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Boundary control by the displacement for the telegraph equation with a variable coefficient and the Neumann boundary condition

Kritskov L. V., Abdukarimov M. F.

Abstract. A problem of the boundary control by the displacement at the point $x = 0$ with the Neumann condition at the point $x = l$ is considered for the process which is described by the telegraph equation with a variable coefficient on the finite interval $0 \leqslant x \leqslant l$. For the critical time period $T = 2l$ a necessary and sufficient condition for the existence of a unique boundary function $\mu(t) = u(0, t)$ which transfers the process from any initial state at $t = 0$ to any terminal state at $t = T$ is given.

Key Words and Phrases: Boundary control, telegraph equation with a variable coefficient

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1. Introduction

In this paper we study a problem of the boundary control by the displacement at one endpoint for the process which is described by the one-dimensional telegraph equation with a variable coefficient

$$\mathcal{L}u \equiv u_{tt}(x,t) - u_{xx}(x,t) - q(x,t)u(x,t) = 0, \quad 0 < x < l, \tag{1}$$

assuming that, on the other endpoint $x = l$, the homogeneous Neumann condition $u_x(l, t) \equiv 0$ holds for all $t \in [0, T]$. The coefficient $q(x, t)$ in (1) is supposed to be a bounded and measurable function in the rectangle $Q_T = [0 \leqslant x \leqslant l] \times [0 \leqslant t \leqslant T]$.

The goal of this paper is to obtain the existence of a unique boundary control at the end-point $x = 0$: $\mu(t) = u(0, t)$ that transfers the process from any initial state $\{u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x)\}$ at $t = 0$ to any terminal state $\{u(x, T) = \varphi_1(x), u_t(x, T) = \psi_1(x)\}$ at $t = T$ in the case when $T = 2l$. It is supposed that $u(x, t)$ satisfies Eq. (1) in the generalized sense (see Section 1) and has a finite energy.

Investigations of a similar problem for the one-dimensional wave equation ($q(x, t) \equiv 0$) in [1-3] showed that the time period $T = 2l$ of the boundary control's action at one endpoint is critical in the following sense. When $T = 2l$ the boundary control is defined uniquely for a rather wide class of initial and terminal data, while in the case $T > 2l$ the boundary control is not unique and in the case $T < 2l$ the control's existence demands the initial and terminal data satisfy restrictive additional conditions.

Note also that certain boundary control problems for Eq. (1) with a constant coefficient $q(x,t) \equiv -c^2$ are studied in [4-6]. Existence of the boundary control for general hyperbolic equations is considered in [7-12] in the case when the time period T exceeds its critical value.

This paper is the development of results announced in [13].

2. Main definitions

In order to define the notion of a generalized solution, we use the classes $\widehat{W}_2^1(Q_T)$ and $\widehat{W}_2^2(Q_T)$ which are quiet natural for hyperbolic equations (see, e.g., [1,2]). Let us consider the following three problems for Eq. (1) in the rectangle Q_T:

– the initial boundary-value problem I with conditions

$$u(0,t) = \mu(t), \;\; u_x(l,t) \equiv 0 \quad \text{for} \quad 0 \leqslant t \leqslant T, \tag{2}$$

$$u(x,0) = \varphi(x), \;\; u_t(x,0) = \psi(x) \quad \text{for} \quad 0 \leqslant x \leqslant l \tag{3}$$

where $\mu(t) \in W_2^1[0,T]$, $\varphi(x) \in W_2^1[0,l]$, $\psi(x) \in L_2[0,l]$ and the compatibility condition $\mu(0) = \varphi(0)$ is satisfied;

– the initial boundary-value problem II with conditions (2) for $0 \leqslant t \leqslant T$ and conditions

$$u(x,T) = \varphi_1(x), \;\; u_t(x,T) = \psi_1(x) \quad \text{for} \quad 0 \leqslant x \leqslant l, \tag{4}$$

where $\mu(t) \in W_2^1[0,T]$, $\varphi_1(x) \in W_2^1[0,l]$, $\psi_1(x) \in L_2[0,l]$ and the compatibility condition $\mu(T) = \varphi_1(0)$ is satisfied;

– the boundary control problem III with the Neumann condition $u_x(l,t) \equiv 0$ for $0 \leqslant t \leqslant T$, with the initial data (3) and the terminal data (4) where $\varphi(x), \varphi_1(x) \in W_2^1[0,l]$, $\psi(x), \psi_1(x) \in L_2[0,l]$.

