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Hardy-Littlewood-Stein-Weiss Inequality in the Generalized Morrey Spaces with Variable Exponent

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Abstract. We consider generalized weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$ with variable exponent $p(x)$ and a general function $\omega(x,r)$ defining the Morrey-type norm. In case of bounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the Hardy-Littlewood maximal operator and Calderon-Zygmund singular operators with standard kernel, in such spaces. We also prove a Sobolev-Adams type $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega) \to \mathcal{M}^{q(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$-theorem for the potential operators $I^{\alpha(\cdot)}$, also of variable order. In all the cases the conditions for the boundedness are given it terms of Zygmund-type integral inequalities on $\omega(x,r)$, which do not assume any assumption on monotonicity of $\omega(x,r)$ in r.
Key Words and Phrases: Maximal operator, fractional maximal operator, Riesz potential, singular integral operators, generalized Morrey space, Hardy-Littlewood-Sobolev-Morrey type estimate, BMO space.
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1. Introduction.

Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [45]). They are defined by the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$. In the theory of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ play an important role. Later, Morrey spaces found important applications to Navier-Stokes ([44], [66]) and Schrödinger ([50], [52], [53], [64], [65]) equations, elliptic problems with discontinuous coefficients ([10], [18]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the books [19] and [42].

As is known, over the last two decades there has been an increasing interest in the study of variable exponent spaces and operators with variable parameters in such spaces, we refer the readers to the surveying papers [16], [29], [38], [59], on the progress in this

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field, including topics of Harmonic Analysis and Operator Theory (see also references therein).

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [3] and [47] in the Euclidean setting and in [30] in the setting of metric measure spaces, in case of bounded sets. In [3] there was proved the boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$ and $\lambda(\cdot)$, and for potential operators, under the same log-condition and the assumptions $\inf_{x\in\Omega}\alpha(x) > 0$, $\sup_{x\in\Omega}[\lambda(x) + \alpha(x)p(x)] < n$, there was proved a Sobolev type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$– theorem. In the case of constant α, there was also proved a boundedness theorem in the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha}$, when the potential operator I^α acts from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ into BMO. In [47] the maximal operator and potential operators were considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. P. Hästö in [26] used his new "local-to-global" approach to extend the result of [3] on the maximal operator to the case of the whole space $\mathbb{R}^n$. In [30] there was proved the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces.

Generalized Morrey spaces of such a kind in the case of constant p were studied in [5], [17], [43], [46], [48], [49]. In [22] there was proved the boundedness of the maximal operator, singular integral operator and the potential operators in generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\Omega)$.

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderon-Zygmund singular operators date back to [1] and [51], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [11]; for further results in the case of constant p and λ see, e.g., [6]– [9].

We introduce the generalized variable exponent weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$. Within the frameworks of the spaces $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^\gamma}(\Omega)$, over bounded sets $\Omega \subseteq \mathbb{R}^n$ we consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy,$$

potential type operators

$$I^{\alpha(x)}f(x) = \int_\Omega |x-y|^{\alpha(x)-n} f(y)dy, \ 0 < \alpha(x) < n,$$

the fractional maximal operator

$$M^{\alpha(x)}f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int_{\widetilde{B}(x,r)} |f(y)|dy, \ 0 \le \alpha(x) < n$$

of variable order $\alpha(x)$ and Calderon-Zygmund type singular operator

$$Tf(x) = \int_\Omega K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y)\in\Omega\times\Omega : x\neq y\}$ and satisfying the estimates

$$|K(x,y)|\le C|x-y|^{-n} \text{ for all } x\neq y,$$

$$|K(x,y)-K(x,z)|\le C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}},\ \ \sigma>0,\ \text{ if } |x-y|>2|y-z|,$$

$$|K(x,y)-K(\xi,y)|\le C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}},\ \ \sigma>0,\ \text{ if } |x-y|>2|x-\xi|.$$

We find the condition on the function $\omega(x,r)$ for the boundedness of the maximal operator M and the singular integral operators T in generalized weighted Morrey space $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^{\gamma}}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$. For potential operators, under the same log-condition and the assumptions

$$\inf_{x\in\Omega}\alpha(x)>0,\ \sup_{x\in\Omega}\alpha(x)p(x)<n$$

we also find the condition on $\omega(x,r)$ for the validity of a Sobolev-Adams type $\mathcal{M}^{p(\cdot),\omega,|x-x_0|^{\gamma}}(\Omega)\to\mathcal{M}^{q(\cdot),\omega,,|x-x_0|^{\mu}}(\Omega)$-theorem, which recovers the known result for the case of the classical weighted Morrey spaces with variable exponents, when $\omega(x,r)=r^{\frac{\lambda(x)-n}{p(x)}}$ and $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n-\lambda(x)}$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 3 we introduce the generalized Morrey spaces with variable exponents and recall some facts about generalized Morrey spaces with constant p. In Section 4 we deal with the maximal operator, while potential operators are studied in Section 5. In Section 6 we treat Calderon-Zygmund singular operators.

