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A New Proof of Laguerre's Theorem

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Abstract. Using Möbius transformation and its characterstics we have obtained a different proof of well known Laguerre's theorem on the zeros of the polar derivative of a polynomial.

Key Words and Phrases: polar derivative of a polynomial, zeros, circular region, Laguerre's theorem, Möbius transformation

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1. Introduction

Concerning relative location of zeros of a polynomial and its polar derivative we have the following well known interesting theorem due to Laguerre ([2], [3, p. 49]).

Laguerre's theorem. *If all the zeros z_j of the n^{th} degree polynomial $f(z)$ lie in a circular region C and if Z is any zero of*

$$f_1(z) = nf(z) + (\zeta - z)f'(z),$$

the polar derivative of $f(z)$, then not both points Z and ζ may lie outside of C. Furthermore, if $f(Z) \neq 0$, then any circle K through Z and ζ either passes through all the zeros of $f(z)$ or separates these zeros.

In the literature there exists certain other proof [1] of Laguerre's theorem. In this paper we have used Möbius transformation and its characteristics ([4, chapter 10], [5, chapter 5]) to obtain a different proof of Laguerre's theorem.

2. Lemmas

For the proof of the Laguerre's theorem we require the following lemmas.

Lemma 1. *If each complex number $w_j, j = 1, 2, \ldots, p$, has the properties that $w_j \neq 0$ and*

$$\gamma \leq \ Arg\ w_j < \gamma + \pi,\ \ j = 1, 2, \ldots, p,$$

where γ is a real constant, then their sum $w = \sum_{j=1}^{p} w_j$ can not vanish.

This lemma is due to Marden [3, Theorem (1,1)].

Lemma 2. *If each complex number $w_j, j = 1, 2, \dots, p$, has the properties that $w_j \neq 0$ and*

$$\gamma < \ Arg\ w_j < \gamma + \pi,\ \textit{for at least one } j, j = 1, 2, \dots, p, \tag{1}$$

with

$$\gamma \leq \ Arg\ w_j \leq \gamma + \pi,\ \textit{for remaining } j\,'s, j = 1, 2, \dots, p, \tag{2}$$

where γ is a real constant, then their sum $w = \sum_{j=1}^{p} w_j$ can not vanish.

Proof of Lemma 2. We begin with the case

$$\gamma = 0.$$

Then

$$Im\ w_j > 0,\ \text{for at least one } j, j = 1, 2, \dots, p,\ (\text{by } (1)),$$

with

$$Im\ w_j \geq 0,\ \text{for remaining } j\,'s, j = 1, 2, \dots, p,\ (\text{by } (2)).$$

Therefore

$$Im\ w > 0$$

and accordingly

$$w \neq 0.$$

In the case that

$$\gamma \neq 0,$$

we may consider the quantities

$$w_j' = w_j e^{-\gamma i}, j = 1, 2, \dots, p.$$

These satisfy inequalities (1) and (2) with

$$\gamma = 0$$

and consequently their sum w' does not vanish. As

$$w' = e^{-i\gamma} w,$$

it follows that

$$w \neq 0.$$

This completes the proof of Lemma 2.

3. Proof of Laguerre's theorem

To prove the first part of the theroem we can assume that

$$f(Z) \neq 0 \tag{3}$$

(as for the possibility

$$f(Z) = 0,$$

proof is trivial). As

$$0 = f_1(Z) = (\zeta - Z)f'(Z) + nf(Z), \tag{4}$$

we get by using (3) that

$$\zeta \neq Z \tag{5}$$

and as

$$f(z) = \prod_{j=1}^{p}(z - z_j)^{m_j}, \sum_{j=1}^{p} m_j = n, \tag{6}$$

we get by using (3), (4) and (5) that

$$0 = \frac{f_1(Z)}{f(Z)(\zeta - Z)} = \frac{f'(Z)}{f(Z)} + \frac{n}{\zeta - Z},$$

i.e.

$$\frac{n}{Z - \zeta} = \sum_{j=1}^{p} \frac{m_j}{Z - z_j} \text{ (by (6)).} \tag{7}$$

By using the symbols

$$w = \frac{1}{Z - \zeta} \tag{8}$$

$$w_j = \frac{1}{Z - z_j}, j = 1, 2, \ldots, p, \tag{9}$$

and (6), (7) can be rewritten as

$$\sum_{j=1}^{p} m_j(w_j - w) = 0. \tag{10}$$

For proving first part of the theorem we have to show that not both points Z and ζ may lie outside of C. On the contrary, we assume that both points Z and ζ are outside of C. We now consider Möbius transformation

$$\tau = g(z) = \frac{1}{Z - z}. \tag{11}$$

Let γ be the boundary of the circular region C and let Γ be the image of γ under the transformation (11). As γ is a straight line or a circle, Γ will also be a straight line or a circle. Accordingly we think of two possibilities:
(i) Γ is a circle. Therefore

$$Z \notin \gamma$$

and as

$$g(Z) = \infty \in \text{ domain (known as exterior of } \Gamma \text{ and represented by the symbol } E(\Gamma))\,, \tag{12}$$