The function $u(x,t)$ is called *the solution from the class $\widehat{W}_2^1(Q_T)$ to the problem* I if it belongs to this class and the identity

$$\int_0^l \int_0^T u(x,t)\mathcal{L}\Phi(x,t)\,dxdt + \int_0^l [\varphi(x)\Phi_t(x,0) - \psi(x)\Phi(x,0)]\,dx - \int_0^T \mu(t)\Phi_x(0,t)\,dt = 0 \tag{5}$$

holds for any test function $\Phi(x,t) \in \widehat{W}_2^2(Q_T)$ which satisfies the conditions $\Phi(0,t) = \Phi_x(l,t) \equiv 0$ for $0 \leqslant t \leqslant T$ and $\Phi(x,T) = \Phi_t(x,T) \equiv 0$ for $0 \leqslant x \leqslant l$.

Analogously the function $u(x,t)$ is called *the solution from the class $\widehat{W}_2^1(Q_T)$ to the problem* II if it belongs to this class and the identity similar to (5) holds* for any test

*The second integral in (5) should be substituted by $-\int_0^l [\varphi_1(x)\Phi_t(x,T) - \psi_1(x)\Phi(x,T)]\,dx$.

function $\Phi(x,t) \in \widehat{W}_2^2(Q_T)$ which satisfies the conditions $\Phi(0,t) = \Phi_x(l,t) \equiv 0$ for $0 \leqslant t \leqslant T$ and $\Phi(x,0) = \Phi_t(x,0) \equiv 0$ for $0 \leqslant x \leqslant l$.

The solution to the initial boundary value problem I $u(x,t)$ is called *the solution from the class* $\widehat{W}_2^1(Q_T)$ *to the boundary control problem* III if the boundary function $\mu(t)$ enables the first terminal condition (4) to hold pointwise and the second terminal condition (4) to hold almost everywhere on $[0,l]$.

It is easy to check (see [4]) that if a function $u(x,t)$ is the solution from $\widehat{W}_2^1(Q_T)$ to the problem I then the function $u_1(x,t) = u(x,T-t)$ gives a solution from the same class to the problem II with the coefficient $q(x,t)$ in (1) substituted by $q(x,T-t)$, the function $\mu(t)$ in (2) – by $\mu(T-t)$ and with $\varphi_1(x) = \varphi(x)$, $\psi_1(x) = -\psi(x)$ in (4).

3. Auxiliary statements

Applying the technique of [13, c.163-165] one can obtain the following

Assertion 1. *Let $T > 0$ and the coefficient $q(x,t)$ in Eq. (1) be bounded and measurable in Q_T. Then both initial boundary-value problems I and II have at most one solution from the class $\widehat{W}_2^1(Q_T)$.*

In what follows we use the notation $\underline{\mu}(t)$ for the function that coincides with $\mu(t)$ for $t \geq 0$ and vanishes for all $t < 0$. Hence $\underline{\mu}(t) \in W_2^1[-\varepsilon, T]$ for any $\varepsilon > 0$.

Assertion 2. *Let $T \leqslant 2l$. Then the unique solution from $\widehat{W}_2^1(Q_T)$ to the problem I with $\varphi(x) \equiv 0$ for $x \in [0,l]$, $\psi(x) = 0$ a.e. on $[0,l]$ and an arbitrary function $\mu(t) \in W_2^1[0,T]$ such that $\mu(0) = 0$, satisfies the relation*

$$u(x,t) = \underline{\mu}(t-x) + \frac{1}{2}\int\limits_0^t \int\limits_{|x-t+\tau|}^{l-|x+t-\tau-l|} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau +$$

$$+ \int\limits_0^{\max\{0,x+t-l\}} \int\limits_{2l-x-t+\tau}^{l} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{6}$$

