end{cases} \tag{1}$$

Let $Lip_0(0,l)$ be a class of Lipshitsz continuous functions $f:(0,l)\to R$ with $f(0)=f(l)=0$. Close this class of functions in a norm

$$\|f\|_{\dot{W}^1_{p(.)}(0,l)} = \|f'\|_{L^{p(\cdot)}(0,l)}.$$

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The obtained variable exponent Sobolev type space denote as $\dot{W}^1_{p(\cdot)}(0,l)$. This is a reflexive Banach space if $1 < p^- := \inf\limits_{(0,l)} p(x), \quad p^+ := \sup\limits_{(0,l)} p(x) < \infty$ (see, e.g. [14, 15])

In space $\dot{W}^1_{p(\cdot)}(0,l)$, consider an eigenvalue problem (1) with Dirichlet conditions in the ends of a finite interval $(0,l)$.

Let λ_1 be the first eigenvalue number of the $p(x)$-Laplace's operator. In other words,

$$\lambda_1 = \inf_{\{y\in AC(0,l),\, y\neq 0,\, y(0)=y(l)=0\}} \frac{\int\limits_0^l |y'(x)|^{p(x)}\,dx}{\int\limits_0^l |y_(x)|^{p(x)}\,dx} \tag{2}$$

It is satisfied

$$\begin{cases} -\frac{d}{dx}\left(\left|\frac{dy_1}{dx}\right|^{p(x)-2}\frac{dy_1}{dx}\right) = \lambda_1 \left|y_1(x)\right|^{p(x)-2} y_1(x), \\ y(x) > 0, \quad 0 < x < l, \\ y(0) = y(l) = 0. \end{cases} \tag{3}$$

for the first eigenvalue λ_1 and the eigenfunction $y_1(x)$ of the problem (2). It has been shown in [3] that there are infinitely many discreet eigenvalues $0 \le \lambda_1 < \lambda_2 ... < \lambda_k ...$ of the problem (3) such that $\lambda_k \to \infty$ as $k \to \infty$. At that, the first eigenvalue may be no strongly positive. In the cited work, it was stated that the first eigenvalue is strongly positive ($\lambda_1 > 0$) if one dimensional case and a monotony exponent function $p(x)$ be considered.

To prove the existence of solution of problem (1), we shall apply a Montain pass theorem due to Ambrosetti and Rabinowitz [2, 1]. In order to carry out this, we need some new variable exponent boundedness and compactness results for Hardy's operator and its conjugate [7, 8, 12, 9].

Theorem 1. *Let $q, p : (0,l) \to (1,\infty)$ be measurable functions such that $1 < p^- \le p(x) \le q(x) \le q^+ < \infty$. Assume that, $\alpha \in (1 - \frac{1}{p^+}, 1)$, and be satisfied the conditions:*

$$\limsup_{x\to 0} |f(x) - f(0)| \ln\frac{1}{x} < \infty, \quad \limsup_{x\to l} |f(x) - f(l)| \ln\frac{1}{l-x} < \infty, \tag{4}$$

moreover,

$$p^+ \le q^- < \frac{1}{\alpha - 1 + \frac{1}{p^+}}. \tag{5}$$

holds.

Then the set of functions $\{y(t) \in AC(0,l) : y(0) = y(l) = 0\}$ with bounded norm

$$\left\|y'(x)\right\|_{L^{p(\cdot)}(0,l)}$$

are compactly embedded into the class of functions with finite norm

$$\left\|\frac{y}{x^\alpha (l-x)^\alpha}\right\|_{L^{q(\cdot)}(0,l)}. \tag{6}$$

For an exact characterization of the Hardy's inequality in variable exponent spaces not using the regularity conditions (4) on the exponent functions see, the recent works [11, 13])

Theorem 2. *Let* $p : (0,l) \to (1,\infty)$ *be measurable function, such that,* $1 < p^- \le p(x) \le p^+ < \infty$*. Assume that,* p *satisfies* (4) *near the origin and* l*. Then it holds an inequality*

$$\left\| \frac{y(x)}{x(l-x)} \right\|_{p(x);(0,l)} \le \frac{C}{l} \left\| y'(x) \right\|_{p(x);(0,l)} \tag{7}$$

for all absolutely continuous functions $u : (0,l) \to R$ *with* $u(0) = u(l) = 0$*. Moreover, a positive constant* C *in* (7) *depends on* p^-, p^+, C_1, C_2*.*

From Theorem 2 one gets easily the following Sobolev type inequality

$$\frac{1}{lC} \|y\|_{L^{p(\cdot)}(0,l)} \le \|y'\|_{L^{p(\cdot)}(0,l)} \tag{8}$$

for any absolutely continuous function y in $(0,l)$ with limits $y(0) = y(l) = 0$.

Theorem 3. *Let* $q, p : (0,l) \to (1,\infty)$ *be measurable functions, such that,* $1 < p^- \le p(x) \le p^+ < q^- \le q(x) \le q^+ < \infty$*, and the conditions* (4) *be satisfied. Let the exponent function* p *be monotony near the origin and* l*. Assume a real positive number* α *satisfies* (5)*. Then there exists a positive solution of the problem* (1) *from space* $\dot{W}^1_{p(\cdot)}(0,l)$ *for any* $\lambda < \lambda_1$ *and* $a(x) \in L^\infty(0,l)$*.*

The proof of the above result relies on the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [1] in the following variant.

Theorem 4. *Let* X *be a real Banach space and let* $F : X \to \mathbb{R}$ *be* C^1*-functional. Suppose that* F *satisfies the Palas-Smale condition and the following geometric assumptions:*

- *1) there exists positive constants* ρ, c_0 *such that* $F(u) \ge c_0$ *for all* $u \in X$ *with* $\|u\| = \rho$*;*
- *2)* $F(0) < c_0$ *and there exists* $v \in X$ *such that* $\|v\| > \rho$ *and* $F(v) < c_0$*.*

Then the functional F *posseses at least a critical point.*

For the multidimensional case $n \ge 3$ and constant exponents $p = 2,\ 2 < q < \frac{2n}{n-2}$, $\alpha = 0$, $a(x) = 1$ we refer to [4], where an enhanced description of nonlinearities and eigenvalue number ranges, enabling multiplicity of solutions for the problem (1) is given applying the Lusternik-Schnirelman category approache in manifold. For the variable exponent setting, we cite [6], where constant exponents q, $\alpha = 0$, $a(x) = 1$ has been considered in case $n \ge 2$.