The main results are given in Theorems 17, 18, 19, 20, 24, 25. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\omega(x,r)$.

N o t a t i o n :
$\mathbb{R}^n$ is the n-dimensional Euclidean space,
$\Omega\subseteq\mathbb{R}^n$ is an open set, $\ell=\text{diam }\Omega$;
$0\in\overline{\Omega}$;
$\chi_E(x)$ is the characteristic function of a set $E\subseteq\mathbb{R}^n$;
$B(x,r)=\{y\in\mathbb{R}^n : |x-y|<r\}),\ \widetilde{B}(x,r)=B(x,r)\cap\Omega$;
by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent Lebesgue and Morrey spaces

Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. An open set Ω is assumed to be bounded throughout the whole paper. We suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{1}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1, \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$

By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 :\ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent. The Hölder inequality is valid in the form

$$\int_\Omega |f(x)||g(x)|dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

For the basics on variable exponent Lebesgue spaces we refer to [63], [41].

The weighted Lebesgue space $L^{p(\cdot),\omega}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot),\omega}(\Omega)} = \inf\left\{\eta > 0 :\ \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \omega(x)dx \le 1\right\}.$$

Definition 1. *By $WL(\Omega)$ (weak Lipschitz) we denote the class of functions defined on Ω satisfying the log-condition*

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2} \ \ x, y \in \overline{\Omega}, \tag{2}$$

where $A = A(p) > 0$ does not depend on x, y.

Theorem 1. *([14]) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy condition* (1)*. Then the maximal operator M is bounded in $L^{p(\cdot)}(\Omega)$.*

Theorem 2. *([36]) Let Ω be bounded and $p \in WL(\Omega)$ satisfy condition* (1), (9)*, $x_0 \in \overline{\Omega}$ and let*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{p'(x_0)}. \tag{3}$$

Then the weighted maximal operator M_β is bounded in $L^{p(\cdot)}(\Omega)$.

The following theorem for bounded sets Ω, but for variable $\alpha(x)$, was proved in [59].

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $p, \alpha \in WL(\Omega)$ satisfy assumption* (1)*, $x_0 \in \overline{\Omega}$ and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \ \sup_{x\in\Omega} \alpha(x)p(x) < n, \tag{4}$$

$$\alpha(x_0)p(x_0) - n < \gamma < n(p(x_0) - 1), \tag{5}$$

$$\mu = \frac{q(x_0)\gamma}{p(x_0)}. \tag{6}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot),|x-x_0|^\gamma}(\Omega)$ to $L^{q(\cdot),|x-x_0|^\mu}(\Omega)$ with

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{7}$$

Singular operators within the framework of the spaces with variable exponents were studied in [15]. From Theorem 4.8 and Remark 4.6 of [15] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition on $p(x)$ at infinity.

Theorem 4. *([15]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $p \in WL(\Omega)$ satisfy condition* (1)*. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

We will also make use of the estimate provided by the following lemma (see [57], Corollary to Lemma 3.22).

Lemma 1. *Let Ω be a bounded domain and p satisfy the assumption $1 \le p_- \le p(x) \le p_+ < \infty$ and condition* (2)*. Let also $\beta \in WL(\Omega)$ and $\sup_{x\in\Omega}[n+\nu(x)p(x)] < 0$, $\sup_{x\in\Omega}[n+\nu(x)p(x) + \beta(x)] < 0$. Then*

$$\| |x-y|^{\nu(x)} \chi_{B(x,r)}(y) \|_{L^{p(\cdot),|\cdot|^{\beta(x)}}} \le C r^{\nu(x)+\frac{n}{p(x)}} (r+|x|)^{\frac{\beta(x)}{p(x)}}, \quad x \in \Omega, \ 0 < r < \ell, \tag{8}$$

where C does not depend on x and r.

Remark 1. *It can be shown that the constant C in* (8) *may be estimated as $C = C_0 \ell^{n\left(\frac{1}{p_-} - \frac{1}{p_+}\right)}$, where C_0 does not depend on Ω.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ and variable weighted Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^\gamma}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot),|\cdot|^{\gamma}}(\Omega)}=\sup_{x\in\Omega,\,t>0}t^{-\frac{\lambda(x)}{p(x)}}\||\cdot|^{\frac{\gamma}{p(\cdot)}}f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively. Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x)=\sup_{r>0}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|f(y)-f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)}(x)=|\widetilde{B}(x,t)|^{-1}\int_{\widetilde{B}(x,t)}f(y)dy$.