we can say that

$$g(C) = \text{ domain (known as interior of } \Gamma \text{ and represented by the symbol } I(\Gamma)) \tag{13}$$

or

$$g(C) = \overline{I(\Gamma)} \tag{14}$$

which imply that

$$w_j \in I(\Gamma), j = 1, 2, \ldots, p, \text{ (by (6), (9) and (11))} \tag{15}$$

or

$$w_j \in \overline{I(\Gamma)}, j = 1, 2, \ldots, p, \text{ (by (6), (9) and (11))} \tag{16}$$

respectively. Further by (12) and by our assumption that both points Z and ζ lie outside of C, we can say that

$$w \in E(\Gamma), \text{ (by (8) and (11))} \tag{17}$$

or

$$w \in \Gamma, \text{ (by (8) and (11)).} \tag{18}$$

(Please note that w can belong to Γ only when (15) happens but (16) does not happen.) (19)

Now by (15), (16), (17), (18) and (19) we can say that there will definitely exist a real number η such that

$$\eta < \ Arg\ (w_j - w) < \eta + \pi,\ j = 1, 2, \ldots, p \tag{20}$$

and therefore by Lemma 1 we can say that

$$\sum_{j=1}^{p} m_j(w_j - w) \neq 0$$

which contradicts the fact represented by (10). Hence our assumption that both points Z and ζ are outside of C should be wrong and we can conclude for the possibility under consideration that not both points Z and ζ may lie outside of C.

(ii) Γ *is a straight line.* Therefore

$$Z \in \gamma \tag{21}$$

and

$$g(C) = \text{domain (known as an open half plane with boundary } \Gamma) . \tag{22}$$

Now (22) helps us to say that

$$\left.\begin{array}{l} \quad w_j \in \text{ half plane } g(C) \text{ with boundary } \Gamma, j = 1, 2, \ldots, p, \\ \text{with} \\ \quad w_j \notin \Gamma, \; j = 1, 2, \ldots, p, \end{array}\right\}, (\text{by } (6), (9) \;\&\; (11)). \tag{23}$$

Further by (21) and by our assumption that both points Z and ζ lie outside of C, we can say that

$w \in$ domain (known as second half plane with boundary Γ and different from half plane $g(C)$),

$$(\text{by } (8) \;\&\; (11)) \tag{24}$$

or

$$w \in \Gamma, \; (\text{by } (8) \text{ and } (11)). \tag{25}$$

By (23), (24) and (25) we can say that there will definitely exist a real number δ such that

$$\delta < \; Arg \; (w_j - w) < \delta + \pi, \; j = 1, 2, \ldots, p$$

and now the proof of the first part of the theorem for the present possibility can be completed similar to the proof of the first part of the theorem for the possibility (i) after expression (20). This completes the proof of the first part of the theorem.

To prove the second part of the theorem we are given that

$$f(Z) \neq 0$$

and therefore (10) is still true. We now assume that a circle K through Z and ζ has at least one z_j in its interior, no z_j in its exterior and the remaining $z_j\,'s$ on its circumference. Under Möbius transformation (11), K will be transformed onto a straight line Γ_0, with

$$g(I(K)) = \text{ an open half plane with boundary } \Gamma_0. \tag{26}$$

(26) and our assumption help us to say that

$$w_j \;\; \in \;\; \text{open half plane } g(I(K)) \text{ with boundary } \Gamma_0, \text{ for}$$

$$\text{at least one } j, 1 \leq j \leq p, \ \text{(by } (6), (9) \ \& \ (11)), \tag{27}$$

with

$$w_j \in \Gamma_0 \text{ for remaining } j\ 's, 1 \leq j \leq p, \ \text{(by } (6), (9) \ \& \ (11)). \tag{28}$$

Further as

$$\zeta \in K,$$

we have

$$w \in \Gamma_0, \ \text{(by } (8) \text{ and } (11)). \tag{29}$$

Now by (27), (28) and (29) we can say that there will exist a real number α such that

$$\alpha < \ Arg\ (w_j - w) < \alpha + \pi, \ \text{for at least one } j, 1 \leq j \leq p,$$

with

$$\alpha \leq \ Arg\ (w_j - w) \leq \alpha + \pi, \ \text{for remaining } j\ 's, 1 \leq j \leq p.$$

(It should be noted here that $(w_j - w)$ may vanish for certain $j\ 's, (1 \leq j \leq p)$). Therefore by Lemma 2 we can say that

$$\sum_{j=1}^{p} m_j(w - w_j) \neq 0$$

which contradicts the fact represented by (10). Hence our assumption that a circle K through Z and ζ has at least one z_j in its interior, no z_j in its exterior and the remaining $z_j\ 's$ on its circumference, should be wrong. One can similarly show that the assumption that a circle K through Z and ζ has at least one z_j in its exterior, no z_j in its interior and the remaining $z_j\ 's$ on its circumference will be wrong. Therefore we conclude that any circle K through Z and ζ must separate $z_j\ 's$ unless it passes through all of them. This completes the proof of the second part of the theorem, thereby completing the proof of the theorem also.

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