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Bushell-Okrasiaski type inequality for pseudo-integrals

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Abstract. In this paper, we prove Bushell-Okrasiaski inequality at decreasing case for two classes of pseudo-integrals. One of them, classes with pseudo-integrals where pseudo-operations are defined via a monotone and continuous generator function. The other one concerns the pseudo-integrals based on a semiring with an idempotent addition and a pseudo-multiplication generator.

Key Words and Phrases: B-O type inequality, Convolution type inequality, Pseudo-addition, Pseudo-multiplication, Pseudo-integral

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1. Introduction

Not long ago, H. Román-Flores et al. analyzed an interesting type of geometric inequalities for the Sugeno integrals with some applications to convex geometry in [12]. More precisely, a Prékopa-Leindler type inequality for fuzzy integrals was proven, and subsequently used for the characterization of some convexity properties of fuzzy measures.

In this paper, we use Pseudo-analysis for the generalization of the classical analysis, where instead of the field of the numbers a semiring is defined on a real interval $[a, b] \subset [-\infty, \infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Thus it would be an interesting topic to generalize an inequality from the classical analysis as special cases. We prove generalizations of the Bushell-Okrasinski's type inequality for pseudo-integrals.

The classical Bushell-Okrasinski [3] is a convolution type inequality. More precisely,

$$0^x (x-t)^{s-1} g(t)^s dt \le \Big(\int_0^x g(t) \Big)^s, \qquad 0 \le x \le b, \tag{1.1}$$

holds for a continuous and increasing function $g : [0,1] \to [0,\infty)$ and $s \geq 1$, $b \leq 1$. This inequality was used by Bushell and Okrasinski [3] in the study of solutions of Volterra integral equations (see also [6]). Later on Walter and Weckesser [16] study some extensions of (1.1) and finally, after the change of variable $t = xs$, Malamud [5] analyze the B-O inequality (1.1) in the following new form:

$$s\int_0^1 (1-t)^{s-1} g(t)^s dt \leq \Big(\int_0^x g(t)\Big)^s.$$

H. Román-Flores et al [11] proved Bushell-Okrasinski type inequality for the Sugeno integrals as the following way:

Theorem 1.1. (Fuzzy B-O inequality). Let $g : [0,1] \to [0,\infty)$ be a continuous and decreasing function. Then

$$s ⨍_0^1 (1-t)^{s-1} g(t)^s dt \geq \Big(⨍_0^1 g(t)dt\Big)^s,$$

holds for all $s \geq 2$.

2. Preliminaries

2.1. Pseudo-integrals

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing(with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a,b]$ is Pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and with a unit element $1 \in [a,b]$, i.e., for each $x \in [a,b]$, $1 \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b], \oplus, \odot)$ is a semiring ([2, 4, 9, 15]). In this paper we will consider semirings with following continuous operations:

Case I. The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II. The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$ (additive generator of $\oplus$), i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \qquad \textit{and} \qquad x \odot y = g^{-1}(g(x).g(y)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators.Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.
If the generator g is increasing (respectively decreasing), the operation $\oplus$ induce the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \preceq g(y)$.
Case III. Both operation are idempotent. We have

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation *sup* induces a usual order ($x \prec y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order($x \preceq y$ if and only if $\inf(x, y) = y$).

2.2. Explicit forms of special Pseudo-integrals

We shall consider the semiring $([a, b], \oplus, \odot)$ for three (with completely different behaviour) cases, namely I(a), II, and III(a). Observe that the cases I(b) and III(b) are linked to the cases I(a) and III(a) by duality. First case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, case then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$X^{\oplus} f \odot dm = g^{-1}\left(\int_X (gof) d(gom)\right), \tag{2.1}$$

Where the integral applied on the right side is the standard Lebesgue integral. In spacial case, when $X = [c,d]$, $\mathcal{A} = \mathcal{B}(X)$ and $m = g^{-1}o\lambda$, λ the standard Lebesgue measure on $[c,d]$, then we use notation

$$\int_{[c,d]}^{\oplus} f(x)dx = \int_{X}^{\oplus} f \odot dm.$$

By (2.1)

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\big(\int_{c}^{d} g(f(x))dx\big),$$

i.e., we have recovered the g-integral (see[8, 9]).

Second case is when the semiring is of the form $([a,b], \sup, \odot)$, case I(a) and III(a). We will consider complete sup-measure m only and $\mathcal{A} = 2^x$, i.e., for any system $(A_i)_i \in I$ of measurable sets,

$$m(\underset{i\in I}{\cup} A_i) = \sup_{i\in I} m(A_i).$$

Recall that if X is countable (especially, if X is finite) then any σ-sup-measure m is complete and, moreover, $m(A) = \sup_{x\in A}\psi(X)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a,b]$ is given by

$$\int_{X}^{\oplus} f \odot dm = \sup_{x\in X}(f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m.

