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On the Boundedness of Commutators Dunkl-type Maximal Operator in the Dunkl-type Morrey Spaces

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Abstract. In this paper we consider the generalized shift operator, associated with the Dunkl operator and we investigate maximal commutators associated with the generalized shift operator. The boundedness of the Dunkl-type maximal commutator $M_{b,\alpha}$ from the Dunkl-type Morrey space $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{p,\lambda,\alpha}(\mathbb{R})$ for all $1 < p < \infty$ when $b \in BMO_\alpha(\mathbb{R})$ are proved.

Key Words and Phrases: commutator, generalized shift operator, Dunkl-type maximal function, Dunkl-type B–Morrey space, BMO_α space.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. On the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. In the works [2, 11, 21, 29] the maximal operator associated with the Dunkl operator on $\mathbb{R}$ were studied. Let b be a locally integrable function on $\mathbb{R}^n$ and T be a Calderon-Zygmund operator. The commutator is defined for smooth functions f as

$$[b, T]f = bT(f) - T(bf).$$

Coifman, Rochberg and Weiss [8] stated that $[b, T]$ is a bounded operator on $L_p(\mathbb{R}^n), 1 < p < \infty$, when b is a BMO function. Chanillo [7] proved that the commutators of the Riesz potentials characterize the function space BMO. In [10] proved that $b \in BMO(\mathbb{R}^n)$ if and only if the maximal commutator M_b is bounded from the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$. In this work, we study the maximal commutator (Dunkl-type Dunkl-type maximal commutator) associated with the Dunkl operator on $\mathbb{R}$. We obtain the necessary and sufficient conditions for the boundedness of the Dunkl-type maximal commutator.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x of radius r.

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Let $f \in L_1^{loc}(\mathbb{R}^n)$. The maximal operator M and the Riesz potential I^β are defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy.$$

The operator M play important role in real and harmonic analysis (see, for example [30]).

In the theory of partial differential equations Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role. They were introduced by C. Morrey in 1938 [26] and defined as follows: *For* $0 \le \lambda \le n$, $1 \le p < \infty$, $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ *if* $f \in L_p^{loc}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$, if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak L_p-space.

F. Chiarenza and M. Frasca [9] studied the boundedness of the maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}$. Their results can be summarized as follows:

Theorem A. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p < \infty$.

1) If $1 < p < \infty$, *then* M *is bounded from* $\mathcal{M}_{p,\lambda}$ *to* $\mathcal{M}_{p,\lambda}$.

2) If $p = 1$, *then* M *is bounded from* $\mathcal{M}_{1,\lambda}$ *to* $W\mathcal{M}_{1,\lambda}$.

2. Definitions, notation and preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(\mathbb{R}, d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_\infty} = ess\sup_{x\in\mathbb{R}} |f(x)| \quad \text{if} \quad p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r \left(\mu_\alpha \{x \in \mathbb{R} \ : \ |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha} \text{ for all } f \in L_{p,\alpha}(\mathbb{R}).$$

Let $B(x,t) = \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-t\}, |x|+t[\ \}$ and $B_t \equiv B(0,t) =]-t,t[$, $t > 0$. Then

$$\mu_\alpha B_t = b_\alpha \, t^{2\alpha+2},$$

where $b_\alpha = \left[2^{\alpha+1}(\alpha+1)\,\Gamma(\alpha+1)\right]^{-1}$.

Let $M_\alpha^\sharp$ be the Dunkl-type sharp maximal function defined by

$$M_\gamma^\sharp f(x) = \sup_{r>0} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)| \, d\mu_\alpha(y),$$

where $f_{B_r}(x) = \frac{1}{\mu_\alpha B_r} \int_{B_r} \tau_x f(y) \, d\mu_\alpha(y)$.