in the case $0 \leqslant t \leqslant l$ and the relation

$$u(x,t) = \underline{\mu}(t-x) + \underline{\mu}(t+x-2l) + \frac{1}{2}\int\limits_0^{t-l} \int\limits_{|2l-x-t+\tau|}^{l-|t-x-l-\tau|} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau +$$

$$+\frac{1}{2}\int\limits_{t-l}^{t} \int\limits_{|x-t+\tau|}^{l-|t+x-l-\tau|} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau + \int\limits_{\max\{0,t-x-l\}}^{x+t-l} \int\limits_{l-x+|l-t+\tau|}^{l} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{7}$$

in the case $l \leqslant t \leqslant T$.

Proof. In order to obtain relations (6) and (7), it is sufficient to construct the solution from the class $\widehat{W}_2^1(Q_T)$ to the following initial boundary-value problem for the inhomogeneous wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in Q_T, \\ u(x,0) = 0, \ u_t(x,0) = 0 \text{ for } 0 \leqslant x \leqslant l, \\ u(0,t) = \mu(t), \ u_x(l,t) = 0 \text{ for } 0 \leqslant t \leqslant T \end{cases} \tag{8}$$

and take $f(x,t) = q(x,t)u(x,t)$.

The solution to (8) is a sum of the solution $\underline{\mu}(t-x) + \underline{\mu}(t+x-2l)$ to the similar problem for the homogeneous wave equation (see Assertion 2 in [3]) and the solution to the problem (8) with zero initial and boundary data. The latter solution is given by the well-known integral $F(x,t) = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\, d\xi d\tau$ in which the integrand coincides with $f(x,t)$ inside Q_T and is the odd extension of $f(x,t)$ over the boundary $x=0$ and its even extension over the boundary $x=l$. One can easily show that if $f(x,t) \in L_2(Q_T)$ then $F(x,t)$ is a unique solution from $\widehat{W}_2^1(Q_T)$ to the corresponding problem. Using the symmetric properties of the continued function $f(x,t)$ one can refine the bounds of integration for $F(x,t)$ that immediately leads to relations (6) and (7). ∎

Assertion 3. *Let $T \leqslant 2l$. Then the solution $u(x,t)$ from $\widehat{W}_2^1(Q_T)$ to the problem I with $\varphi(x) \equiv 0$ for $x \in [0,l]$, $\psi(x) = 0$ a.e. on $[0,l]$ and an arbitrary function $\mu(t) \in W_2^1[0,T]$ such that $\mu(0) = 0$, is defined uniquely; moreover, $u(x,t) \equiv 0$ in the domain $\{(x,t) \mid 0 \leqslant t \leqslant l, t \leqslant x \leqslant l\} \bigcap Q_T$.*

Proof. All the values $T \in (0,2l]$ are discussed similarly. Thus for the sake of brevity let us confine ourselves to the case $T = 2l$. Let the rectangle Q_{2l} be subdivided into seven parts by the characteristic lines starting from its vertices. In these domains:

– the triangle $\Delta_1 = \{(x,t) \mid 0 \leqslant t \leqslant l/2, \ t \leqslant x \leqslant l-t\}$ which is adjacent to the lower base of Q_{2l},

– the triangles $\Delta_2 = \{(x,t) \mid 0 \leqslant t \leqslant l, \ 0 \leqslant x \leqslant l/2 - |t-l/2|\}$ and $\Delta_3 = \{(x,t) \mid 0 \leqslant t \leqslant l, \ l/2 + |t-l/2| \leqslant x \leqslant l\}$ which are adjacent to the lateral sides of Q_{2l} in its lower half,

– the square $\Delta_4 = \{(x,t) \mid l/2 \leqslant t \leqslant 3l/2, \ |t-l| \leqslant x \leqslant l - |t-l|\}$,

– the triangles $\Delta_5 = \{(x,t) \mid l \leqslant t \leqslant 2l, \ 0 \leqslant x \leqslant l/2 - |t-3l/2|\}$ and $\Delta_6 = \{(x,t) \mid l \leqslant t \leqslant 2l, \ l/2 + |t-3l/2| \leqslant x \leqslant l\}$ which are adjacent to the lateral sides of Q_{2l} in its upper half, and