For a solution of problem (1) we call a function $y \in \dot{W}^1_{p(\cdot)}(0,l)$ that satisfies the integral identity

$$\int_0^l |y'|^{p(x)-2} y'v'\,dx - \lambda \int_0^l y_+^{p(x)-1} v\,dx$$
$$- \int_0^l \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \frac{va(x)}{x^\alpha (l-x)^\alpha}\,dx = 0 \tag{9}$$

for any test function $v \in \dot{W}^1_{p(\cdot)}(0,l)$.

Consider in $\dot{W}^1_{p(\cdot)}(0,l)$ the functional $I : \dot{W}^1_{p(\cdot)}(0,l) \to R$ defined as

$$I(y) = \int_0^l \frac{1}{p(x)} |y'|^{p(x)}\,dx - \int_0^l \frac{\lambda}{p(x)} y_+^{p(x)}\,dx - \int_0^l \frac{a(x)}{q(x)} \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)} dx, \tag{10}$$

where $y_+ = \max\,(y(x), 0)\,.$

Correct setting a solution notion. Verify correctness of the solution notion and the functional $I(y)$ setled in $E := \dot{W}^1_{p(\cdot)}(0,l)$. The first integral in (9) is well defined by virtue of Holder's inequality and $y, v \in \dot{W}^1_{p(\cdot)}(0,l)$. By virtue of (7) and Holder's inequalities second and third integrals are well-defined:

$$\int_0^l |y_+|^{p(x)-2} |y_+ v|\,dx \le c_0 \big\| |y_+|^{p(x)-1} \big\|_{L^{p'(\cdot)}(0,l)} \cdot \big\| v \big\|_{L^{p(\cdot)}(0,l)}$$
$$\le c_0 \Big(1 + \|y_+\|^{p^+-1}_{L^{p(\cdot)}(0,l)}\Big) \big\| v \big\|_{L^{p(\cdot)}(0,l)}$$
$$c_0 l \Big(1 + C l^{p^+-1} \|y'\|^{p^+-1}_{L^{p(\cdot)}(0,l)}\Big) \|v'\|_{L^{p(\cdot)}(0,l)}.$$

For the third integral by use of Young's inequality, it follows

$$\int_0^l \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \Big|\frac{v}{x^\alpha (l-x)^\alpha}\Big|\,dx$$

$$\int_0^l \frac{a(x)}{q'(x)} \big(\frac{y_+}{x^\alpha (l-x)^\alpha}\big)^{q(x)}\,dx + \int_0^l \frac{a(x)}{q(x)} \big|\frac{v}{x^\alpha (l-x)^\alpha}\big|^{q(x)}\,dx = i_1 + i_2$$

For every summand here we have the inequalities

$$i_1 \le \int_0^l \frac{K^{q(x)}}{q^-} \big(\frac{y_+}{K x^\alpha (l-x)^\alpha}\big)^{q(x)}\,dx \le \frac{1 + K^{q^+}}{q^-} \int_0^l \big(\frac{y_+}{K x^\alpha (l-x)^\alpha}\big)^{q(x)}\,dx$$

$$\le \frac{1+K^{q^+}}{q^-}\Big(1+\big(\varepsilon\|y'\|_{L^{p(\cdot)}(0,l)}+C_\varepsilon\|y\|_{L^{p(\cdot)}(0,l)}\big)^{q^+-1}\Big)$$

$$\le \frac{1}{q^-}+\frac{1}{q^-}\big(\varepsilon\|y'\|_{L^{p(\cdot)}(0,l)}+C_\varepsilon l^2\|\frac{y}{x(l-x)}\|_{L^{p(\cdot)}(0,l)}\big)^{q^+-1}$$

$$\le \frac{1}{q^-}+\frac{1}{q^-}\big(\varepsilon+C_2C_\varepsilon l\big)^{q^+-1}\|y'\|^{q^+-1}_{L^{p(\cdot)}(0,l)}\le C_3\big\|y\big\|_{\dot W^1_{p(\cdot)}(0,l)}$$

with

$$K=\Big\|\frac{y_+}{x^\alpha(l-x)^\alpha}\Big\|_{L^{q(\cdot)}(0,l)},\quad \varepsilon>0.$$

Notice, here it has been used the inequality

$$\big\|y\big\|_Y\le\varepsilon\big\|y\big\|_X+C_\varepsilon\big\|y\big\|_Z \tag{11}$$

for a triple Banach spaces $Y\subset X\subset Z$ with the imbedding $Y\subset\subset X$ to be compactly [10] and Theorem 1 and Theorem 2.

Same chain of inequalities hold for the i_2 too.

The Gatox derivative of $I(y)$ and its continuity.

Show that the functional $I(u)$ has a continuous Gatox derivative $I'(u)\in E^*$ and for every $v\in E$ it holds

$$<\ I'(u),v>=\int_0^l |y'|^{p(x)-2}y'v'\ dx-\lambda\int_0^l |y|^{p(x)-2}yv\,dx$$

$$-\int_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-2}\frac{y_+}{x^\alpha(l-x)^\alpha}\cdot\frac{v}{x^\alpha(l-x)^\alpha}\,dx. \tag{12}$$

Derivatives of $J(y)$. For a functional $J(u)=\int_0^l|y'|^{p(x)}\,dx$ number r, a function $v\in E$ using the mean value theorem, and Lebesgue's limit theorem tending $r\to0$, it follows

$$\frac{J(y+rv)-J(y)}{r}=\int_0^l\frac{1}{p(x)}\frac{1}{r}\Big(|y'(x)+rv'(x)|^{p(x)}-|y'(x)|^{p(x)}\Big)\,dx$$

$$=\int_0^l|y'(x)+\theta rv'(x)|^{p(x)-2}y'(x)v'(x)\,dx\to\int_0^l|y'(x)|^{p(x)-2}y'(x)v'(x)\,dx, \tag{13}$$

where $\theta\in(0,1)$ depends on $x,y(x)$.