Definition 2. *We define the $BMO_{|\cdot|^\beta}(\Omega)$ space as the set of all locally integrable functions f with the finite norm*

$$\|f\|_{BMO_{|\cdot|^\beta}}=\sup_{x\in\Omega}|x|^\beta M^\sharp f(x)=\|M^\sharp f\|_{L_{\infty,|\cdot|^\beta}}.$$

The following statements are known.

Theorem 5. *([3]) Let Ω be bounded and $p\in WL(\Omega)$ satisfy condition (1) and let a measurable function λ satisfy the conditions*

$$0\le\lambda(x),\qquad\sup_{x\in\Omega}\lambda(x)<n. \tag{9}$$

Then the maximal operator M is bounded in $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Theorem 5 was extended to unbounded domains in [26].

Note that the boundedness of the maximal operator in Morrey spaces with variable $p(x)$ was studied in [30] in the more general setting of quasimetric measure spaces.

Theorem 6. *([3]) Let Ω be bounded, $p,\alpha,\lambda\in WL(\Omega)$ and p satisfy condition (1). Let also $\lambda(x)\ge 0$ and the conditions (4), (7) be fulfilled. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{q(\cdot),\mu(\cdot)}(\Omega)$, where*

$$\frac{\mu(x)}{q(x)}=\frac{\lambda(x)}{p(x)}. \tag{10}$$

Theorem 7. *([3]) Let Ω be bounded, $p,\alpha,\lambda\in WL(\Omega)$ and p satisfy condition (1). Let also $\lambda(x)\ge 0$ and the conditions*

$$\inf_{x\in\Omega}\alpha(x)>0,\ \sup_{x\in\Omega}[\lambda(x)+\alpha(x)p(x)]<n \tag{11}$$

hold. Then the operator $I^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $\mathcal{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$, where

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha}{n-\lambda(x)}. \tag{12}$$

Theorem 8. *([3]) Let Ω be bounded and $p, \alpha, \lambda \in WL(\Omega)$ satisfy conditions* (1) *and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \ \lambda(x) + \alpha(x)p(x) = n$$

hold. Then the operator $M^{\alpha(\cdot)}$ is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $L^\infty(\Omega)$.

Theorem 9. *([3]) Let Ω be bounded and $p, \lambda \in WL(\Omega)$ satisfy conditions* (1) *and let $0 < \alpha < n$, $0 \le \lambda(x)$, $\sup \lambda(x) < n - \alpha$,*

$$p(x) = \frac{n - \lambda(x)}{\alpha}.$$

Then the operator I^α is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ to $BMO(\Omega)$.

3. Variable exponent generalized Morrey spaces

Throughout this paper the functions $\omega(x,r)$, $\omega_1(x,r)$ and $\omega_2(x,r)$ are non-negative measurable functions on $\Omega \times (0,\ell)$, $\ell = diam\,\Omega$.

We find it convenient to define the generalized Morrey spaces as follows.

Definition 3. *Let $1 \le p < \infty$. The generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ is defined by the norms*

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega,|\cdot|^\gamma}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))}.$$

According to this definition, we recover the space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$:

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)\Big|_{\omega(x,r)=r^{\frac{\lambda(x)-n}{p(x)}}}.$$

In the sequel we assume that

$$\inf_{x\in\Omega, r>0} \omega(x,r) > 0 \tag{13}$$

which makes the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ nontrivial. Note that when p is constant, in the case of $w(x,r) \equiv const > 0$, we have the space $L^\infty(\Omega)$.

3.1. Preliminaries on Morrey spaces with constant exponents p

In [20], [21], [46] and [48] sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular operator T from $\mathcal{M}^{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{p,\omega_2}(\mathbb{R}^n)$ were obtained. In [48] the following condition was imposed on $w(x,r)$:

$$c^{-1}\omega(x,r) \le \omega(x,t) \le c\,\omega(x,r) \tag{14}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, with

$$\int_r^\infty \omega(x,t)^p \frac{dt}{t} \le C\,\omega(x,r)^p \tag{15}$$

for tmaximal or singular operator and

$$\int_r^\infty t^{\alpha p}\omega(x,t)^p \frac{dt}{t} \le C\,r^{\alpha p}\omega(x,r)^p \tag{16}$$

for potential or fractional maximal operator, where $C(>0)$ does not depend on r and $x \in \mathbb{R}^n$.

Remark 2. *The left-hand side inequality in* (14) *is satisfied for any non-negative function* $w(x,r)$ *such that there exists a number* $a \in \mathbb{R}^1$ *such that the function* $r^a w(x,r)$ *is almost increasing in* r *uniformly in* x*:*

$$t^a w(x,t) \le c r^a w(x,r) \qquad \textit{for all} \qquad 0 < t \le r < \infty$$

where $c \ge 1$ *does not depend on* x, r, t.

