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Oscillatory Integral Operators and Their Commutators on Vanishing Generalized Morrey Spaces with Variable Exponent

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Abstract. We consider the generalized Morrey spaces $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ with variable exponent $p(x)$ and a general function $\varphi(x,r)$ defining the Morrey-type norm. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the conditions in terms of Calderón-Zygmund-type integral inequalities for oscillatory integral operators and its commutators in the vanishing generalized weighted Morrey spaces with variable exponent.

Key Words and Phrases: maximal operator, singular integral operators, Calderón-Zygmund-type integral inequalities for oscillatory integral operators, generalized weighted Morrey space with variable exponent, BMO space.

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1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [51] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [20, 22, 24, 51]. Mizuhara [52] and Nakai [55] introduced generalized Morrey spaces. Later, Guliyev [24] defined the generalized Morrey spaces $M^{p,\varphi}$ with normalized norm.

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [18, 40, 43, 59], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [11, 12, 13, 15, 16, 17, 42, 45].

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [2] and [53] in the Euclidean setting and in [41] in the setting of metric measure spaces, in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey

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spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [2]. In [54] the maximal operator was considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. P. Hästö in [35] used his new "local-to-global" approach to extend the result of [2] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [41].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [4], [46], [52], [55]. In the case of bounded sets the boundedness of the maximal operator, singular integral operators and the potential operator in generalized variable exponent Morrey type spaces was proved in [29], [30], [31] and in the case of unbounded sets in [32], see also [36, 37, 56].

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [58], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [14]; for further results in the case of constant p and λ (see, for instance, [3]– [5]).

We consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy.$$

A distribution kernel $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if} \quad |x-y| > 2|x-\xi|$$

Calderón-Zygmund type singular operator and the oscillatory integral operator are defined by

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy, \tag{1}$$

$$Sf(x) = \int_{\Omega} e^{P(x,y)} K(x,y)f(y)dy, \tag{2}$$

where $P(x,y)$ is a real valued polynomial defined on $\Omega \times \Omega$. Lu and Zhang [50] used L^2-boundedness of T to get L^p- boundedness of S with $1 < p < \infty$.

Let

$$T^* f(x) = \sup_{\varepsilon>0} |T_{\varepsilon} f(x)|$$

be the maximal singular operator, where $T_{\varepsilon} f(x)$ is the usual truncation

$$T_{\varepsilon} f(x) = \int_{\{y\in\Omega:|x-y|\geq\varepsilon\}} K(x,y)f(y)dy.$$

We find the condition on the Morrey function $\varphi(x,r)$ for the boundedness of the oscillatory integral operator in generalized weighted Morrey space $\mathcal{M}^{p(\cdot),\varphi}_{\omega}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces. In Section 3 we treat oscillatory integral operators and its commutators in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$.

The main results are given in Theorems 7, 8, 9, 11, 12, 13. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\varphi(x,r)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega\subset\mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E\subseteq\mathbb{R}^n$, $B(x,r)=\{y\in\mathbb{R}^n : |x-y|<r\})$, $\widetilde{B}(x,r)=B(x,r)\cap\Omega$, by c,C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line. By $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2. Preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces

We refer to the book [16] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1<p_-\le p(x)\le p_+<\infty, \tag{3}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f)=\int_\Omega |f(x)|^{p(x)}dx<\infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)}=\inf\left\{\eta>0:\ I_{p(\cdot)}\left(\frac{f}{\eta}\right)\le 1\right\},$$

this is a Banach function space. By $p'(\cdot)=\frac{p(x)}{p(x)-1}$, $x\in\Omega$, we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{f(x):\left|\int_\Omega f(y)g(y)dy\right|<\infty\ \ for\ all\ \ g\in L^{p'(\cdot)}(\Omega)\right\} \tag{4}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}}\le 1} \left| \int_\Omega f(y)g(y)dy \right| \tag{5}$$

see [47, Proposition 2.2], see also [44, Theorem 2.3], or [60, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [61], [44].
$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1,\infty)$;
$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \ \ |x-y| \le \frac{1}{2} \ \ x, y \in \Omega, \tag{6}$$

where $A = A(p) > 0$ does not depend on x, y;
$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \to \mathbb{R}^n$ satisfying the condition (6);
$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \le p_+ < \infty$;
for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \le \frac{A_\infty}{\ln(2+|x|)}, \ \ x \in \mathbb{R}^n, \tag{7}$$

where $p_\infty = \lim\limits_{x\to\infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [16], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\theta_p(x,r)}, \qquad x \in \Omega, \ p \in \mathbb{P}^{log}_\infty(\Omega), \tag{8}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, \ r \le 1, \\ \frac{n}{p(\infty)}, \ r \ge 1. \end{cases}$