Theorem 2.1. Let m be a sup-measure on $([0,\infty], \mathfrak{B}([0,\infty]))$, where $\mathfrak{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A) = esssup_{\mu}(\psi(x)|x \in A)$, where $\psi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure on $([0,\infty[, \mathfrak{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in]0,\infty[$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.

For any continuous function $f : [0,\infty] \to [0,\infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [7].

Theorem 2.2. Let $([0,\infty], \sup, \odot)$ be a semiring with $\odot$ generated by some increasing generator g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a,b]$. Let m be the same as in Theorem 2.1. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in]0,\infty[$, such that for every continuous function $f : [0,\infty] \to [0,\infty]$

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1}\big(\int g^\lambda(f(x))dx\big).$$

Now we recall generalization of the Jensen inequality for pseudo-integral that proved by E. Pap et al. on [10].

Theorem 2.3. Let $\Phi : [a, b] \to [a, b]$ be a convex and nondecreasing function. If a generator $g : [a, b] \to [a, b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then for any measurable function $f : [0, 1] \to [a, b]$ we have

$$\Phi\Big(\int_{[0,1]}^{\oplus} f(x)dx\Big) \leq \int_{[0,1]}^{\oplus} \Phi(f(x))dx.$$

Theorem 2.4. Let $\Phi : [a, b] \to [a, b]$ be a convex and nondecreasing function, and the pseudo-multiplication $\odot$ is represented by a convex and increasing generator g. Let m be the same as in Theorem 2.1. Then for any continuous function $f : [0, 1] \to [a, b]$ we have

$$\Phi\Big(\int_{[0,1]}^{\sup} f \odot dm\Big) \leq \int_{[0,1]}^{\sup} \Phi(f) \odot dm.$$

Theorem 2.5. Let $u, v : [0, 1] \to [a, b]$ be two measurable functions and let a generator $g : [a, b] \to [0, \infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone functions, then the inequality

$$\int_{[0,1]}^{\oplus} (u \odot v)dx \geq \Big(\int_{[0,1]}^{\oplus} udx\Big) \odot \Big(\int_{[0,1]}^{\oplus} vdx\Big),$$

holds and the reserve inequality also holds whenever u and v are countermonotone functions.

3. Main results

In this section, we prove Bushell-Okrasiaski inequality for pseudo-integrals.

Theorem 3.1. (Pseudo Bushell-Okrasiaski inequality) Let $f : [0, 1] \to]a, b[$ be a continuous and decreasing function. If a generator $g :]a, b[\to]a, b[$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then

$$\int_{[0,1]}^{\oplus} (1-t)^{s-1} \odot f^s(t)dt \geq \frac{1}{s} \odot \Big(\int_{[0,1]}^{\oplus} f(t)dt\Big)^s,$$

holds for all $s \geq 2$.

Proof. By the definition of pseudo-integral and pseudo-operations we have

$$\begin{aligned}\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt &= g^{-1}\Big(\int_0^1 g\big[(1-t)^{s-1}\odot f^s(t)\big]dt\Big)\\ &= g^{-1}\Big(\int_0^1 g\big[g^{-1}(g((1-t)^{s-1})g(f^s(t))\big]dt\Big)\\ &= g^{-1}\Big(\int_0^1 g((1-t)^{s-1})g(f^s(t))dt\Big).\end{aligned}$$

By classic Chebyshev's integral inequality ([14]), we have;

$$\begin{aligned}g^{-1}\Big(\int_0^1 g((1-t)^{s-1})g(f^s(t))dt\Big) &\geq g^{-1}\Big[\Big(\int_0^1 g((1-t)^{s-1})dt\Big(\int_0^1 g(f^s(t))dt\Big)\Big]\\ &= g^{-1}\Big[gg^{-1}\Big(\int_0^1 g((1-t)^{s-1})dt\Big)gg^{-1}\Big(\int_0^1 g(f^s(t))dt\Big)\Big]\\ &= g^{-1}\Big[g\Big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\Big)g\Big(\int_{[0,1]}^{\oplus}f^s(t)dt\Big)\Big]\\ &= \Big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\Big)\odot\Big(\int_{[0,1]}^{\oplus}f^s(t)dt\Big).\end{aligned}$$

By using the Theorem 2.3,

$$[0,1]^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \Big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\Big)\odot\Big(\int_{[0,1]}^{\oplus}f(t)dt\Big)^s, \tag{3.1}$$

in the other hand by using the classic Jensen inequality ([13]), we can show that

$$\begin{aligned}\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt &= g^{-1}\Big(\int_0^1 g((1-t)^{s-1})dt\Big)\\ &\geq g^{-1}\Big(g\int_0^1(1-t)^{s-1}dt\Big)\end{aligned}$$

$$= \int_0^1 (1-t)^{s-1}dt = \frac{1}{s} \tag{3.2}$$

so by (3.1) and (3.2) we obtain that:

$$\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \frac{1}{s}\odot\Big(\int_{[0,1]}^{\oplus}f(t)dt\Big)^s.$$