Note that integral conditions of type (15) after the paper [4] of 1956 are often referred to as Bary-Stechkin or Zygmund-Bary-Stechkin conditions, see also [25]. The classes of almost monotonic functions satisfying such integral conditions were later studied in a number of papers (see [28], [54], [55] and references therein), where the characterization of integral inequalities of such a kind was given in terms of certain lower and upper indices known as Matuszewska-Orlicz indices. Note that in the cited papers the integral inequalities were studied as $r \to 0$. Such inequalities are also of interest when they allow to impose different conditions as $r \to 0$ and $r \to \infty$; such a case was dealt with in [56], [40].

In [48] the following statements were proved.

Theorem 10. *[48] Let* $1 < p < \infty$ *and* $\omega(x,r)$ *satisfy conditions* (14)-(15)*. Then the operators* M *and* T *are bounded in* $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$.

Theorem 11. *[48] Let* $1 < p < \infty, 0 < \alpha < \frac{n}{p}$*, and* $\omega(x,t)$ *satisfy conditions* (14) *and* (16)*. Then the operators* M^α *and* I^α *are bounded from* $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ *to* $\mathcal{M}^{q,\omega}(\mathbb{R}^n)$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The following statement, containing the results in [46], [48], was proved in [20] (see also [21]). Note that Theorems 12 and 13 do not impose condition (14).

Theorem 12. *[20] Let $1 < p < \infty$ and $\omega_1(x,r)$, $\omega_2(x,r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty \omega_1(x,t)\frac{dt}{t} \le c_1\,\omega_2(x,r) \tag{17}$$

with $c_1 > 0$ not depending on $x \in \mathbb{R}^n$ and $t > 0$. Then the operators M and T are bounded from $\mathcal{M}^{p,\omega_1(\cdot)}(\mathbb{R}^n)$ to $\mathcal{M}^{p,\omega_2(\cdot)}(\mathbb{R}^n)$.

Theorem 13. *[20] Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\omega_1(x,r)$, $\omega_2(x,r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty t^\alpha \omega_1(x,t)\frac{dt}{t} \le c_1\,\omega_2(x,r). \tag{18}$$

Then the operators M^α and I^α are bounded from $\mathcal{M}^{p,\omega_1(\cdot)}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\omega_2(\cdot)}(\mathbb{R}^n)$.

Theorem 14. *[22] Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p \in WL(\Omega)$ satisfy assumption (1) and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_r^\ell \omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{19}$$

where C does not depend on x and t. Then the maximal operators M and T are bounded from the space $\mathcal{M}^{p(\cdot),\omega_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\omega_2}(\Omega)$.

Theorem 15. *[22] Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p, q \in WL(\Omega)$ satisfy assumption (1), $\alpha(x), q(x)$ satisfy the conditions (4), (7) and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_r^\ell t^{\alpha(x)}\omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{20}$$

where C does not depend on x and r. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 16. *[22] Let $p \in WL(\Omega)$ satisfy assumption (1), $\alpha(x)$ fulfill the condition (4) and let $\omega(x,t)$ satisfy condition (19) and the conditions*

$$\omega(x,r) \le \frac{C}{r^{\frac{\alpha(x)}{1-\frac{p(x)}{q(x)}}}}, \tag{21}$$

$$\int_r^\ell t^{\alpha(x)-1}\,\omega(x,t)dt \le C\omega(x,r)^{\frac{p(x)}{q(x)}}, \tag{22}$$

where $q(x) \ge p(x)$ and C does not depend on $x \in \Omega$ and $r \in (0,\ell]$. Suppose also that for almost every $x \in \Omega$, the function $w(x,r)$ fulfills the condition

$$\text{there exist an } a = a(x) > 0 \text{ such that } \omega(x,\cdot) : [0,\ell] \to [a,\infty) \text{ is surjective.} \tag{23}$$

Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega(\cdot)}(\Omega)$.

4. The maximal operator in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot),|\cdot|^\gamma}(\Omega)$

Theorem 17. *Let Ω be bounded and $p \in WL(\Omega)$ satisfy condition (1), $x_0 \in \overline{\Omega}$ and (3). Then*

$$\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}$$
$$\leq Ct^{\frac{n}{p(x)}}(t+|x-x_0|)^{\frac{\beta}{p(x)}}\int_t^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,s))}ds,\ 0<t<\frac{\ell}{2}, \quad (24)$$

for every $f \in L^{p(\cdot),|x-x_0|^\beta}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \quad t > 0. \quad (25)$$

Then

$$\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))} \leq \|Mf_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))} + \|Mf_2\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}.$$