A locally integrable function $\omega : \Omega \to (0,\infty)$ is called a weight. We say that $\omega \in A_p(\Omega)$, $1 < p < \infty$, if there is a constant $C > 0$ such that

$$\left(\frac{1}{|\widetilde{B}(x,t)|} \int_{\widetilde{B}(x,t)} \omega(x)dx \right) \left(\frac{1}{|\widetilde{B}(x,t)|} \int_{\widetilde{B}(x,t)} \omega^{1-p'}(x)dx \right)^{p-1} \le C,$$

where $1/p + 1/p' = 1$. We say that $\omega \in A_1(\Omega)$ if there is a constant $C > 0$ such that $M\omega(x) \le C\omega(x)$ almost everywhere.

The extrapolation theorems (Lemma 1 and Lemma 2 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [12]. Here we use the form in [16], see Theorem 7.2.1 and Theorem 7.2.3 in [16].

Lemma 1. *([16]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < \infty$, every $(f, g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\int_\Omega |f(x)|^{p_0}\omega(x)dx \le C_0 \int \Omega |g(x)|^{p_0}\omega(x)dx.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \le p_-$. If maximal operator is bounded on $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 2. *([16]). Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < q_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_1$*

$$\left(\int_\Omega |f(x)|^{q_0}\omega(x)dx\right)^{\frac{1}{q_0}} \le C_0 \left(\int_\Omega |g(x)|^{p_0}\omega^{\frac{p_0}{q_0}}(x)dx\right)^{\frac{1}{p_0}}.$$

Let $p(\cdot) \in P(\Omega)$ with $p_0 \le p_-$ and $\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{p_+}$, and define $q(x)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

If maximal operator is bounded on $L^{\left(\frac{q(\cdot)}{q_0}\right)'}(\Omega)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{q(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

Singular operators within the framework of the spaces with variable exponents were studied in [17]. From Theorem 4.8 and Remark 4.6 of [17] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

Theorem 1. *([17, Theorem 4.8]) Let $\Omega \subset \mathbb{R}^n$ be a unbounded open set and $p \in \mathbb{P}^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0,n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

respectively.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1}\int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x,t)|^{-1}\int_{\widetilde{B}(x,t)} f(y)dy$.

Definition 1. *We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO} = \sup_{x\in\Omega} M^\sharp f(x) = \sup_{x\in\Omega,\, r>0} |B(x,r)|^{-1}\int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy.$$

Definition 2. *We define the $BMO_{p(\cdot)}(\Omega)$ space as the set of all locally integrable functions f with finite norm*

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}.$$

Theorem 2. *[47] Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, then the norms $\|\cdot\|_{BMO_{p(\cdot)}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

Lemma 3. *[34] Let $b \in BMO(\Omega)$. Then there is a constant $C > 0$ such that*

$$\left| b_{\widetilde{B}(x,r)} - b_{\widetilde{B}(x,t)} \right| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are non-negative measurable functions on $\Omega \times (0,\infty)$. We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

Definition 3. *Let $1 \le p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norms*

$$\|f\|_{\mathcal{M}^{p(\cdot),\varphi}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)t^{\theta_p(x,t)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

respectively.

According to this definition, we recover the space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\varphi(x,r) = r^{\theta_p(x,r) - \frac{\lambda(x)}{p(x)}}$:

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\varphi(\cdot)}(\Omega)\Big|_{\varphi(x,r)=r^{\theta_p(x,r)-\frac{\lambda(x)}{p(x)}}}.$$

Definition 4. *(Vanishing generalized weighted Morrey space) The vanishing generalized weighted Morrey space $V\mathcal{M}^{p(\cdot),\varphi}_{\omega}(\Omega)$ is defined as the space of functions $f \in \mathcal{M}^{p(\cdot),\varphi}_{\omega}(\Omega)$ such that*

$$\lim_{r\to 0} \sup_{x\in\Omega} \frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_{\omega}(\Omega)} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \inf\limits_{x\in\Omega} \varphi(x,t)} = 0. \tag{9}$$

and

$$\sup_{0<r<\infty} \frac{1}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \inf\limits_{x\in\Omega} \varphi(x,t)} = 0. \tag{10}$$

which makes the spaces $V\mathcal{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ non-trivial, because bounded functions with compact support belong then to this space.