Thereby, the theorem is proved.◀

Example 3.2. Let $g(x)=e^x$. The corresponding pseudo-operations are $x\oplus y=\ln(e^x+e^y)$ and $x\odot y=x+y$, the Theorem 3.1 reduces on the following inequality,

$$\ln\Big(\int_0^1 e^{(1-t)^{s-1}+f^s(t)}dt\Big)\geq\frac{1}{s}+\Big(\ln\big(\int_0^1 e^{f(t)}dt\big)\Big)^s.$$

In the sequel, we generalize the Bushell-Okrasiaski inequality by the semiring $([0,1], max, \odot)$, where $\odot$ is generated.

Theorem 3.3. (Pseudo Bushell-Okrasiaski inequality) Let $f:[0,1]\to]a,b[$ be a continuous and decreasing function, and $\odot$ is represented by a convex and increasing multiplication generator g and m be the same as in Theorem 2.1, then

$$\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot f^s(t)\odot dm \geq \frac{1}{s}\odot\big(\int_{[0,1]}^{\sup} f(t)dt\big)^s,$$

holds for all $s\geq 2$.

Proof. By Theorem 2.2 we have:

$$\begin{aligned}\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot f^s(t)\odot dm &= \lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda}(1-t)^{s-1}\odot f^s(t)\odot dm_\lambda\\ &= \lim_{\lambda\to\infty}(g^\lambda)^{-1}\big(\int_0^1 g^\lambda((1-t)^{s-1}\odot f^s(t))dt\big).\end{aligned}$$

Using the Theorem 2.5 so we have

$$\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot f^s(t)\odot dm \geq \lim_{\lambda\to\infty}\big[(g^\lambda)^{-1}\big(\int_0^1 g^\lambda((1-t)^{s-1})dt\big)\odot(g^\lambda)^{-1}\big(\int_0^1 g^\lambda(f^s(t))dt\big)\big]$$

$$= \big[\lim_{\lambda\to\infty}(g^\lambda)^{-1}\int_0^1 g^\lambda((1-t)^{s-1})dt\big)\odot\big(\lim_{\lambda\to\infty}(g^\lambda)^{-1}\int_0^1 g^\lambda((f^s(t))dt\big]$$

$$= \big(\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot dm\big)\odot\big(\int_{[0,1]}^{\sup} f^s(t)\odot dm\big).$$

Applying the Theorem 2.4, we obtain that:

$$\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot f^s(t)\odot dm \geq \big(\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot dm\big)\odot\big(\int_{[0,1]}^{\sup} f(t)\odot dm\big)^s. \tag{3.3}$$

Also we have:

$$\begin{aligned}\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot dm &= \lim_{\lambda\to\infty}\big(\int_{[0,1]}^{\oplus_\lambda}(1-t)^{s-1}\odot dm_\lambda\big)\\ &= \lim_{\lambda\to\infty}(g^\lambda)^{-1}\big(\int_0^1 g^\lambda((1-t)^{s-1})dt\big)\\ &\geq \lim_{\lambda\to\infty}(g^\lambda)^{-1}\big(g^\lambda\int_0^1((1-t)^{s-1})dt\big)\end{aligned}$$

$$= \lim_{\lambda\to\infty}\int_0^1((1-t)^{s-1})dt = \frac{1}{s}. \tag{3.4}$$

From (3.3) and (3.4) we have $\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot f^s(t)\odot dm \geq \frac{1}{s}\odot\big(\int_{[0,1]}^{\sup} f(t)dt\big)^s$. ◄

Example 3.4. Let $g^\lambda = e^{\lambda x}$ and $\psi(x)$ be from Theorem 2.1, then

$$x \odot_\lambda y = x + y \quad \text{and} \quad \lim_{\lambda \to \infty} \big(\frac{1}{\lambda} \ln(e^{\lambda x} + e^{\lambda y})\big) = max(x, y).$$

Therefore B-O type inequality from Theorem 3.3 reduces on

$$\underset{x \in [0,1]}{sup} \big[\big((1-x)^{s-1} + f^s(x)\big) + \psi(x)\big] \geq \frac{1}{s} + \big[\underset{x \in [0,1]}{sup} \big(f(x) + \psi(x)\big)\big]^s.$$

Note that third important case $\oplus$ =max and $\odot$ =min has been studied in [11] and the Pseudo-integrals in such a case yields the Sugeno integral.

4. Conclusion

We have proved the B-o integral type inequality for the pseudo-integral for two characteristic cases: generated and max-plus. For further investigation we continue to explore other integral inequalities for fuzzy integrals.
Open problem: Dose B-O type inequalities hold for the Chaquet integral?

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