By the Theorem 2 we obtain

$$\begin{aligned}\|Mf_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))} &\leq \|Mf_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\Omega)}\\ &\leq C\|f_1\|_{L^{p(\cdot),|x-x_0|^\beta}(\Omega)} = C\|f\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,2t))},\end{aligned} \quad (26)$$

where C does not depend on f. We assume for simplicity that $x_0 = 0$. From (26) we obtain

$$\begin{aligned}\|Mf_1\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,t))} &\leq Ct^{\frac{n}{p(x)}}(t+|x-x_0|)^{\frac{\beta}{p(x)}}\int_{2t}^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds\\ &\leq Ct^{\frac{n}{p(x)}}(t+|x-x_0|)^{\frac{\beta}{p(x)}}\int_{t}^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds\end{aligned} \quad (27)$$

easily obtained from the fact that $\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,2t))}$ on the right-hand side of (26) is dominated by the right-hand side of (27). Note that this "complication" of estimate in comparison with (26) is done because the term Mf_2 will be estimated below in a similar way (see (29)).

To estimate Mf_2, we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|f(y)|dy \leq C\int_t^\ell s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds, \quad 0<t<\ell. \quad (28)$$

To this end, we choose $\delta > \frac{n}{p_-}$ and proceed as follows:

$$\begin{aligned}\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy &\le \delta \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^{\ell} s^{-\delta-1}ds \\ &= \delta \int_t^{\ell} s^{-\delta-1}ds \int_{\{y\in\Omega: 2t\le|x-y|\le s\}} |x-y|^{-n+\delta}|f(y)|dy \\ &\le C\int_t^{\ell} s^{-\delta-1}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}\||x-y|^{-n+\delta}\|_{L^{p'(\cdot),|\cdot|^{\beta/(1-p(x))}}(\widetilde{B}(x,s))}ds.\end{aligned}$$

We then make use of Lemma 1 and obtain (28).

For $z\in\widetilde{B}(x,t)$ we get

$$\begin{aligned}Mf_2(z) &= \sup_{r>0}|B(z,r)|^{-1}\int_{\widetilde{B}(z,r)}|f_2(y)|dy \\ &\le C\sup_{r\ge 2t}\int_{(\Omega\setminus\widetilde{B}(x,2t))\cap\widetilde{B}(z,r)}|y-z|^{-n}|f(y)|dy \\ &\le C\sup_{r\ge 2t}\int_{(\Omega\setminus\widetilde{B}(x,2t))\cap\widetilde{B}(z,r)}|x-y|^{-n}|f(y)|dy \\ &\le C\int_{\Omega\setminus\widetilde{B}(x,2t)}|x-y|^{-n}|f(y)|dy.\end{aligned}$$

Then by (28)

$$\begin{aligned}Mf_2(z) &\le C\int_{2t}^{\ell} s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds, \\ &\le C\int_{t}^{\ell} s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds,\end{aligned}$$

where C does not depend on x, r. Thus, the function $Mf_2(z)$, with fixed x and t, is dominated by the expression not depending on z. Then

$$\|Mf_2\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,t))} \le C\int_t^{\ell} s^{-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^\beta}(\widetilde{B}(x,s))}ds\,\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\beta}(\Omega)}. \tag{29}$$

Since $\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\beta}(\Omega)} \le Ct^{\frac{n}{p(x)}}(t+|x|)^{\frac{\beta}{p(x)}}$ by Lemma 1, we then obtain (24) from (27) and (29).

The following theorem extends Theorem 15 to the case of generalized weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,|\cdot|^\beta}(\Omega)$.

Theorem 18. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded set and $p\in WL(\Omega)$ satisfy assumption* (1)*, $x_0\in\overline{\Omega}$,* (3) *and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_t^{\ell}(r+|x-x_0|)^{-\frac{\beta}{p(x)}}\omega_1(x,r)\frac{dr}{r} \le C\,(t+|x-x_0|)^{-\frac{\beta}{p(x)}}\omega_2(x,t). \tag{30}$$

Then the maximal operator M is bounded from the space $\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)$. We have

$$\|Mf\|_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} = \sup_{x\in\Omega,\, t\in(0,\ell)} \omega_2^{-1}(x,t)t^{-\frac{n}{p(x)}}\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}.$$

The estimation is obvious for $\frac{\ell}{2} \le t \le \ell$ in view of (13). For

$$\|Mf\|^{\sim}_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} = \sup_{x\in\Omega,\, t\in\left(0,\frac{\ell}{2}\right)} \omega_2^{-1}(x,t)t^{-\frac{n}{p(x)}}\|Mf\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,t))}$$

by Theorem 17 we obtain

$$\|Mf\|^{\sim}_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} \le C \sup_{x\in\Omega,\, 0<t\le\ell} \omega_2^{-1}(x,t)(t+|x-x_0|)^{\frac{\beta}{p(x)}}$$

$$\int_t^\ell r^{-\frac{n}{p(x)}-1}(r+|x-x_0|)^{-\frac{\beta}{p(x)}}\|f\|_{L^{p(\cdot),|x-x_0|^\beta}(\widetilde{B}(x,r))}dr.$$