We have used that $|y'(x)+\theta rv'(x)|^{p(x)-2}\to|y'(x)|^{p(x)-2}$ as $r\to0$ a.e. $x\in(0,l)$. We have also used that there exists an integrable majorant function for all $r\in(-1,1)$ in order to apply the Legesgue theorem:

$$\Big||y'(x)+\theta rv'(x)|^{p(x)-2}y'(x)v'(x)\Big|$$

$$\leq \Big(|y'(x)|+|r||v'(x)|\Big)^{p(x)-2}\Big(\frac{|y'|+|v'|}{2}\Big)^2$$

$$\leq \Big(|y'(x)|+|r||v'(x)|\Big)^{p(x)} \leq 2^{p(x)-1}\Big(|y'(x)|^{p(x)}+|r|^{p(x)}|v'(x)|^{p(x)}\Big).$$

Therefore, the upper passage to the limit in (13) is legitimately.

The continuity of derivatives $J(y)$. Show $J\in C^1(E,E^*)$. Let $y_n\to y$ in E. Then for a $v\in E$ we have

$$|<J'(y_n)-J'(y),v>| = \Big|\int_0^1 \big(|y_n'|^{p(x)-2}y_n' - |y'|^{p(x)-2}y'\big)v'\,dx\Big|$$

Using Egorov's theorem, there is a set $A\subset(0,l)$ with $|A|<\delta$ such that $y_n'\to y'$ uniformly in $(0,l)\setminus A$. Let $N(\varepsilon)\in N$ be such that $|y_n'(x)-y'(x)|<\varepsilon, \quad x\in(0,l)\setminus A$ as $n>N(\varepsilon)$. Then

$$|<J'(y_n)-J'(y),v>| \leq \int_{(0,1)\setminus A} \big||y_n'|^{p(x)-2}y_n' - |y'|^{p(x)-2}y'\big||v'|\,dx$$

$$+\int_A \big||y_n'|^{p(x)-1}+|y'|^{p(x)-1}\big||v'|\,dx$$

$$\leq C\varepsilon\|v\|_{\dot W^1_{p(\cdot)}(0,l)} + c_0\|v\|_{\dot W^1_{p(\cdot)}(0,l)}\Big(\|y_n'\|^{p^+}_{L_{p(\cdot)}(A)} + \|y'\|^{p^-}_{L_{p(\cdot)}(A)}\Big)$$

Therefore and since $y_n'\to y'$ in $L_{p(\cdot)}(0,1)$,

$$\|J(y_n)-J(y)\|_{E^*} \leq C\varepsilon + c_0\|y_n'\|_{L_{p(\cdot)}(A)} + c_0\|y'\|_{L_{p(\cdot)}(A)}$$

$$\leq (C+1)\varepsilon + 2c_0\|y'\|_{L_{p(\cdot)}(A)} < \epsilon$$

choosing sufficiently small $\delta>0$ and ε.

Derivatives of $F(y)$. For a functional

$$F(y)=\int_0^l y_+^{p(x)}\,dx, \quad \text{where} \quad y_+(x)=\max\{y(x),0\},$$

show that

$$<F'(y),v> = \int_0^l y_+^{p(x)-2}y_+v\,dx.$$

By the same way, as above,

$$\frac{F(y+rv)-F(y)}{r} = \int_0^l \frac{1}{p(x)}\cdot\frac{(y+rv)_+^{p(x)}-y_+^{p(x)}}{r}\,dx$$

$$= \int_0^l \zeta_+^{p(x)-1} v\,dx \to \int_0^l y_+^{p(x)-1} v\,dx \quad \text{as} \quad r \to 0,$$

where ζ is a number between y_+ and $(y+rv)_+$.

Continuity of derivatives of $F(y)$. To show $F \in C^1(E, E^*)$ let $y_n \to y$ in E. From Theorem 2 it follows $y_n \to y$ in $L^{p(\cdot)}(0,l)$. For a fixed $v \in E$ we have

$$| < F'(y_n) - F'(y), v > | = \Big| \int_0^l \left((y_n)_+^{p(x)-1} - y_+^{p(x)-1} \right) v\,dx \Big|$$

Since $y_n \to y$ in $L^{p(\cdot)}(0,l)$ there exists a subsequence y_{n_k} converging y almost everywhere in $(0,l)$. Denote it again y_n. Using Egorov's theorem there exists a set $|A| < \delta$ with any small $\delta > 0$, such that, the convergence y_n to y is uniformly on $(0,l) \setminus A$.

Then since $|(y_n)_+ - y_+| \le |y_n - y|$, it follows

$$| < F'(y_n) - F'(y), v > | = \Big| \int_{(0,1)\setminus A} \left((y_n)_+^{p(x)-1} - y_+^{p(x)-1} \right) v\,dx \Big|$$

$$+ \Big| \int_A \left((y_n)_+^{p(x)-1} - y_+^{p(x)-1} \right) v\,dx \Big|$$

$$\le \varepsilon \int_{(0,1)\setminus A} |v|\,dx + \int_A (y_n)_+^{p(x)-1} |v|\,dx + \int_A y_+^{p(x)-1} |v|\,dx$$

Applying Holder's inequality here one gets

$$| < F'(y_n) - F'(y), v > |$$

$$\le \left(C\varepsilon + \|(y_n)_+^{p(x)-1}\|_{L^{p'(\cdot)}(A)} + \|y_+^{p(x)-1}\|_{L^{p'(\cdot)}(A)} \right) \|v\|_{L^{p(\cdot)}(0,l)} \tag{14}$$

Applying for any $g \in L^{p(\cdot)}$ the inequality

$$\|g^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \le \|g\|_{L^{p(\cdot)}}^{p^+-1} + \|g\|_{L^{p(\cdot)}}^{p^--1}$$

in the right hand side (14) one gets

$$| < F'(y_n) - F'(y), v > |$$

$$\left((C+1)\varepsilon + 3\|y\|_{L^{p(\cdot)}(A)}^{p^--1} \right) \|v\|_{L^{p(\cdot)}(0,l)}$$

Choosing sufficiently small $\delta > 0$ and applying inequality (11) this is exceeded

$$(C+2)C_1\varepsilon\|v\|_E.$$

Hence

$$\|F(y_n) - F(y)\|_{E^*} \le (C+2)C_1\varepsilon,$$

which proves the continuity of derivative of functional F.