– the triangle $\Delta_7 = \{(x,t) \mid 3l/2 \leqslant t \leqslant 2l, \ 2l-t \leqslant x \leqslant t-l\}$ which is adjacent to the upper base of Q_{2l},

let us assign $u_j(x,t) = u(x,t)$ for $(x,t) \in \Delta_j$, $j = \overline{1,7}$, and consider Eqs. (6) and (7) successively for the points $(x,t) \in \Delta_1, \Delta_2, ..., \Delta_7$ as integral equations for obtaining $u_1(x,t)$, $u_2(x,t)$, ..., $u_7(x,t)$.

In the case $j=1$ Eq. (6) leads to the following equation for $u_1(x,t)$:

$$u_1(x,t) = \frac{1}{2} \iint\limits_{D_1} q(\xi,\tau) u_1(\xi,\tau)\, d\xi d\tau \tag{9}$$

where $D_1(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant t,\ x-t+\tau \leqslant \xi \leqslant x+t-\tau\}$.

Rewriting Eq. (9) in the operator form $u_1(x,t) = [\mathcal{N}_1 u_1](x,t)$ one can note that the operator $\mathcal{N}_1$ is bounded in $L_\infty(\Delta_1)$ and its powers $\mathcal{N}_1^k$ satisfy the estimate†

$$\left|[\mathcal{N}_1^k \chi](x,t)\right| \leqslant \|q\|_\infty^k \frac{t^{2k}}{(2k)!} \sup_{(x,t)\in\Delta_1} |\chi(x,t)|, \quad (x,t) \in \Delta_1. \tag{10}$$

Thus Eq. (9) is a homogeneous Volterra-type integral equation of the second kind and has only the trivial solution.

As $u_1(x,t) \equiv 0$ and $\underline{\mu}(t-x) \equiv 0$ for $(x,t) \in \Delta_3$, the function $u_3(x,t)$ satisfies the equation similar to (9):

$$u_3(x,t) = \frac{1}{2}\left(\iint\limits_{D_3'} + \iint\limits_{D_3''}\right) q(\xi,\tau) u_3(\xi,\tau)\, d\xi d\tau \tag{11}$$

where the quadrangle $D_3'(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant t,\ (l+x-t)/2 + |\tau-(l-x+t)/2| \leqslant \xi \leqslant (l+x+t-\tau)/2 - |(\tau-x-t+l)/2|\}$ and the triangle $D_3''(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant x+t-l,\ (3l-x-t)/2 + |\tau-(x+t-l)/2| \leqslant \xi \leqslant l\}$ lay entirely in Δ_3.

Since the integral operator $\mathcal{N}_3$ in the right-hand side of (11) is bounded in $L_\infty(\Delta_3)$ and satisfies the estimate

$$\left|[\mathcal{N}_3^k \chi](x,t)\right| \leqslant (2\|q\|_\infty)^k \frac{t^{2k}}{(2k)!} \sup_{(x,t)\in\Delta_3} |\chi(x,t)|, \quad (x,t) \in \Delta_3, \tag{12}$$

one similarly obtains that $u_3(x,t) \equiv 0$.

As in the domain Δ_2: $u_1(x,t) \equiv 0$, Eq. (6) there takes the form

$$u_2(x,t) = \mu(t-x) + \frac{1}{2}\iint\limits_{D_2} q(\xi,\tau) u_2(\xi,\tau)\, d\xi d\tau \tag{13}$$

where $D_2(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant t,\ |x-t+\tau| \leqslant \xi \leqslant (x+t)/2 - |\tau-(x+t)/2|\} \subset \Delta_2$. As one rewrites Eq. (13) in the operator form $u_2(x,t) = \mu(t-x) + [\mathcal{N}_2 u_2](x,t)$, it is clear that $\mu(t-x)$ is bounded in Δ_2 and the bounded in $L_\infty(\Delta_2)$ operator $\mathcal{N}_2$ satisfies the estimate

$$\left|[\mathcal{N}_2^k \chi](x,t)\right| \leqslant (\|q\|_\infty/2)^k \frac{t^{2k}}{(2k-1)!!} \sup_{(x,t)\in\Delta_2} |\chi(x,t)|, \quad (x,t) \in \Delta_2. \tag{14}$$

Therefore Eq. (13) has the solution which equals the absolutely convergent Neumann series $u_2(x,t) = (I + \sum_{k=1}^{\infty} \mathcal{N}_2^k)\mu(t-x)$ and is bounded in Δ_2.