Hence

$$\|Mf\|^{\sim}_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\beta}(\Omega)} \le$$

$$\le C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)} \sup_{x\in\Omega,\, t\in(0,\ell)} \omega_2^{-1}(x,t)(t+|x-x_0|)^{\frac{\beta}{p(x)}}$$

$$\int_t^\ell \omega_1(x,r)(r+|x-x_0|)^{-\frac{\beta}{p(x)}}\frac{dr}{r} \le C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\beta}(\Omega)},$$

by (30), which completes the proof.

5. Riesz potential operator in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot),|\cdot|^\gamma}(\Omega)$

5.1. Spanne type result

Theorem 19. *Let $p \in WL(\Omega)$ satisfy conditions* (1) *and let* (3), $x_0 \in \overline{\Omega}$, (5), (6), $\alpha(x), q(x)$ *satisfy the conditions* (4) *and* (7). *Then*

$$\|I^{\alpha(\cdot)}f\|_{L^{q(\cdot),|x-x_0|^\mu}(\widetilde{B}(x,t))}$$

$$\le Ct^{\frac{n}{q(x)}}(t+|x-x_0|)^{\frac{\gamma}{p(x)}}\int_t^l s^{-\frac{n}{q(x)}-1}(s+|x-x_0|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,s))}ds, \quad 0<t\le\frac{\ell}{2} \tag{31}$$

where t is an arbitrary number in $\left(0,\frac{\ell}{2}\right)$ and C does not depend on f, x and t.

Proof. As in the proof of Theorem 17, we represent function f in form (25) and have

$$I^{\alpha(\cdot)}f(x) = I^{\alpha(\cdot)}f_1(x) + I^{\alpha(\cdot)}f_2(x).$$

By Theorem 3 we obtain

$$\begin{aligned}\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\widetilde{B}(x,t))} &\le \|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\Omega)} \\ &\le C\|f_1\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\Omega)} = C\|f\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,2t))}.\end{aligned}$$

Then

$$\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,2t))},$$

where the constant C is independent of f.

We assume for simplicity that $x_0 = 0$. Taking into account that

$$\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,2t))} \le Ct^{\frac{n}{q(x)}}(t+|x|)^{\frac{\gamma}{p(x)}}\int_t^l s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds,$$

we get

$$\|I^{\alpha(\cdot)}f_1\|_{L^{q(\cdot),|\cdot|^{\mu}}(\widetilde{B}(x,t))} \le Ct^{\frac{n}{q(x)}}(t+|x|)^{\frac{\gamma}{p(x)}}\int_t^l s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds. \tag{32}$$

When $|x-z| \le t$, $|z-y| \ge 2t$, we have $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$\begin{aligned}|I^{\alpha(\cdot)}f_2(x)| &\le \int_{\Omega\setminus\widetilde{B}(x,2t)} |z-y|^{\alpha(y)-n}|f(y)|dy \\ &\le C\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{\alpha(x)-n}|f(y)|dy.\end{aligned}$$

We choose $\beta > \frac{n}{q(x)}$ and obtain

$$\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{\alpha(x)-n}|f(y)|dy$$

$$\begin{aligned}&= \beta\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{\alpha(x)-n+\beta}|f(y)|\left(\int_{|x-y|}^l s^{-\beta-1}ds\right)dy \\ &= \beta\int_{2t}^l s^{-\beta-1}\left(\int_{\{y\in\Omega:2t\le|x-y|\le s\}} |x-y|^{\alpha(x)-n+\beta}|f(y)|dy\right)ds \\ &\le C\int_{2t}^l s^{-\beta-1}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}\||x-y|^{\alpha(x)-n+\beta}\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(x))}}(\widetilde{B}(x,s))}ds \\ &\le C\int_{2t}^l s^{\alpha(x)-\frac{n}{p(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds.\end{aligned}$$

Hence

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot),|\cdot|^{\mu}}(\widetilde{B}(x,t))} \leq C\int_{2t}^{l} s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds\|\chi_{\widetilde{B}(x,t)}\|_{L^{q(\cdot),|\cdot|^{\mu}}(\Omega)}.$$

Therefore

$$\|I^{\alpha(\cdot)}f_2\|_{L^{q(\cdot),|\cdot|^{\mu}}(\widetilde{B}(x,t))} \leq Ct^{\frac{n}{q(x)}}(t+|x|)^{\frac{\gamma}{p(x)}}\int_{2t}^{l} s^{-\frac{n}{q(x)}-1}(s+|x|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,s))}ds \tag{33}$$

which together with (32) yields (31).