Derivatives of $G(y)$. By the same way, find the Gatox derivative of the functional

$$G(u) = \int_0^l \frac{a(x)}{q(x)} \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)} dx$$

in E and show its continuity. Show that

$$< G'(y), v >= \int_0^l a(x) \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \cdot \frac{v}{x^\alpha (l-x)^\alpha} dx. \qquad (15)$$

By the same way, as above,

$$\frac{G\,(y+rv) - G(y)}{r} = \int_0^l \frac{a(x)}{q(x)} \cdot \frac{1}{r}\Big(\Big(\frac{(y+rv)_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)} - \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)}\Big) v\, dx$$

$$= \int_0^l \frac{a(x)}{q(x)} \cdot \frac{1}{x^\alpha (l-x)^\alpha} \Big(\frac{(y+rv)_+^{q(x)} - y_+^{q(x)}}{r}\Big) v\, dx$$

Using the mean value formula this equals

$$\int_0^l a(x) \cdot \frac{1}{x^\alpha (l-x)^\alpha} \theta^{q(x)-1} v\, dx,$$

where θ is a quantity ranged between y_+ and $(y+rv)_+$. Tending $r \to 0$ and applying Lebesgue convergence theorem from this one gets (15). For this, it has been used that $a \in L^\infty$ and $v, \theta \in L^{q(\cdot)}(0,l)$. The last inclusion follows from Holder's inequality and Theorem 2:

$$\|\theta\|_{L^{q(\cdot)}(0,l)} \le \|(y+rv)_+\|_{L^{q(\cdot)}(0,l)} + \|y_+\|_{L^{q(\cdot)}(0,l)}$$

$$\le 2\|y\|_{L^{q(\cdot)}(0,l)} + r\|v\|_{L^{q(\cdot)}(0,l)}$$

$$\le 2l^{2\alpha} \|\frac{y}{x^\alpha (l-x)^\alpha}\|_{L^{q(\cdot)}(0,l)} + r l^{2\alpha} \|\frac{v}{x^\alpha (l-x)^\alpha}\|_{L^{q(\cdot)}(0,l)}.$$

Applying the compact embedding result from Theorem 2 by using inequality (11) from here we get

$$\|\theta\|_{L^{q(\cdot)}(0,l)} \le \varepsilon 2 l^{2\alpha} C_1 \Big(\|y'\|_{L^{p(\cdot)}(0,l)} + r\|v'\|_{L^{p(\cdot)}(0,l)}\Big) +$$

$$C_\varepsilon 2 l^{2\alpha} \Big(\|y\|_{L^{p(\cdot)}(0,l)} + 2 l^{2\alpha} r \|v\|_{L^{p(\cdot)}(0,l)}\Big).$$

This guaranties the limiting prosses using Lebesgue Theorem.

Continuity of derivatives of $G(y)$. Show the continuity of derivative of the functional G. Let $y_n \to y$ in E. Show that $G'(y_n) \to G'(y)$ in E^*. In this way, let $v \in E$ be any function.

We have

$$|< G'(y_n) - G'(y), v >|$$

$$= \Big| \int_0^l a(x) \Big(\Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} - \Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} \Big) \cdot \frac{v}{x^\alpha (l-x)^\alpha}\, dx \Big|$$

As the preceding estimates since $|(y_n)_+ - y_+| \le |y_n - y|$, we have

$$|< G'(y_n) - G'(y), v >| = \Big| \int_{(0,1)\setminus A} \frac{|a(x)|}{(x^\alpha (l-x)^\alpha)^{q(x)}} \Big((y_n)_+^{q(x)-1} - y_+^{q(x)-1} \Big) v\, dx \Big|$$

$$+ \Big| \int_A \frac{|a(x)|}{(x^\alpha (l-x)^\alpha)^{q(x)}} \Big((y_n)_+^{q(x)-1} - y_+^{q(x)-1} \Big) v\, dx \Big|$$

$$\le \varepsilon \int_{(0,1)\setminus A} |a(x)| \cdot \frac{|v|}{x^\alpha (l-x)^\alpha}\, dx + \int_A \frac{|a(x)|\, (y_n)_+^{q(x)-1} |v|}{(x^\alpha (l-x)^\alpha)^{q(x)}}\, dx$$

$$+ \int_A \frac{|a(x)|\, y_+^{q(x)-1} |v|}{(x^\alpha (l-x)^\alpha)^{q(x)}}\, dx$$

(we have included a little neighborhoods of origin and l to the set A).

Applying Holder's inequality in the preceding inequality, one gets

$$|< G'(y_n) - G'(y), v >|$$

$$\le \Big[C\varepsilon + \Big\| \Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} \Big\|_{L^{q'(\cdot)}(A)}$$

$$+ \Big\| \Big(\frac{y_+}{x^\alpha (l-x)^\alpha} \Big)^{q(x)-1} \Big\|_{L^{q'(\cdot)}(A)} \Big] \cdot \Big\| \frac{v}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}(0,l)} \tag{16}$$

Applying in the case $g(x) = \Big(\frac{y_+(x)}{x^\alpha (l-\alpha)} \Big)^{q(x)-1}$ and $p(x) = q(x)$ the inequality

$$\|g^{p(\cdot)-1}\|_{L^{p'(\cdot)}} \le \|g\|_{L^{p(\cdot)}}^{p^+-1} + \|g\|_{L^{p(\cdot)}}^{p^--1}$$

in the right hand side (16) one gets

$$|< G'(y_n) - G'(y), v >|$$

$$\le \Big((C+1)\varepsilon + 3 \Big\| \frac{y}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}(A)}^{q^--1} \Big) \Big\| \frac{v}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}(0,l)}$$

Choosing sufficiently small $\delta > 0$ and applying inequality (11) this is exceeded

$$(C+2)C_1\varepsilon\|v\|_E.$$

This entails

$$\|G'(y)-G'(y_n)\|_{E^*}\le C_1\varepsilon,$$

which proves the continuity of functional G'.