Let $L_v^\infty(\mathbb{R}_+)$ be the weighted L^∞-space with the norm

$$\|g\|_{L_v^\infty(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t).$$

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}, \quad t \in (0,\infty).$$

The following theorem was proved in [3].

Theorem 3. *Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{v_1}^\infty(\mathbb{R}_+)$ to $L_{v_2}^\infty(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|_{L_\infty(0,\cdot)}^{-1} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \;\; H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [26, 27].

Theorem 4. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 5. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{11}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (11).

3. Oscillatory integral operators and its commutators in $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderón [6, 7] studied a kind of commutators, appearing in Cauchy integral problems of Lipschitz curve. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b,K]f = K(bf) - b\,Kf$ is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [9], [10], [19], [20], [22]).

Lemma 4. *(see [49]). If K is a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$, then for any real polynomial $P(x,y)$ and $\omega \in A_p$ $(1<p<\infty)$, there exists constants $C>0$ independent of the coefficients of P such that*

$$\|Sf\|_{L^p_\omega(\Omega)} \le C \|f\|_{L^p_\omega(\Omega)}.$$

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.*

Proof. By the Lemma 1 and Lemma 4, we derive the operator S is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following local estimates are valid.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and $f \in L^{p(\cdot)}(\Omega)$. Then*

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}, \tag{12}$$

where C does not depend on f, $x \in \Omega$ and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \quad t>0, \tag{13}$$

and have

$$\|Sf\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By the Theorem 6 we obtain

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|Sf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)},$$

so that

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

we get

$$\|Sf_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{14}$$

To estimate $\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|Sf_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|Sf_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy,$$

To estimate Sf_2, we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{15}$$

To this end, we choose $\delta > 0$ and proceed as follows

$$\begin{aligned}
&\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \le \delta \int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy \int_{|x-y|}^\infty s^{-\delta-1}ds \\
&\le C \int_t^\infty s^{-n}\frac{ds}{s} \int_{\{y\in\Omega:2t\le|x-y|\le s\}} |f(y)|dy \le C \int_t^\infty s^{-n}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \|\chi_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\Omega)} \frac{ds}{s} \\
&\le C \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}.
\end{aligned} \tag{16}$$

Hence by inequality (16), we get

$$\|Sf_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \leq C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}$$

$$= Ct^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{17}$$

From (14) and (17) we arrive at (12).

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $\varphi_1(x,t)$ and $\varphi_2(x,r)$ fulfill condition*

$$\int_t^\infty \frac{\operatorname*{ess\,inf}_{s<r<\infty} \varphi_1(x,r) r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}} \frac{ds}{s} \leq C\varphi_2(x,t), \tag{18}$$

where C does not depend on $x \in \Omega$ and t. Then the singular integral operators T and T^* are bounded from the space $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$. As usual, when estimating the norm

$$\|Sf\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} = \sup_{x\in\Omega,\, t>0} \varphi_2(x,t)^{-1} t^{-\theta_p(x,t)} \|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}. \tag{19}$$

We estimate $\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}$ in (19) by means of Theorem 7 and obtain

$$\begin{aligned} &\|Sf\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} \\ &\leq C \sup_{x\in\Omega,\, t>0} \frac{t^{\theta_p(x,t)}}{\varphi_2(x,t)t^{\theta_p(x,t)}} \int_t^\infty s^{-\theta_p(x,s)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \\ &\leq C \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_1(x,t)t^{\theta_p(x,t)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)}. \end{aligned}$$

It remains to make use of condition (18).

Theorem 9. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$ and $\varphi_1(x,t)$ and $\varphi_2(x,r)$ fulfill satisfy the conditions* (18) *and*

$$C_\gamma := \int_t^\infty \frac{\operatorname*{ess\,inf}_{s<r<\infty} \varphi_1(x,r) r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}} \frac{ds}{s} < \infty \tag{20}$$

for every γ.

Then the singular integral operators S is bounded from the space $V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 7, so we only have to prove that if

$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)t^{\theta_p(x,t)}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0\Rightarrow$$
$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)t^{\theta_p(x,t)}}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0 \tag{21}$$

otherwise.