We denote by $BMO_\alpha(\mathbb{R})$ (Dunkl-type BMO space) the set of locally integrable functions f with finite norm

$$\|f\|_{BMO_\alpha} = \sup_{r>0,\ x\in\mathbb{R}} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)| \, d\mu_\alpha(y) < \infty$$

or

$$\|f\|_{BMO_\alpha} = \inf_C \sup_{r>0,\ x\in\mathbb{R}} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - C| \, d\mu_\alpha(y).$$

$BMO\,(X,\nu)$ space is defined as the space of locally integrable functions f with the following finite norm

$$\|f\|_* = \sup_{t>0,\ x\in X} \nu(B(x,t))^{-1} \int\limits_{B(x,t)} |f(y) - f_{B(x,t)}| d\nu(y) < \infty,$$

where $f_{B(x,t)} = \nu(B(x,t))^{-1} \int_{B(x,t)} f(y) d\nu(y)$.

Theorem 1. *[24] Let $f \in BMO\,(X,\nu)$ and ν doubling measure. For any $r > 0$, then*

$$\nu\left|\left\{y \in B(x,t) \ : \ |f(x) - f_{B(x,t)}| > r\right\}\right| \le C\nu(B(x,t)) e^{\frac{-cr}{\|f\|_*}},$$

where the constants C and c are independent of f and r.

It is clear that $BMO(X,\nu) = BMO_p(X,\nu)$ if the John-Nirenberg inequality holds. The following theorem holds.

Theorem 2. *1) Let $f \in L_{1,\alpha}^{loc}(\mathbb{R})$. If*

$$\sup_{t>0,\ x\in\mathbb{R}} \left(\mu_\alpha(B_t)^{-1} \int\limits_{B_t} |\tau_y f(x) - f_{B_t}|^p d\mu_\alpha(y) \right)^{1/p} = \|f\|_{BMO_{p,\alpha}} < \infty$$

then for any $1 < p < \infty$,

$$\|f\|_{BMO_\alpha} \le \|f\|_{BMO_{p,\alpha}} \le A_p \|f\|_{BMO_\alpha},$$

where the constant A_p depends only on p.

2) Let $f \in BMO_\alpha(\mathbb{R})$. Then, there is a constant $C > 0$ such that

$$|f_{B_r} - f_{B_t}| \le C\|f\|_{BMO_\alpha} \ln \frac{t}{r}, \ \ 0 < 2r < t,$$

where C is independent of f, x, r and t.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (X, ρ, ν). By this we mean a topological space $X = \mathbb{R}$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu B(x, 2r) \le C_0 \nu B(x, r) \tag{1}$$

with a constant C_0 being independent of x and $r > 0$. Here $B(x, r) = \{y \in X : \rho(x, y) < r\}$, $\rho(x, y) = |x - y|$. Let (X, ρ, ν) be a space of homogeneous type, where $X = \mathbb{R}$, $d\nu(x) = d\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (1).

Since

$$\int\limits_{B_r} \tau_y |f(x)| d\mu_\alpha(y) \approx \int\limits_{B(x,r)} |f(y)| \ d\nu(y)$$

we get

$$\sup_{t>0,\ x\in\mathbb{R}} \mu(B_t)^{-1} \int\limits_{B_t} |\tau_y f(x) - C| d\mu_\alpha(y)$$

$$\begin{aligned} &\approx \sup_{t>0,\ x\in X} \nu(B(x,t))^{-1} \int\limits_{B(x,t)} |f(y) - C| \ d\nu(y) \\ &= \|f\|_{BMO(X,\nu)} \approx \|f\|_{BMO_p(X,\nu)} \approx \|f\|_{BMO_{p,\alpha}}. \end{aligned}$$

Similarly, we can prove

$$|f_{B_r} - f_{B_t}| \le C\|f\|_{BMO_\alpha} \ln \frac{t}{r}, \ \ 0 < 2r < t.$$

For all $x,\, y,\, z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1-\sigma_{x,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } \ x,\, y \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{otherwise} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } \ |z| \in A_{x,y}, \\ 0, & \text{otherwise}, \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [||x|-|y||, |x|+|y|]$.