Weak lower semi continuity of $I(y)$.

Lower semi continuity of $J(y)$. First show the weak lower semi continuity (w.l.s.c.) of $J(y)$. (In order to show this, some people use the fact from [5] asserting that a convex functional is w.l.s.c. if it is a strongly lower semi continuous).

Show that $J(y)$ is convex in E. For any $\theta\in(0,1)$ and $y,z\in E$ we have

$$J\big(\theta y+(1-\theta)z\big)=\int_0^l \big|\theta y'(x)+(1-\theta)z'(x)\big|^{p(x)}\,dx,$$

by convexity of the function x^p,

$$\le\theta\int_0^l \big|y'(x)\big|^{p(x)}+(1-\theta)\int_0^l \big|z'(x)\big|^{p(x)}\,dx$$

To show the strong lower semi continuity of $J(y)$ in E set $y_n\to y$. We have

$$\int_0^l |y_n'|^{p(x)}\,dx-\int_0^l |y'|^{p(x)}\,dx=\int_0^l \frac{d}{dt}|y'+t(y_n'-y')|^{p(x)}\,dx$$

$$=\int_0^l p(x)|y'+t(y_n'-y')|^{p(x)-2}\big(y'+t(y_n'-y')\big)(y_n'-y')\,dx$$

$$=\int_0^l p(x)\Big(|y'+t(y_n'-y')|^{p(x)-2}\big(y'+t(y_n'-y')\big)-|y'|^{p(x)-2}y'\Big)\big(y'+t(y_n'-y')-y'\big)\,\frac{dx}{t}$$

$$+\int_0^l |y'|^{p(x)-2}y'(y_n'-y')\,dx,$$

since the first integral is positive by the convexity it holds an inequality, $\big(|a|^{p-2}a-|b|^{p-2}\big)(a-b)\ge 0$, for any $a,b\in\mathbb{R}$ that entails $|b|^{p-2}b\ge|a|^{p-2}a+p|a|^{p-2}a(b-a)$, therefore,

$$\ge\int_0^l |y'|^{p(x)-2}y'(y_n'-y')\,dx.$$

Now, it remains to take a limit in the preseeding inequality, in order to show that $J(y)$ is weakly lower semi continues in E :

$$\liminf_{n\to\infty}\int\limits_0^l |y_n'|^{p(x)}\,dx \geq \int\limits_0^l |y'|^{p(x)}\,dx + \liminf_{n\to\infty}\int\limits_0^l |y'|^{p(x)-2}y'(y_n'-y')\,dx$$

$$\geq \int\limits_0^l |y'|^{p(x)}\,dx,$$

i.e.

$$\liminf_{n\to\infty} J(y_n) \geq J(y)$$

Lower semi continuity of $I(y)$. Let $\{y_n\}\subset E$ be a weakly convergent subsequence of E tending to $y\in E$, i.e. $y_n \rightharpoonup y$. Show that $\liminf\limits_{n\to\infty} I(y_n)\geq I(y)$. By Theorem 1 the space E compactly imbedded into the class (6). By this, there exists a subsequence y_{n_k} that converges strongly to y in the norm $\left\|\Big(x(l-x)\Big)^{-\alpha}\cdot\right\|_{L^{q(\cdot)}(0,l)}$. and $\|\|_{L^{p(\cdot)}(0,l)}$ This means

$$\liminf_{n\to\infty} I(y_{n_k}) = \liminf_{n\to\infty}\int\limits_0^l \frac{1}{p(x)}\,|y_{n_k}'|^{p(x)}\,dx$$

$$-\lim_{n\to\infty}\int\limits_0^l \frac{\lambda}{p(x)}\,|(y_{n_k})_+|^{p(x)}\,dx - \lim_{n\to\infty}\int\limits_0^l \frac{a(x)}{q(x)}\left|\frac{(y_{n_k})_+}{x^\alpha(l-x)^\alpha}\right|^{q(x)}dx$$

$$\geq \int\limits_0^l \frac{1}{p(x)}\,|y'|^{p(x)}\,dx - \int\limits_0^l \frac{\lambda}{p(x)}\,|y_+|^{p(x)}\,dx$$

$$-\int\limits_0^l \frac{a(x)}{q(x)}\left|\frac{y_+}{x^\alpha(l-x)^\alpha}\right|^{q(x)}dx = I(y)$$

Therefore,

$$\liminf_{n\to\infty} I(y_{n_k}) \geq I(y),$$

that proves lower semi continuity of $I(y)$.

Palas-Smale condition (PS). Recall the notion of PS -condition.

Let $\{y_n\}\subset E$ be a sequence such that

1) $I(y_n)$ is bounded ;

2) $I'(y_n)\to I'(y)$ in E^*.

Then there exists a subsequence y_{n_k} that converges to y strongly in E. Since $I(y_n)$ is bounded, we may assume that $I(y_{n_k}) \to c$ by some real number $c \in R$. To save simplicity, denote y_{n_k} as y_n.

Boundedness of y_n' *in* E . From condition 1) it follows that there exists an $M > 0$ not depending on n such that $|I(y_n)| \le M$, i.e.

$$\int_0^l \frac{1}{p(x)} \left(|y_n'|^{p(x)} - \lambda (y_n)_+^{p(x)}\right) dx - \int_0^l \frac{a(x)}{q(x)} \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)} dx \le M,$$

or

$$\int_0^l \frac{a(x)}{q(x)} \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)} dx \ge \int_0^l \frac{1}{p(x)} \left(|y_n'|^{p(x)} - \lambda (y_n)_+^{p(x)}\right) dx - M.$$

Then by assumption $\lambda_1 > 0$ it follows that

$$\int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)} dx \ge \frac{q^-}{p^+} \int_0^l |y_n'|^{p(x)} dx - \int_0^l \frac{\lambda q^-}{p^+} (y_n)_+^{p(x)} dx - Mq^-. \tag{17}$$

On other hand, from condition 2) it follows that

$$| < I'(y_n), v > | \le o(1) \|v\|_{W^1_{p(\cdot)}(0,l)},$$

i.e.