Now let $(x,t) \in \Delta_4$. As $u_1(x,t) \equiv 0$, $u_3(x,t) \equiv 0$, Eqs. (6) and (7) yield

$$u_4(x,t) = \mu(t-x) + \frac{1}{2}\iint\limits_{D_4'} q(\xi,\tau) u_2(\xi,\tau)\, d\xi d\tau + \frac{1}{2}\iint\limits_{D_4} q(\xi,\tau) u_4(\xi,\tau)\, d\xi d\tau \tag{15}$$

†By $\|q\|_\infty$ we denote the norm $\operatorname{ess\,sup}_{(x,t)\in Q_{2l}} |q(x,t)|$.

where $D_4'(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (l+t-x)/2,\ |x-t+\tau| \leqslant \xi \leqslant l/2 - |\tau - l/2|\}$, $D_4(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant t,\ (l-t+x)/2 + |(l+t-x)/2 - \tau| \leqslant \xi \leqslant (t+x)/2 - |(t+x)/2 - \tau|\}$. Since $D_4' \subset \Delta_2$, the first integral term on the right-hand side of (15) is the known bounded function. As $D_4 \subset \Delta_4$, Eq. (15) can be treated as an integral equation for $u_4(x,t)$ in which the operator $[\mathcal{N}_4\chi](x,t) \equiv \frac{1}{2}\iint_{D_4} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ in the right-hand side is bounded in $L_\infty(\Delta_4)$ and satisfies the estimate

$$\left|[\mathcal{N}_4^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k \,\frac{(t-x)^k}{2k!} \sup_{(x,t)\in\Delta_4} |\chi(x,t)|, \quad (x,t) \in \Delta_4. \tag{16}$$

Similar to (13), the estimates (16) yield that Eq. (15) has a bounded in Δ_4 solution $u_4(x,t)$.

The solution $u(x,t)$ in the remaining domains Δ_5, Δ_6, Δ_7 is constructed using the same approach. Let us list only the related integral equations and corresponding estimates.

In the triangle Δ_5, one obtains the equation

$$u_5(x,t) = \mu(t-x) + \frac{1}{2}\iint\limits_{D_5'} q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau + \frac{1}{2}\iint\limits_{D_5} q(\xi,\tau)u_5(\xi,\tau)\,d\xi d\tau \tag{17}$$

where $D_5'(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (l+t+x)/2,\ (t-x-l)/2 + |(t-x+l)/2 - \tau| \leqslant \xi \leqslant (t+x)/2 - |(t+x)/2 - \tau|\} \subset \Delta_4$, $D_5(x,t) = \{(\xi,\tau) \mid (t-x+l)/2 \leqslant \tau \leqslant t,\ |x-t+\tau| \leqslant \xi \leqslant (t+x-l)/2 - |(t+x+l)/2 - \tau|\} \subset \Delta_5$, and the integral operator $[\mathcal{N}_5\chi](x,t) \equiv \frac{1}{2}\iint_{D_5} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ satisfies the estimate

$$\left|[\mathcal{N}_5^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/4)^k \,\frac{(t-x-l)^k}{k!} \sup_{(x,t)\in\Delta_5} |\chi(x,t)|, \quad (x,t) \in \Delta_5. \tag{18}$$