Properties 1. *(see Rösler [32]) The signed kernel W_α is even with respect to all variables and satisfies the following properties*

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\; d\mu_\alpha(z) \le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x,y,z)\, d\mu_\alpha(z) & \text{if } \ x,\, y \in \mathbb{R}\setminus\{0\}, \\ d\delta_x(z) & \text{if } \ y=0, \\ d\delta_y(z) & \text{if } \ x=0. \end{cases}$$

Definition 1. *For $x,\, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)\, d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see [32])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left((x,y)_\theta\right)\, h_1(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta$$

$$+c_\alpha \int_0^\pi f_o\left((x,y)_\theta\right)\, h_2(x,y,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $(x,y)_\theta = \sqrt{x^2+y^2-2|xy|\cos\theta}$, $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with

$$c_\alpha \equiv \left(\int_0^\pi (\sin\theta)^{2\alpha}\, d\theta\right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$

$$h_1(x,y,\theta) = 1 - sgn(xy)\cos\theta$$

and

$$h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-sgn(xy)\cos\theta]}{(x,y)_\theta}, & \text{if } \ xy \neq 0, \\ 0, & \text{if } \ xy = 0. \end{cases}$$

Using the change of variable $z = (x,y)_\theta$, we have also (see [4])

$$\tau_x f(y) = c_\alpha \int_0^\pi \{f\left((x,y)_\theta\right) + f\left(-(x,y)_\theta\right) + \frac{x+y}{(x,y)_\theta}\left[f\left((x,y)_\theta\right) - f\left(-(x,y)_\theta\right)\right]\}(1-\cos\theta)(\sin\theta)^{2\alpha}\,d\theta.$$

Now we define the Dunkl-type fractional maximal function by

$$M_{\beta,\alpha}f(x) = \sup_{r>0}\big(\mu_\alpha B_r\big)^{\frac{\beta}{2\alpha+2}-1}\int_{B_r}\tau_x|f|(y)\,d\mu_\alpha(y), \quad 0 \le \beta < 2\alpha+2.$$

If $\beta = 0$, then $M_\alpha \equiv M_{0,\alpha}$ is the Hardy-Littlewood maximal operator associated with the Dunkl operator (see [2, 11, 21, 29]).

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ to be bounded from the spaces $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$, $1 < p < q < \infty$ and from the spaces $L_{1,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\alpha}(\mathbb{R})$, $1 < q < \infty$.

Theorem 3. *([12]) Let $0 < \beta < 2\alpha + 2$ and $1 \le p \le \frac{2\alpha+2}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_{\beta,\alpha}$ from $L_{p,\alpha}(\mathbb{R})$ to $L_{q,\alpha}(\mathbb{R})$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_{\beta,\alpha}$ from $L_{1,\alpha}(\mathbb{R})$ to $WL_{q,\alpha}(\mathbb{R})$.

3) If $p = \frac{2\alpha+2}{\beta}$, then $M_{\beta,\alpha}$ is bounded from $L_{p,\alpha}(\mathbb{R})$ to $L_\infty(\mathbb{R})$.

Theorem 4. *[11]*

1. If $f \in L_{1,\omega,\alpha}(\mathbb{R})$ and $\omega \in A_{1,\alpha}(\mathbb{R})$, then $M_\alpha f \in WL_{1,\omega,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{WL_{1,\omega,\alpha}} \le C_{1,\alpha}\|f\|_{L_{1,\omega,\alpha}},$$

where $C_{1,\alpha}$ depends only on α.

2. If $f \in L_{p,\omega,\alpha}(\mathbb{R})$, $1 < p < \infty$ and $\omega \in A_{p,\alpha}(\mathbb{R})$, then $M_\alpha f \in L_{p,\omega,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{L_{p,\omega,\alpha}} \le C_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}},$$

where $C_{p,\alpha}$ depends only on p, α.