$$\int_0^l |y_n'|^{p(x)-2} y_n' v' \, dx - \lambda \int_0^l (y_n)_+^{p(x)-1} v \, dx$$

$$- \int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)-1} \cdot \frac{v}{x^\alpha (l-x)^\alpha} dx$$

$$= o(1) \|v'\|_{L^{p(\cdot)}(0,l)}.$$

Inserting here $v = y_n$ this yields

$$\int_0^l |y_n'|^{p(x)} dx - \lambda \int_0^l (y_n)_+^{p(x)} dx$$

$$- \int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)} dx = o(1) \|y_n'\|_{L^{p(\cdot)}(0,l)}$$

or

$$\int_0^l a(x) \left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)} dx \le \int_0^l \left(|y_n'|^{p(x)} - \lambda (y_n)_+^{p(x)}\right) dx$$

$$+o(1)\|y_n'\|_{L^{p(\cdot)}(0,l)}. \tag{18}$$

From (18) and (17) and the assumption $q^- > p^+$ it follows that

$$\Big(\frac{q^-}{p^+}-1\Big)\int\limits_0^l |y_n'|^{p(x)}\,dx \le \lambda\Big(\frac{q^-}{p^+}-1\Big)\int\limits_0^l (y_n)_+^{p(x)}\,dx + Mq^- + o(1)\|y_n'\|_{L^{p(\cdot)}(0,l)}$$

or

$$\int\limits_0^l |y_n'|^{p(x)}\,dx \le \lambda\int\limits_0^l (y_n)_+^{p(x)}\,dx + \frac{Mq^-p^+}{q^- - p^+} + o(1)\|y_n'\|_{L^{p(\cdot)}(0,l)}$$

Now assuming $\lambda < \lambda_1$ and a strong positivity of the first eigenvalue λ_1 in (2), (3) from this it follows

$$\int\limits_0^l |y_n'|^{p(x)}\,dx \le O(1).$$

The bounded ness of y_n in E has been proved.

Now, after establishment of the bounded ness $\{y_n\}$ in E, we may apply the the weak convergence for some subsequence $\{y_{n_k}\}$. Moreover, show the strong convergence $y_n \to y$ in E. Remaining the notation y_n in place of y_{n_k}, the weak convergence $y_n \to y$ in E, we have the equality for PS-sequence:

$$\int\limits_0^l |y_n'|^{p(x)-2}y_n'v'\,dx - \lambda\int\limits_0^l (y_n)_+^{p(x)-1}v\,dx$$

$$-\int\limits_0^l a(x)\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\cdot\frac{v}{x^\alpha(l-x)^\alpha}\,dx$$

$$= o(1)\|v'\|_{L^{p(\cdot)}(0,l)}. \tag{19}$$

Inserting in (19) $v = y_n - y$, we get

$$\int\limits_0^l |y_n'|^{p(x)-2}y_n'(y_n'-y')\,dx - \lambda\int\limits_0^l (y_n)_+^{p(x)-1}(y_n-y)\,dx$$

$$-\int\limits_0^l a(x)\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\cdot\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx$$

$$= o(1)\|y_n'-y'\|_{L^{p(\cdot)}(0,l)}. \tag{20}$$

From (20), we easily get

$$\int_0^l \Big(|y_n'|^{p(x)-2}y_n' - |y'|^{p(x)-2}y'\Big)(y_n' - y')\,dx + \int_0^l |y'|^{p(x)-2}y'(y_n' - y')$$

$$= \lambda \int_0^l \big((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\big)(y_n - y)\,dx + \lambda \int_0^l y_+^{p(x)-1}(y_n - y)$$

$$+ \int_0^l a(x)\Big[\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1} - \Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\Big]\cdot \frac{y_n - y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+ \int_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\frac{y_n - y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+o(1)\|y_n' - y'\|_{L^{p(\cdot)}(0,l)}. \tag{21}$$

Now, since $y_n \to y$ weakly in E , we see that the additional terms in (21) tend to zero: those are

$$\lim_{n\to\infty} \int_0^l |y'|^{p(x)-2}y'(y_n' - y') = 0 \tag{22}$$

that is implied from the fact that for $y \in E$ it is $|y'|^{p(x)-2}y' \in E^*$ (that is $|y'|^{p(x)-2}y' \in L^{p'}$).

The convergence

$$\lim_{n\to\infty} \int_0^l y_+^{p(x)-1}(y_n - y) = 0 \tag{23}$$

follows from the fact that $y_+^{p(x)-1} \in E^*$, and $y_n \to y$ weakly in E since

$$\Big|\int_0^l y_+^{p(x)-1}(y_n - y)\,dx\Big| \le C(l)\int_0^l \Big(\frac{y_+}{x(l-x)}\Big)^{p(x)-1}\Big|\frac{y_n - y}{x(l-x)}\Big|\,dx, \tag{24}$$

where $C(l) = l^2 \max\big\{l^{p^+-1},\, l^{p^--1}\big\}$. Applying inequality (7) to the expression (24) we find that is exceeded

$$\le C(l)\Big\|\frac{y_n - y}{x(l-x)}\Big\|_{L^{p(\cdot)}}\Big\|\frac{y_+^{p(x-1)}}{x(l-x)}\Big\|_{L^{p'(\cdot)}}$$

$$\le C_2^2 C(l)\big\|y_n' - y'\big\|_{L^{p(\cdot)}}\Big(\|y'\|_{L^{p(\cdot)}}^{p^+-1} + \|y'\|_{L^{p(\cdot)}}^{p^--1}\Big) \le C_3\big\|y_n' - y'\big\|_E.$$