In the triangle Δ_6, one obtains the equation

$$u_6(x,t) = \mu(t-x) + \mu(t+x-2l) + \frac{1}{2}\left(\iint\limits_{D_{62}'} + \iint\limits_{D_{62}''}\right) q(\xi,\tau)u_2(\xi,\tau)\,d\xi d\tau +$$

$$+\frac{1}{2}\left(\iint\limits_{D_{64}'} + \iint\limits_{D_{64}''}\right) q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau + \frac{1}{2}\left(\iint\limits_{D_6'} + \iint\limits_{D_6''}\right) q(\xi,\tau)u_6(\xi,\tau)\,d\xi d\tau \tag{19}$$

where $D_{62}'(x,t) = \{(\xi,\tau) \mid (t+x-2l)/2 \leqslant \tau \leqslant (t+x-l)/2,\ |x+t-2l-\tau| \leqslant \xi \leqslant l/2 - |l/2 - \tau|\} \subset \Delta_2$, $D_{62}''(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (l+t-x)/2,\ |x-t+\tau| \leqslant \xi \leqslant l/2 - |l/2-\tau|\} \subset \Delta_2$, $D_{64}'(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant (2l-x+t)/2,\ (l+x-t)/2 + |(l-x+t)/2 - \tau| \leqslant \xi \leqslant l - |l-\tau|\} \subset \Delta_4$, $D_{64}''(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant (x+t)/2,\ (3l-x-t)/2 + |(x+t-l)/2 - \tau| \leqslant \xi \leqslant l - |l-\tau|\} \subset \Delta_4$, $D_6'(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant x+t-l,\ (4l-x-t)/2 + |(x+t)/2 - \tau| \leqslant \xi \leqslant l\} \subset \Delta_6$, $D_6''(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant$

$t,\ (2l+x-t)/2+|(2l-x+t)/2-\tau| \leqslant \xi \leqslant (l+x+t-\tau/)2-|(\tau-x-t+l)/2|\} \subset \Delta_6$, and the integral operator $[\mathcal{N}_6\chi](x,t) \equiv \frac{1}{2}(\iint_{D_6'} + \iint_{D_6''})q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ satisfies the estimate

$$\left|[\mathcal{N}_6^k\chi](x,t)\right| \leqslant (l\|q\|_\infty)^k \,\frac{(x+t-2l)^k}{k!} \sup_{(x,t)\in\Delta_6} |\chi(x,t)|, \quad (x,t)\in\Delta_6. \tag{20}$$

And finally, in the triangle Δ_7, the following equation holds:

$$\begin{aligned} u_7(x,t) = \mu(t-x) + \mu(t+x-2l) + \frac{1}{2}\iint\limits_{D_{72}} q(\xi,\tau)u_2(\xi,\tau)\,d\xi d\tau + \\ +\frac{1}{2}\left(\iint\limits_{D_{74}'} + \iint\limits_{D_{74}''}\right) q(\xi,\tau)u_4(\xi,\tau)\,d\xi d\tau + \frac{1}{2}\iint\limits_{D_{75}} q(\xi,\tau)u_5(\xi,\tau)\,d\xi d\tau + \\ +\frac{1}{2}\left(\iint\limits_{D_{76}'} + \iint\limits_{D_{76}''}\right) q(\xi,\tau)u_6(\xi,\tau)\,d\xi d\tau + \frac{1}{2}\iint\limits_{D_7} q(\xi,\tau)u_7(\xi,\tau)\,d\xi d\tau \end{aligned} \tag{21}$$