Definition 2. *Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha + 2$. We denote by $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ Dunkl-type Morrey space ($\equiv$ D-Morrey space) as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\lambda,\alpha}} = \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int_{B_t} [\tau_x |f(y)|\,]^p d\mu_\alpha(y) \right)^{1/p}.$$

Theorem 5. *[13]*

1. If $f \in \mathcal{M}_{1,\lambda,\alpha}(\mathbb{R})$, $0 \le \lambda < 2\alpha + 2$, then $M_\alpha f \in W\mathcal{M}_{1,\lambda,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{W\mathcal{M}_{1,\lambda,\alpha}} \le C_{1,\lambda,\alpha} \|f\|_{\mathcal{M}_{1,\lambda,\alpha}},$$

where $C_{1,\lambda,\alpha}$ depends only on λ,α and n.

2. If $f \in \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p < \infty$,$0 \le \lambda < 2\alpha + 2$, then $M_\alpha f \in \mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{\mathcal{M}_{p,\lambda,\alpha}} \le C_{p,\lambda,\alpha} \|f\|_{\mathcal{M}_{p,\lambda,\alpha}},$$

where $C_{p,\lambda,\alpha}$ depends only on p,λ,α and n.

Theorem 6. *[13] Let $0 < \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2 - \beta$ and $1 \le p < \frac{2\alpha+2-\lambda}{\beta}$.*

1) If $1 < p < \frac{2\alpha+2-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness $M_{\beta,\alpha}$ from $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R})$ to $\mathcal{M}_{q,\lambda,\alpha}(\mathbb{R})$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness $M_{\beta,\alpha}$ from $\mathcal{M}_{1,\lambda,\alpha}(\mathbb{R})$ to $W\mathcal{M}_{q,\lambda,\alpha}(\mathbb{R})$.

For $1 \le p, \theta \le \infty$, $0 \le \lambda \le 2\alpha + 2$ and $0 < s < 1$, the Besov-Morrey space for the Dunkl operators on $\mathbb{R}$ (Besov-Morrey-Dunkl space) $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ consists of all functions f in $L_{p,\lambda,\alpha}(\mathbb{R})$ so that

$$\|f\|_{B^s_{p\theta,\lambda,\alpha}} = \|f\|_{L_{p,\lambda,\alpha}} + \left(\int_{\mathbb{R}} \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{L_{p,\lambda,\alpha}}}{|x|^{2\alpha+2+s\theta}} d\mu_\alpha(x) \right)^{1/\theta} < \infty. \tag{2}$$

Besov spaces in the setting of the Dunkl operators studied by C. Abdelkefi and M. Sifi [3, 4], R. Bouguila, M.N. Lazhari and M. Assal [5], L. Kamoun [19], Y.Y. Mammadov [22] and V.S. Guliyev, Y.Y. Mammadov [12]. In the following theorem we prove the boundedness of the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ in the Dunkl-type Besov spaces.

Theorem 7. *([12]) For $1 < p < q < \infty$, $0 \le \lambda \le 2\alpha + 2$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2-\lambda}$, $1 \le \theta \le \infty$ and $0 < s < 1$ the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ is bounded from $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ to $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$. More precisely, there is a constant $C > 0$ such that*

$$\|M_{\beta,\alpha} f\|_{B^s_{q\theta,\lambda,\alpha}} \le C \|f\|_{B^s_{p\theta,\lambda,\alpha}}$$

hold for all $f \in B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$.

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) \tag{3}$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem :

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R}$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel [6, 27, 33] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where j_α is the normalized Bessel function of the first kind and order α [34], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\,\alpha(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

We can write for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ (see Rösler [32], p. 295)

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}(1-t)\, e^{i\lambda x t}\, dt.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

The Dunkl transform $\mathcal{F}_\alpha$ of a function $f \in L_{1,\alpha}(\mathbb{R})$, is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x)\, f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix| \leq 1$ for every $x \in \mathbb{R}$ [32], p. 295.

Note that $\mathcal{F}_{-1/2}$ agrees with the classical Fourier transform $\mathcal{F}$, given by:

$$\mathcal{F} f(\lambda) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\lambda x}\, f(x) dx, \quad \lambda \in \mathbb{R}.$$

Proposition 1. *(see Soltani [28])*

(i) If f is an even positive continuous function, then $\tau_x f$ is positive.