Theorem 20. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $p, q \in WL(\Omega)$ satisfy assumptions* (1), (5), (6), (3), *$x_0 \in \overline{\Omega}$, $\alpha(x), q(x)$ satisfy the conditions* (4), (7) *and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_{r}^{\ell} t^{\alpha(x)}(t+|x-x_0|)^{-\frac{\gamma}{p(x)}}\omega_1(x,t)\frac{dt}{t} \leq C\,(r+|x-x_0|)^{-\frac{\gamma}{p(x)}}\omega_2(x,r). \tag{34}$$

Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega_1(\cdot),|x-x_0|^{\gamma}}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega_2(\cdot),|x-x_0|^{\mu}}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega,|x-x_0|^{\gamma}}(\Omega)$. As usual, when estimating the norm

$$\|I^{\alpha(\cdot)}f\|_{\mathcal{M}^{q(\cdot),\omega_2,|x-x_0|^{\mu}}(\Omega)} = \sup_{x\in\Omega,\, t>0}\frac{t^{-\frac{n}{q(x)}}}{\omega_2(x,t)}\|I^{\alpha(\cdot)}f\chi_{\widetilde{B}(x,t)}\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\Omega)}, \tag{35}$$

it suffices to consider only the values $t \in \left(0, \frac{\ell}{2}\right)$, thanks to condition (13). We estimate $\|I^{\alpha(\cdot)}f\chi_{\widetilde{B}(x,t)}\|_{L^{q(\cdot),|x-x_0|^{\mu}}(\Omega)}$ in (35) by means of Theorem 19 and obtain

$$\begin{aligned}
&\|I^{\alpha(\cdot)}f\|_{\mathcal{M}^{q(\cdot),\omega_2,|x-x_0|^{\mu}}(\Omega)}\\
&\leq C\sup_{x\in\Omega,\, t>0}\frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)}\int_{t}^{\ell} r^{-\frac{n}{q(x)}-1}(r+|x-x_0|)^{-\frac{\gamma}{p(x)}}\|f\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,r))}dr\\
&\leq C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^{\gamma}}(\Omega)}\sup_{x\in\Omega,\, t>0}\frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)}\int_{t}^{\ell}\frac{r^{\alpha(x)}(r+|x-x_0|)^{-\frac{\gamma}{p(x)}}\omega_1(x,r)}{r}dr.
\end{aligned}$$

It remains to make use of condition (34).

Theorem 21. *Let $p \in WL(\Omega)$ satisfy assumption* (1), *$x_0 \in \overline{\Omega}$, γ, μ satisfy conditions*

$$0 \leq \gamma < \frac{n}{p'(x_0)}, \;\; \mu = \frac{\gamma}{p(x_0)}, \tag{36}$$

$\inf\limits_{x\in\Omega}\alpha(x) > 0$ and let $\omega(x,t)$ satisfy condition

$$r^{\alpha(x)}\omega(x,r) \leq C. \tag{37}$$

Then the operator $M^{\alpha(\cdot)}$ is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot),|x-x_0|^{\gamma}}(\Omega)$ to $L^{\infty,|x-x_0|^{\mu}}(\Omega)$.

Proof. Let $x \in \Omega$ and $r > 0$. We assume for simplicity that $x_0 = 0$. By the Hölder inequality we get successively

$$
\begin{aligned}
& r^{\alpha(x)-n} \int_{\widetilde{B}(x,r)} |f(y)|dy \\
& \le C r^{\alpha(x)-n} r^{\frac{n}{p(x)}} \omega(x,r) r^{-\frac{n}{p(x)}} \omega^{-1}(x,r) \|f\|_{L^{p(\cdot),|\cdot|^{\gamma}}(\widetilde{B}(x,r))} \|\chi_{\widetilde{B}(x,r)}\|_{L^{p'(\cdot),|\cdot|^{\gamma/(1-p(\cdot))}}} \\
& \le C r^{\alpha(x)} \omega(x,r) |x|^{-\frac{\gamma}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C |x|^{-\frac{\gamma}{p(x)}} \|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.
\end{aligned}
$$

We again refer to the logarithmic condition for $p(x)$ which provides the equivalence

$$
|x|^{\frac{\gamma}{p(x)}} \sim |x|^{\frac{\gamma}{p(0)}}.
$$

Theorem 22. *Let $p \in WL(\Omega)$ satisfy assumption (1), $x_0 \in \overline{\Omega}$, γ, μ satisfy condition (36), $0 < \alpha < n$ and let $\omega(x,t)$ satisfy condition (37).*
Then the operator I^{α} is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot),|x-x_0|^{\gamma}}(\Omega)$ to $BMO_{|x-x_0|^{\mu}}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega(\cdot),|x-x_0|^{\gamma}}(\Omega)$. In [1] it was proved that

$$
M^{\sharp}(I^{\alpha} f)(x) \le C M^{\alpha} f(x). \tag{38}
$$

The proof of Theorem 22 follows from the Theorem 21 and inequality (38).