To show that $\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)t^{\theta_p(x,t)}}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$ for small r, we split the right-hand side of (12):

$$\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|Sf\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_{\omega}(\Omega)}\le C_0\left(I_{1,\gamma}(x,r)+I_{2,\gamma}(x,r)\right), \tag{22}$$

where $\gamma>0$ will be chosen as shown below (we may take $\gamma<1$),

$$I_{1,\gamma}(x,r):=\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^{\gamma_0}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

$$I_{2,\gamma}(x,r):=\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_{\gamma_0}^{\infty}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

and it is supposed that $r<\gamma$. Now we choose any fixed $\gamma>0$ such that

$$\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_{\omega}(\Omega)}<\frac{\varepsilon}{2CC_0},\ \text{for all } 0<t<\gamma,$$

where C and C_0 are constants from (18) and (22), which is possible since $f\in V\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)$. Then

$$\sup_{x\in\Omega}CI_{1,\gamma}(x,r)<\frac{\varepsilon}{2},\ \ 0<r<\gamma,$$

by (21).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (10). We have

$$I_{2,\gamma}(x,r)\le C_\gamma\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)},$$

where C_γ is the constant from (20). Then, by (10) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}<\frac{\varepsilon}{2CC_\gamma\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)}}$$

which completes the proof of (21).

Lemma 5. *(see [62]). If K is a standard Calderón-Zygmund kernel and the Calderón-Zygmund singular integral operator T is of type $(L^2(\Omega), L^2(\Omega))$, then for any real polynomial $P(x,y)$ and $\omega \in A_p$ $(1 < p < \infty)$, there exists constants $C > 0$ independent of the coefficients of P such that*

$$\|[b,S]f\|_{L^p_\omega(\Omega)} \le C\|b\|_*\|f\|_{L^p_\omega(\Omega)}.$$

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $b \in BMO(\Omega)$, $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the commutator operator $[b,S]$ is bounded on the space $L^{p(\cdot)}(\Omega)$.*

Proof. By Lemma 1 and Lemma 5, we derive the operator $[b,S]$ is bounded in the space $L^{p(\cdot)}(\Omega)$.

The following weighted local estimates are valid.

Theorem 11. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and $b \in BMO(\Omega)$. Then*

$$\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} C\|b\|_*\|t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{23}$$

for every $f \in L^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent function f as in (13) and have

$$\|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le \|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} + \|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorem 10 we obtain

$$\begin{aligned} \|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} &\le \|[b,S]f_1\|_{L^{p(\cdot)}(\Omega)} \\ &\le C\|b\|_*\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|b\|_*\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}, \end{aligned} \tag{24}$$

where C does not depend on f. From (24) we obtain

$$\|[b,S]f_1\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \le C\|b\|_* t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \tag{25}$$

easily obtained from the fact that $\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,2t))}$ on the right-hand side of (24) is dominated by the right-hand side of (25). To estimate $\|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$, we observe that

$$|[b,S]f_2(z)| \le C \int\limits_{\Omega\setminus B(x,2t)} |b(z)-b(y)| \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, and therefore

$$|[b,S]f_2(z)| \le C \int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n} |b(z)-b(y)|\,|f(y)|dy.$$

To estimate $[b,S]f_2$, we first prove the following auxiliary inequality

$$\begin{aligned}&\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(z)-b(y)||f(y)|dy\\&\le C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}.\end{aligned}\tag{26}$$

To estimate $[b,S]f_2(z)$, we observe that for $z\in\widetilde{B}(x,t)$ we have

$$\begin{aligned}&\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(z)-b(y)||f(y)|dy\\&\le\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy\\&+\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(z)-b_{\widetilde{B}(x,t)}||f(y)|dy=J_1+J_2.\end{aligned}$$

To this end, we choose $\delta>0$, by Theorem 2 and Lemma 3 we obtain

$$\begin{aligned}J_1=&\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy\\&\le\delta\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n+\delta}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy\int_{|x-y|}^\infty s^{-\delta-1}ds\\&\le C\int_t^\infty s^{-n-1}\int_{\{y\in\Omega:2t\le|x-y|\le s\}}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dyds\\&\le C\int_t^\infty s^{-n-1}\|b(\cdot)-b_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds\\&+C\int_t^\infty s^{-n-1}|b_{\widetilde{B}(x,t)}-b_{\widetilde{B}(x,s)}|\int_{\widetilde{B}(x,s)}|f(y)|dyds\\&\le C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)-n-1}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds\\&+C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)-n-1}\ln\frac{s}{t}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds\\&\le C\|b\|_*\int_t^\infty s^{-\theta_p(x,s)}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}.\end{aligned}$$