(ii) For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}. \tag{4}$$

(iii) For all x, $\lambda \in \mathbb{R}$ and $f \in L_{1,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_\alpha(i\lambda x)\, \mathcal{F}_\alpha f(\lambda).$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y)\, g(y)\, d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The generalized convolution $*_\alpha$ is associative and commutative [32]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

Proposition 2. *(see Soltani [28])*

(i) If f is an even positive function and g a positive function with compact support, then $f *_\alpha g$ is positive.

(ii) Assume that p, q, $r \in [1, +\infty[$ satisfying $1/p + 1/q = 1 + 1/r$ (the Young condition). Then the map $(f, g) \mapsto f *_\alpha g$, defined on $\mathcal{E}_c \times \mathcal{E}_c$, extends to a continuous map from $L_{p,\alpha}(\mathbb{R}) \times L_{q,\alpha}(\mathbb{R})$ to $L_{r,\alpha}(\mathbb{R})$, and we have

$$\|f *_\alpha g\|_{r,\alpha} \le 4\|f\|_{p,\alpha}\, \|g\|_{q,\alpha}.$$

(ii) For all $f \in L_{1,\alpha}(\mathbb{R})$ and $g \in L_{2,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha\, (f *_\alpha g) = (\mathcal{F}_\alpha f)\; (\mathcal{F}_\alpha g)\,.$$

3. Maximal commutators in $L_{p,\lambda,\alpha}(\mathbb{R})$

The commutator generated by the Dunkl-type maximal operator M_α, for given a measurable function b is formally defined by

$$[M_\alpha, b]f = M_\alpha(bf) - bM_\alpha(f)$$

and for given a measurable function b, the Dunkl-type maximal commutator is defined by

$$M_{b,\alpha}(f)(x) := \sup_{r>0} \mu_\alpha(B_r)^{-1} \int\limits_{B_r} \tau_y |(b(x) - b(y)) f(x)| d\mu_\alpha(y),$$

for all $x \in \mathbb{R}$.

Lemma 1. *Let $1 < s < \infty$, $b \in BMO(\mathbb{R})$. Then, there exists $C > 0$ such that for all $x \in \mathbb{R}$*

$$M^\sharp_\alpha(M_{b,\alpha} f)(x) \le C\|b\|_{BMO_\alpha} \left((M_\alpha(M_\alpha f)^s)^{\frac{1}{s}}\,(x) + M_\alpha\,(M_\alpha|f|^s)^{\frac{1}{s}}\,(x) \right)$$

holds.

Proof. From the boundedness of the Dunkl-type maximal operator M_α and the pointwise inequality we have

$$M^\sharp_\alpha(M_{b,\alpha} f)(x) \le 2M_\alpha(M_{b,\alpha} f)(x),\ x \in \mathbb{R}.$$