6. Singular operators in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot),|\cdot|^{\gamma}}(\Omega)$

Theorem 23. *[33] Let Ω be bounded, $p \in WL(\Omega)$ and p satisfy conditions (1) and (3), $x_0 \in \overline{\Omega}$. Then the operators T and T^* are bounded in the space $L^{p(\cdot),|x-x_0|^{\gamma}}(\Omega)$.*

Theorem 24. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $p \in WL(\Omega)$ satisfy conditions (1), (3), $x_0 \in \overline{\Omega}$ and $f \in L^{p(\cdot),|x-x_0|^{\gamma}}(\Omega)$. Then*

$$
\begin{aligned}
& \|Tf\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,t))} \\
& \le C t^{\frac{n}{p(x)}} (t+|x-x_0|)^{\frac{\gamma}{p(x)}} \int_t^{\ell} r^{-\frac{n}{p(x)}-1} (r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|x-x_0|^{\gamma}}(\widetilde{B}(x,r))} dr, \quad 0 < t \le \frac{\ell}{2},
\end{aligned} \tag{39}
$$

where C does not depend on f and t.

Proof. We represent function f as in (25) and have

$$\|Tf\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \le \|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} + \|Tf_2\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))}.$$

By the Theorem 23 we obtain

$$\|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \le \|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)} \le C\|f_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)},$$

so that

$$\|Tf_1\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,2t))}.$$

We assume for simplicity that $x_0 = 0$. Taking into account the inequality

$$\|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le$$

$$Ct^{\frac{n}{p(x)}}(t+|x|)^{\frac{\gamma}{p(x)}} \int_{2t}^{\ell} r^{-\frac{n}{p(x)}-1}(r+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} dr, \quad 0 < t \le \frac{\ell}{2},$$

we get

$$\|Tf_1\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le Ct^{\frac{n}{p(x)}}(t+|x|)^{\frac{\gamma}{p(x)}} \int_{t}^{\ell} r^{-\frac{n}{p(x)}-1}(r+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} dr. \tag{40}$$

To estimate $\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))}$, we observe that

$$|Tf_2(z)| \le C \int\limits_{\Omega\backslash B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|Tf_2(z)| \le C \int_{\Omega\backslash\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy.$$

Hence by estimate (8) (with $\nu(x) \equiv 0$) and inequality (28), we get

$$\|Tf_2\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,t))} \le C \int_{t}^{\ell} r^{-\frac{n}{p(x)}-1}(r+|x|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|\cdot|^\gamma}(\widetilde{B}(x,r))} dr\, \|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|\cdot|^\gamma}(\Omega)}. \tag{41}$$

From (40) and (41) we arrive at (39).

Theorem 25. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $p \in WL(\Omega)$ satisfy condition* (1), (3), *$x_0 \in \overline{\Omega}$ and $\omega_1(x,t)$ and $\omega_2(x,r)$ fulfill conditions* (30). *Then the singular integral operator T is bounded from the space $\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\gamma}(\Omega)$.*

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)$. As usual, when estimating the norm

$$\|Tf\|_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\gamma}(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{t^{-\frac{n}{p(x)}}}{\omega_2(x,t)} \|Tf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)}, \tag{42}$$

it suffices to consider only the values $t \in \left(0, \frac{\ell}{2}\right)$, thanks to condition (13). We estimate $\|Tf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot),|x-x_0|^\gamma}(\Omega)}$ in (42) by means of Theorem 24 and obtain

$$\begin{aligned}
&\|Tf\|_{\mathcal{M}^{p(\cdot),\omega_2,|x-x_0|^\gamma}(\Omega)} \\
&\le C \sup_{x\in\Omega,\, t>0} \frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)} \int_t^\ell r^{-\frac{n}{p(x)}-1}(r+|x-x_0|)^{-\frac{\gamma}{p(x)}} \|f\|_{L^{p(\cdot),|x-x_0|^\gamma}(\widetilde{B}(x,r))} dr \\
&\le C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1,|x-x_0|^\gamma}(\Omega)} \sup_{x\in\Omega,\, t>0} \frac{(t+|x-x_0|)^{\frac{\gamma}{p(x)}}}{\omega_2(x,t)} \int_t^\ell \frac{(r+|x-x_0|)^{-\frac{\gamma}{p(x)}}\omega_1(x,r)}{r} dr.
\end{aligned}$$

It remains to make use of condition (30).

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