The convergence

$$\lim_{n\to\infty}\int_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx=0 \tag{25}$$

follows from the fact that $\frac{a(x)}{x^\alpha(l-x)^\alpha}\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\in E^*$, since

$$\Big|\int_0^l a(x)\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx\Big|$$

$$\leq C_1(l)\|a\|_{L^\infty}\cdot\Big\|\frac{y_n-y}{x(l-x)}\Big\|_{L^{q(\cdot)}}\cdot\Big\|\Big(\frac{y_+}{x(l-x)}\Big)^{q(x)-1}\Big\|_{L^{q'(\cdot)}}$$

$$\leq C_1(l)C_2^2\|a\|_{L^\infty}\cdot\Big\|y_n'-y'\Big\|_{L^{p(\cdot)}}\cdot\Big(\|y'\|_{L^{p(\cdot)}}^{q^+-1}+\|y'\|_{L^{p(\cdot)}}^{q^--1}\Big)\leq C_4\big\|y_n-y\big\|_E,$$

where $C_1(l)=\max\big\{l^{2(1-\alpha)q^+},\,l^{2(1-\alpha)q^-}\big\}$. Applying the limits (22), (23), (25) it follows from (21) that

$$\int_0^l\Big(|y_n'|^{p(x)-2}y_n'-|y'|^{p(x)-2}y'\Big)(y_n'-y')\,dx$$

$$=\lambda\int_0^l\big((y_n)_+^{p(x)-1}-y_+^{p(x)-1}\big)(y_n-y)\,dx$$

$$+\int_0^l a(x)\Big[\Big(\frac{(y_n)_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}-\Big(\frac{y_+}{x^\alpha(l-x)^\alpha}\Big)^{q(x)-1}\Big]\cdot\frac{y_n-y}{x^\alpha(l-x)^\alpha}\,dx$$

$$+o(1)\|y_n'-y'\|_{L^{p(\cdot)}(0,l)}+o(1). \tag{26}$$

We need the following two inequalities for $a,b\in R$ (see e.g., [?] in the case of n -dimensional vectors)

$$\Big(|a|^{p-1}a-|b|^{p-2}\Big)(a-b)\geq\gamma_1(p)|a-b|^p\quad\text{if}\qquad p\geq 2,$$

$$\Big(|a|^{p-1}a-|b|^{p-2}\Big)(a-b)\geq\gamma_2(p)\frac{|a-b|^2}{\Big(|a|+|b|\Big)^{2-p}}\quad\text{if}\quad p\leq 2. \tag{27}$$

In order to finish the proof of convergence $y_n\to y$ in E , we shall use Egorov's theorem in order to show a convergence to zero of the first summand in the right hand side (26), and compact imbedding theorem, to show the convergence of second summand.

For $\lambda \geq 0$, $p \geq 2$ using (27) it follows from (21) that

$$\gamma_1(p) \int\limits_0^l |y_n' - y'|^{p(x)}\, dx \leq \lambda \int\limits_0^l \left((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\right)(y_n - y)\, dx$$

$$+ \int\limits_0^l a(x) \left[\left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)-1} - \left(\frac{y_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)-1}\right] \cdot \frac{y_n - y}{x^\alpha (l-x)^\alpha}\, dx$$

$$+ o(1) \|y_n' - y'\|_{L^{p(\cdot)}(0,l)} + o(1). \tag{28}$$

Using mean value theorem, the last integral (28) is estimated as

$$\left| \int\limits_0^l a(x) \left[\left(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)-1} - \left(\frac{y_+}{x^\alpha (l-x)^\alpha}\right)^{q(x)-1}\right] \cdot \frac{y_n - y}{x^\alpha (l-x)^\alpha}\, dx \right|$$

$$\leq (q^+ - 1) \|a(x)\|_{L^\infty} \cdot \int\limits_0^l \left(\frac{y_n - y}{x^\alpha (l-x)^\alpha}\right)^2 \cdot \frac{|y_n|^{q(x)-2} + |y|^{q(x)-2}}{\left(x^\alpha (l-x)^\alpha\right)^{q(x)-2}}\, dx$$

Further, applying Holder's inequality in the right hand side it is exceeded

$$\leq (q^+ - 1) \|a(x)\|_{L^\infty} \left\| \frac{y_n - y}{x^\alpha (l-x)^\alpha} \right\|_{L^{q(\cdot)}}^2 \cdot \left(\left\| \frac{|y_n| + |y|}{x^\alpha (l-x)^\alpha} \right\|_{L^{q(\cdot)}} \right)^{q^+ - 2} \to 0 \tag{29}$$

as $n \to \infty$ by using the compact embedding E into the weighted class (6) in Theorem 2.

Using Egorov's theorem there exists a set $|A| < \delta$ with any small $\delta > 0$, such that the convergence y_n to y is uniformly on $A^c = (0, l) \setminus A$. Applying that, and the Holder inequality, we see

$$\int\limits_0^l \left((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\right)(y_n - y)\, dx$$

$$\leq \varepsilon \int\limits_{A^c} \left((y_n)_+^{p(x)-1} + y_+^{p(x)-1}\right) dx \tag{30}$$

$$+ \int\limits_A \left(|y_n|^{p(x)} + |y_n|^{p(x)-2}|y|^2 + |y_n|^2 |y|^{p(x)-2} + |y|^{p(x)}\right) dx < (M + 4)\varepsilon$$

choosing sufficiently small $\delta > 0$ and large n.

Inserting in (28) the estimates (30), (29) we get the strong convergence $y_n \to y$ in E for the case $p(x) \geq 2$.

It remains to consider the case $p < 2$. Inserting the second inequality (27) in (28) and applying the Holder inequality, we get

$$\gamma_2(p) \int_0^l \frac{|y_n' - y'|^2}{\Big(|y_n'| + |y'|\Big)^{2-p}} \, dx \le \lambda \int_0^l \big((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\big)(y_n - y) \, dx$$

$$+ \int_0^l a(x) \left[\Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} - \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \right] \cdot \frac{y_n - y}{x^\alpha (l-x)^\alpha} \, dx \tag{31}$$

$$+ o(1) \|y_n' - y'\|_{L^{p(\cdot)}(0,l)} + o(1)$$

The second integral in the right hand side (31) is estimated as

$$\Big| \int_0^l a(x) \left[\Big(\frac{(y_n)_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} - \Big(\frac{y_+}{x^\alpha (l-x)^\alpha}\Big)^{q(x)-1} \right] \cdot \frac{y_n - y}{x^\alpha (l-x)^\alpha} \, dx \Big|$$

$$\le \|a(x)\|_{L^\infty} \cdot \int_0^l \Big(\frac{y_n - y}{x^\alpha (l-x)^\alpha} \Big) \cdot \frac{|y_n|^{q(x)-1} + |y|^{q(x)-1}}{\big(x^\alpha (l-x)^\alpha\big)^{q(x)-1}} \, dx$$

On base of Holder's inequality

$$\le \Big\| \frac{y_n - y}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}} \cdot \left[\Big\| \frac{y_n}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}}^{q^+ - 1} + \Big\| \frac{y}{x^\alpha (l-x)^\alpha} \Big\|_{L^{q(\cdot)}}^{q^+ - 1} \right] \to 0$$

as $n \to \infty$ on base of compactness Theorem 2.