Since $M_{b,\alpha}(f)(y)=\sup\limits_{t>0}M_{b,t,\alpha}(f)(y)$ then, we get

$$\begin{aligned}
M_{b,t,\alpha}(f)(y)&=(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y(|b(y)-b(z)||f(z)|)d\mu_\alpha(z)\\
&\le (\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y(|b(z)-b_{B_t}||f(z)|)d\mu_\alpha(z)\\
&+ |b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z)\\
&\le (\mu_\alpha B_t)^{-1}\left(\int\limits_{B_t}[\tau_y|b(z)-b_{B_t}|]^{s'}d\mu_\alpha(z)\right)^{\frac{1}{s'}}\left(\int\limits_{B_t}[\tau_y|f(z)|]^{s}d\mu_\alpha(z)\right)^{\frac{1}{s}}\\
&+ |b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z)\\
&\le C\|b\|_{BMO_\alpha}\left(M_\alpha|f|^s\right)^{\frac{1}{s}}(y)+|b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z).
\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}
&(\mu_\alpha B_r)^{-1}\int\limits_{B_r}\tau_x\left[|b(y)-b_{B_t}|(\mu_\alpha B_t)^{-1}\left(\int\limits_{B_t}\tau_y|f(z)|d\mu_\alpha(z)\right)\right]d\mu_\alpha(y)\\
\le\ &(\mu_\alpha B_r)^{-1}\int\limits_{B_r}\tau_x\left[|b(y)-b_{B_t}|M_\alpha f(y)\right]d\mu_\alpha(y)\\
\le\ &(\mu_\alpha B_r)^{-1}\left(\int\limits_{B_r}[\tau_x|b(y)-b_{B_r}|]^{s'}d\mu_\alpha(y)\right)^{\frac{1}{s'}}\left(\int\limits_{B_r}\tau_x(M_\alpha f)^s(y)d\mu_\alpha(y)\right)^{\frac{1}{s}}\\
+\ &(\mu_\alpha B_r)^{-1}\int\limits_{B_r}\tau_x\left[|b_{B_t}-b_{B_r}|M_\alpha f(y)\right]d\mu_\alpha(y)\\
\le\ &C\|b\|_{BMO_\alpha}\left(M_\alpha(M_\alpha f)^s\right)^{\frac{1}{s}}(x).
\end{aligned}$$

Therefore

$$M_\alpha(M_{b,\alpha}f)(x)=\sup_{r>0}(\mu_\alpha B_r)^{-1}\int_{B_r}\tau_x(M_{b,\alpha}f)(y)d\mu_\alpha(y)$$

$$\le C\|b\|_{BMO_\alpha}\left((M_\alpha(M_\alpha f)^s)^{\frac{1}{s}}(x)+\sup_{r>0}(\mu_\alpha B_r)^{-1}\int_{B_r}\tau_x\left(M_\alpha|f|^s\right)^{\frac{1}{s}}(y)d\mu_\alpha(y)\right)$$

$$\le C\|b\|_{BMO_\alpha}\left((M_\alpha(M_\alpha f)^s)^{\frac{1}{s}}(x)+M_\alpha\left(M_\alpha|f|^s\right)^{\frac{1}{s}}(x)\right). \qquad (5)$$

Proposition 3. *[18] For all weights ω and all nonnegative function f satisfying $\nu(\{x\in X: f(x)>\beta\})<\infty$ for all $\beta>0$, there exists a positive constant C such that*

1. If $\nu(X)=\infty$, then

$$\int_X f(y)g(y)d\nu(y)\le C\int_X M^\sharp f(y)Mg(y)d\nu(y).$$

2. If $\nu(X)<\infty$, then

$$\int_X f(y)g(y)d\nu(y)\le C\int_X M^\sharp f(y)Mg(y)d\nu(y)+Cg(X)\nu_X(f),$$

where g is nonnegative function, $g(X)=\int_X g(x)d\nu(x)$, $\nu_X(f)=\frac{1}{\nu(X)}\int_X f(y)d\nu(y)$.

Lemma 2. *Let $1<p<\infty$, $\omega\in A_{p,\alpha}(\mathbb{R})$. Then*

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\le C\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}$$

where a constant $C>0$ is independent of f.

Proof. Let (X,ν) be a space of homogeneous type. According to Proposition 1, we have

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\le C\sup_{\|g\|_{L_{p',\gamma}}\le 1}\left|\int_{\mathbb{R}} f(y)g(y)\omega^{\frac{1}{p}}(y)d\mu_\alpha(y)\right|$$

$$=C\sup_{\|g\|_{L_{p'}}\le 1}\left|\int_X f(y)g(y)\omega^{\frac{1}{p}}(y)d\nu(y)\right|\le C\sup_{\|g\|_{L_{p'}}\le 1}\left|\int_X M^\sharp f(y)M(g\omega^{\frac{1}{p}})(y)d\nu(y)\right|.$$

Hence

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\le C\sup_{\|g\|_{L_{p',\gamma}}\le 1}\left|\int_{\mathbb{R}} M_\alpha^\sharp f(y)M_\alpha(g\omega^{\frac{1}{p}})(y)d\mu_\alpha(y)\right|.$$