To estimate J_2, by (15), we have

$$\begin{aligned}J_2=&|b(z)-b_{\widetilde{B}(x,t)}|\int_{\Omega\setminus\widetilde{B}(x,t)}|x-y|^{-n}|f(y)|dy\\&\le C|B(x,t)|^{-1}\int_{\widetilde{B}(x,t)}|b(z)-b(y)|dy\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}\\&\le CM_b\chi_{B(x,t)}(z)\int_t^\infty s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},\end{aligned}$$

where C does not depend on x, t.

Hence by inequality (26), we get

$$\|[b,S]f_2\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \lesssim \|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\|b\|_*$$
$$\times \int_t^\infty \left(1+\ln\frac{s}{t}\right) s^{-\theta_p(x,s)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$
$$= \|b\|_* t^{\theta_p(x,t)} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \quad (27)$$

From (25) and (27) we arrive at (23).

Theorem 12. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\int_t^\infty \left(1+\ln\frac{s}{t}\right) \frac{\operatorname*{ess\,inf}_{s<r<\infty} \varphi_1(x,r) r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}}\frac{ds}{s} \le C\varphi_2(x,t). \quad (28)$$

Then the operator $[b,S]$ is bounded from the space $\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$. We have

$$\|[b,S]f\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_2(x,t) t^{\theta_p(x,t)}} \|[b,S]f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By (28), Theorems 4 and 11 we obtain

$$\|[b,S]f\|_{\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)}$$
$$\le C\|b\|_* \sup_{x\in\Omega,\, t>0} \frac{t^{\theta_p(x,t)}}{\varphi_2(x,t) t^{\theta_p(x,t)}} \int_t^\infty s^{-\theta_p(x,s)} \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}$$
$$\le C\|b\|_* \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_1(x,t) t^{\theta_p(x,t)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|b\|_*\|f\|_{\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)}$$

which completes the proof.

Theorem 13. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the conditions (28) and*

$$C_{\delta_0} := \int_t^\infty \left(1+\ln\frac{t}{s}\right) \frac{\operatorname*{ess\,inf}_{s<r<\infty} \varphi_1(x,r) r^{\theta_p(x,r)}}{s^{\theta_p(x,s)}}\frac{ds}{s} < \infty \quad (29)$$

for every δ_0.

Then the operator $[b,S]$ is bounded from the space $V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$ to the space $V\mathcal{M}^{p(\cdot),\varphi_2}(\Omega)$.

Proof. The norm inequalities follow from Theorem 11, so we only have to prove that if

$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0\Rightarrow$$
$$\lim_{r\to 0}\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|[b,S]f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}=0 \tag{30}$$

otherwise.

To show that $\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|[b,S]f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}<\varepsilon$ for small r, we split the right-hand side of (23):

$$\sup_{x\in\Omega}\frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|[b,S]f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}\le C_0\left(I_{1,\delta_0}(x,r)+I_{2,\delta_0}(x,r)\right), \tag{31}$$

where $\delta_0>0$ will be chosen as shown below (we may take $\delta_0<1$),

$$I_{1,\delta_0}(x,r):=\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^{\delta_0}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

$$I_{2,\delta_0}(x,r):=\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_{\delta_0}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{q(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},$$

and it is supposed that $r<\delta_0$. Now we choose any fixed $\delta_0>0$ such that

$$\sup_{x\in\Omega}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}<\frac{\varepsilon}{2CC_0\|b\|_*},\ \text{for all } 0<t<\delta_0,$$

where C and C_0 are constants from (28) and (31), which is possible since $f\in V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)$. Then

$$\sup_{x\in\Omega}CI_{1,\delta_0}(x,r)<\frac{\varepsilon}{2},\ \ 0<r<\delta_0,$$

by (30).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (10). We have

$$I_{2,\delta_0}(x,r)\le C_{\delta_0}\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)},$$

where C_{δ_0} is the constant from (29). Then, by (10) it suffices to choose r small enough such that

$$\frac{\varphi_2(x,r)}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}<\frac{\varepsilon}{2CC_\delta\|b\|_*\|f\|_{V\mathcal{M}^{p(\cdot),\varphi_1}(\Omega)}}$$

which completes the proof of (30).

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