By the same way, it is not difficult to show that

$$\int_0^l \big((y_n)_+^{p(x)-1} - y_+^{p(x)-1}\big)(y_n - y) \; dx \to 0 \quad \text{as} \quad n \to \infty$$

Therefore,

$$\int_0^l \frac{|y_n' - y'|^2}{\Big(|y_n'| + |y'|\Big)^{2-p}} \; dx = o(1) \quad \text{as} \quad n \to \infty.$$

Applying it Holder's inequality, we get

$$\|y_n' - y'\|_{L^{p(\cdot)}}^4 \le c_0 \left(\int_0^l \frac{|y_n - y|^2}{(|y_n| + |y|)^{2-p}} \; dx \right) \; \Big(\|y_n\|_{L^{p(\cdot)}} + \|y_n\|_{L^{p(\cdot)}} \Big)^{2-p^+} = o(1).$$

as $n \to \infty$.

This proves the PS-property of the functional $I(y)$. Now, we are ready to the application of Mountain pass theorem in order to get an existence result for the problem (1).

Mountain pass theorem. Let y be a fixed function in E. Inserting ty in place of y we see that

$$I(ty) = \int_0^l \frac{t^{p(x)}}{p(x)} |y'| \, dx - \lambda \int_0^l \frac{t^{p(x)}}{p(x)} y_+^{p(x)} \, dx - \int_0^l \frac{t^{q(x)}}{q(x)} \frac{y_+^{q(x)}}{\big(x^\alpha (l-x)^\alpha\big)^{q(x)}} \, dx$$

For sufficiently large $t > 0$ we have the estimation

$$I(ty_0) \le \frac{t^{p^+}}{p^-} \int_0^l |y'|^{p(x)} \, dx - \lambda \frac{t^{p^-}}{p^+} \int_0^l y_+^{p(x)} \, dx - \frac{t^{q^-}}{q^+} \int_0^l \frac{y_+^{q(x)}}{\big(x^\alpha (l-x)^\alpha\big)^{q(x)}} \, dx$$

Using the condition $q^- > p^+$ from this it follows $I(y) < 0$ for sufficiently large $t > 0$.

On other hand, $I(y) > 0$ for sufficiently small norm $\|y'\|_{L^{p(\cdot)}}$. Indeed, for such $y \in E$ it holds the estimates

$$I(y) \ge C_1 \int_0^l \Big(\frac{|y'|}{\|y'\|_{L^{p(\cdot)}}}\Big)^{p(x)} \|y'\|_{L^{p(\cdot)}}^{p(x)} \, dx - \int_0^l \frac{1}{q^-} \Big(\frac{y_+^{q(x)}}{N x^\alpha (l-x)^\alpha}\Big)^{q(x)} N^{q(x)} \, dx$$

$$\ge C_1 \|y'\|_{L^{p(\cdot)}}^{p^+} \int_0^l \Big(\frac{|y'|}{\|y'\|_{L^{p(\cdot)}}}\Big)^{p(x)} dx - \frac{N^{q^+}}{q^-} \int_0^l \Big(\frac{y_+^{q(x)}}{x^\alpha (l-x)^\alpha}\Big)^{q(x)} dx$$

$$\ge C_1 \|y'\|_{L^{p(\cdot)}}^{p^+} - \frac{N^{q^+}}{q^-},$$

where $N = \|\frac{y_+}{x^\alpha (l-x)^\alpha}\|_{L^{q(\cdot)}}$, using the Theorem 2, $N \le C \|y'\|_{L^{p(\cdot)}}$,

$$\ge C_1 \|y'\|_{L^{p(\cdot)}}^{p^+} - \frac{1}{q^-} \|y'\|_{L^{p(\cdot)}}^{q^-} \ge \frac{C_1}{2} \|y'\|_{L^{p(\cdot)}}^{p^+}$$

choosing $\|y'\|_{L^{p(\cdot)}} = \Big(\frac{q^- C_1}{2}\Big)^{\frac{1}{q^- - p^+}}$.

Therefore, all conditions of Mountain pass theorem is satisfied by the sphere $\|y\|_E = \rho$ with $\rho = \Big(\frac{q^- C_1}{2}\Big)^{\frac{1}{q^- - p^+}}$. Then there exists a point $y_0 \in E$ such that $I(\hat{y}) = c = \inf I(y)$ and $c = \inf \sup I(y)$ and such that $I'(\hat{y}) = 0$, i.e. for any $v \in E$ it holds

$$0 =< I'(\hat{y}), v >= \int_0^l |\hat{y}'|^{p(x)-2} \hat{y}' v' \, dx - \lambda \int_0^l \hat{y}_+^{p(x)-1} v \, dx$$

$$-\int_0^l a(x)\left(\frac{\hat{y}_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\cdot\frac{v}{x^\alpha(l-x)^\alpha}\,dx.$$

that is a solution of the problem (1). It remains to show that y_0 is positive. Insert in the preceding equality $v = y_- := (-y)_+$. Then

$$0 = \int_0^l |\hat{y}'|^{p(x)-2}\hat{y}'\hat{y}'_-\,dx - \lambda\int_0^l \hat{y}_+^{p(x)-1}\hat{y}_-\,dx$$

$$-\int_0^l a(x)\left(\frac{\hat{y}_+}{x^\alpha(l-x)^\alpha}\right)^{q(x)-1}\cdot\frac{\hat{y}_-}{x^\alpha(l-x)^\alpha}\,dx.$$

$$= \int_0^l |\hat{y}'_-|^{p(x)}\,dx$$

Therefore, $\hat{y}_- = 0$, i.e $\hat{y}$=0, then $\hat{y}$ is a positive solution of the problem (1).

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