Finally by using the Hölder inequality and Theorem 4, we get

$$\|f\omega^{\frac{1}{p}}\|_{L_{p,\alpha}}\le C\sup_{\|g\|_{L_{p',\gamma}}\le 1}\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}\|\omega^{-\frac{1}{p}}M_\alpha(g\omega^{\frac{1}{p}})\|_{L_{p',\gamma}}$$

$$\le C\sup_{\|g\|_{L_{p',\gamma}}\le 1}\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}\|g\|_{L_{p',\gamma}}\le C\|\omega^{\frac{1}{p}}M_\alpha^\sharp f\|_{L_{p,\alpha}}.$$

Theorem 8. *Let $b \in BMO_\alpha(\mathbb{R})$, $1 < p < \infty$, $\omega \in A_{p,\alpha}(\mathbb{R})$. Then $M_{b,\alpha}$ is bounded on the space $L_{p,\omega,\alpha}(\mathbb{R})$.*

Proof. By using Lemma 1, Lemma 2 and Theorem 4, we have $M_{b,\alpha}$ is bounded on the space $L_{p,\omega,\gamma}(\mathbb{R})$.

Operators $M_{b,\alpha}$ and $[M_\alpha, b]$ are essentially different from each other. For example, $M_{b,\alpha}$ is a positive and sublinear operator, but $[M_\alpha, b]$ is neither positive nor sublinear. However, if b satisfy some additional conditions, then operator $M_{b,\alpha}$ is controled by $[M_\alpha, b]$.

Theorem 9. *Let $1 < p < \infty$, $0 \le \lambda < 2\alpha + 2$. Then the commutator $M_{b,\alpha}$ is bounded on $L_{p,\lambda,\alpha}(\mathbb{R})$ if and only if $b \in BMO_\alpha(\mathbb{R})$.*

Proof. Sufficiency: Let $1 < p < \infty$, $0 \le \lambda < 2\alpha + 2$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$. We have

$$\int_{B_t} \tau_y \left[M_{b,\alpha} f\right]^p(x) d\mu_\alpha(y) \le \int_{\mathbb{R}} \tau_y \left[M_{b,\alpha} f\right]^p(x) (M_\alpha \chi_{B_t}(y))^\delta d\mu_\alpha(y), \ \ x \in \mathbb{R}.$$

Taking into account the properties of $A_{p,\alpha}(\mathbb{R})$, we can easily see that $(M_\alpha \chi_{B_t})^\delta \in A_{p,\alpha}(\mathbb{R})$, for any $0 < \delta < 1$. Then by using Lemma 2 and Theorem 8 we obtain

$$\begin{aligned}
\int_{B_t} \tau_y \left[M_{b,\alpha} f\right]^p(x) d\mu_\alpha(y) &\le C \ \|b\|^p_{BMO_\alpha} \int_{\mathbb{R}} \tau_y |f(x)|^p (M_\alpha \chi_{B_r}(y))^\theta d\mu_\alpha(y) \\
&\le C \ \|b\|^p_{BMO_\alpha} \int_{B_r} \tau_y |f(x)|^p d\mu_\alpha(y) \\
&+ C \|b\|^p_{BMO_\alpha} \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \tau_y |f(x)|^p (M_\alpha \chi_{B_r}(y))^\theta d\mu_\alpha(y) \\
&\le C \ \|b\|^p_{BMO_\alpha} \int_{B_r} \tau_y |f(x)|^p d\mu_\alpha(y) \\
&+ C \|b\|^p_{BMO_\alpha} \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \tau_y |f(x)|^p \frac{r^{(2\alpha+2)\theta}}{(|y| + r)^{(2\alpha+2)\theta}} d\mu_\alpha(y) \\
&\le C \ \|b\|^p_{BMO_\alpha} \|f\|^p_{L_{p,\lambda,\alpha}} \left(r^\lambda + \sum_{j=1}^{\infty} \frac{1}{(2^j + 1)^{(2\alpha+2)\theta}} (2^{j+1} r)^\lambda \right) \\
&\le C \ r^\lambda \|b\|^p_{BMO_\alpha} \|f\|^p_{L_{p,\lambda,\alpha}}.
\end{aligned}$$

Necessity: Let $M_{b,\alpha}$ be bounded from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p < \infty$.

Obviously,

$$\|f\|_{L_{p,\lambda,\alpha}} = \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int_{B_t} \tau_y |f(x)|^p d\mu_\alpha(y) \right)^{1/p}.$$

Now we consider $f = \chi_{B_r}$. It is easy to compute that

$$\begin{aligned}
\|\chi_{B_r}\|_{L_{p,\lambda,\alpha}} &\approx \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int\limits_{B_t} \tau_y \chi_{B_r}(x) d\mu_\alpha(y) \right)^{1/p} \\
&\approx \sup_{t>0,\, x\in\mathbb{R}} \left(t^{-\lambda} \int\limits_{B_t} \chi_{B_r}(y) d\mu_\alpha(y) \right)^{1/p} \\
&\approx \sup_{B_t\subset B_r} \left(t^{-\lambda} \mu_\alpha(B_t \cap B_r) \right)^{1/p} \le r^{\frac{2\alpha+2-\lambda}{p}}.
\end{aligned}$$

Then

$$\frac{1}{(\mu_\alpha B_t)} \int_{B_t} |\tau_z b(x) - f_{B_t}|\, d\mu_\alpha(z)$$

$$\begin{aligned}
&= \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \left| \tau_z b(x) - \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \tau_z b(y) d\mu_\alpha(y) \right| d\mu_\alpha(z) \\
&\le \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \frac{1}{(\mu_\alpha B_t)} \int_{B_t} |\tau_z b(x) - \tau_z b(y)|\, d\mu_\alpha(y) d\mu_\alpha(z) \\
&\le \frac{1}{(\mu_\alpha B_t)} \int_{B_t} \frac{1}{(\mu_\alpha B_t)} \int_{B_t} |\tau_z (b(x) - b(y))|\, d\mu_\alpha(y) d\mu_\alpha(z) \\
&\le \frac{1}{(\mu_\alpha B_t)} \int_{B_t} M_{b,\alpha} \chi_{B_t}(z) d\mu_\alpha(z) \\
&\le Ct^{-2\alpha-2+\lambda} \|M_{b,\alpha}\chi_{B_t}\|_{L_{p,\lambda,\alpha}} \|\chi_{B_t}\|_{L_{p',\lambda,\alpha}} \le Ct^{\frac{2\alpha+2-\lambda}{p'} + \frac{2\alpha+2-\lambda}{p} - 2\alpha-2+\lambda} \le C.
\end{aligned}$$

Theorem 10. *Let $0 \le \lambda < 2\alpha+2$, $b \in BMO_\alpha(\mathbb{R})$. Then the commutator $M_{b,\alpha}$ is bounded from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{1,\lambda,\alpha}(\mathbb{R})$.*

Proof. Let $0 \le \lambda < 2\alpha + 2$, $f \in L_{1,\lambda,\alpha}(\mathbb{R})$. This assertion is easily obtained from $f(x) \le M_\alpha f(x)$. Finally, by using (5) and Theorem 5, we get

$$\begin{aligned}
\|M_{b,\alpha} f\|_{WL_{1,\lambda,\alpha}} &\le \|M_\alpha(M_{b,\alpha} f)\|_{WL_{1,\lambda,\alpha}} \\
&\le \|b\|_{BMO_\alpha} \| \left(M_\alpha(M_\alpha f)^s\right)^{\frac{1}{s}} + M_\alpha \left(M_\alpha |f|^s\right)^{\frac{1}{s}} \|_{WL_{1,\lambda,\alpha}} \\
&\le C\, \|b\|_{BMO_\alpha} \|f\|_{L_{1,\lambda,\alpha}}.
\end{aligned}$$

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