where $D_{72}(x,t) = \{(\xi,\tau) \mid (t+x-2l)/2 \leqslant \tau \leqslant (x+t-l)/2,\ |x+t-2l-\tau| \leqslant \xi \leqslant l/2-|l/2-\tau|\} \subset \Delta_2$, $D_{74}'(x,t) = \{(\xi,\tau) \mid l/2 \leqslant \tau \leqslant (x+t)/2,\ (3l-x-t)/2+|(x+t-l)/2-\tau| \leqslant \xi \leqslant l-|l-\tau|\} \subset \Delta_4$, $D_{74}''(x,t) = \{(\xi,\tau) \mid (t-x)/2 \leqslant \tau \leqslant (3l)/2,\ (t-x-l)/2+|(t-x+l)/2-\tau| \leqslant \xi \leqslant l-|l-\tau|\} \subset \Delta_4$, $D_{75}(x,t) = \{(\xi,\tau) \mid (t-x+l)/2 \leqslant \tau \leqslant (2l-x+t)/2,\ |x-t+\tau| \leqslant \xi \leqslant l/2-|3l/2-\tau|\} \subset \Delta_5$, $D_{76}'(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant (x+t+l)/2,\ l/2+|(3l)/2-\tau| \leqslant \xi \leqslant (l+x+t-\tau)/2-|(t+x-l-\tau)/2|\} \subset \Delta_6$, $D_{76}''(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant x+t-l,\ (4l-x-t)/2+|(x+t)/2-\tau| \leqslant \xi \leqslant l\} \subset \Delta_6$, $D_7(x,t) = \{(\xi,\tau) \mid 3l/2 \leqslant \tau \leqslant t,\ (2l+x-t)/2+|(2l+t-x)/2-\tau| \leqslant \xi \leqslant (x+t-l)/2-|(x+t+l)/2-\tau|\} \subset \Delta_7$, and the integral operator $[\mathcal{N}_7\chi](x,t) \equiv \frac{1}{2}\iint_{D_7} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ satisfies the estimate

$$\left|[\mathcal{N}_7^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/4)^k \,\frac{(t+x-2l)^k}{k!} \sup_{(x,t)\in\Delta_7} |\chi(x,t)|, \quad (x,t)\in\Delta_7. \tag{22}$$

Thus it is proved that Eqs. (6) and (7) have the bounded solution $u(x,t)$ in the rectangle Q_T. The next step is to study its smoothness.

As all the terms in the right-hand sides of (6) and (7) are continuous with respect to (x,t) in Q_T, the obtained solution $u(x,t)$ is also continuous in Q_T. For the sake of simplicity, let us introduce the bounded function $U(x,t) \equiv q(x,t)u(x,t)$ in Q_T and make its odd extension over $x=0$ and its even extension over $x=l$. Then Eqs. (6) and (7) can be coupled into one equation

$$u(x,t) = \underline{\mu}(t-x) + \underline{\mu}(t+x-2l) + \frac{1}{2}\int\limits_0^t \int\limits_{x-t+\tau}^{x+t-\tau} U(\xi,\tau)\,d\xi d\tau. \tag{23}$$

By the straightforward differentiation, Eq. (23) yields that, a.e. in Q_T,

$$u_x(x,t) = -\underline{\mu}'(t-x) + \underline{\mu}'(t+x-2l) + \frac{1}{2}\int_0^t [U(x+t-\tau,\tau) - U(x-t+\tau,\tau)]\,d\tau, \quad (24)$$

$$u_t(x,t) = \underline{\mu}'(t-x) + \underline{\mu}'(t+x-2l) + \frac{1}{2}\int_0^t [U(x+t-\tau,\tau) + U(x-t+\tau,\tau)]\,d\tau, \quad (25)$$

and therefore the derivatives $u_x(x,t)$ and $u_t(x,t)$ belong to $L_2(0 \leqslant x \leqslant l)$ for all $t \in [0,2l]$ and to $L_2(0 \leqslant t \leqslant 2l)$ for all $x \in [0,l]$. ∎

Remark. The direct analysis of the Neumann series for the solutions of Eqs. (9), (11), (13), (15), (17), (19), (21) and the inequalities (10), (12), (14), (16), (18), (20), (22) lead to the estimate

$$\max_{(x,t)\in Q_T} |u(x,t) - u^*(x,t)| \leqslant C\|q\|_\infty \quad (26)$$

where $u^*(x,t) = \underline{\mu}(t-x)+\underline{\mu}(t+x-2l)$ is the solution from $\widehat{W}_2^1(Q_T)$ to the initial boundary-value problem I for the homogeneous wave equation with zero initial data in the case when $T \leqslant 2l$ (see the proof of Assertion 2).

Together with Eqs. (24) and (25), it means that

$$\|u_x - u_x^*\|_{L_2(Q_T)} + \|u_t - u_t^*\|_{L_2(Q_T)} \leqslant C\|q\|_\infty. \quad (27)$$

Thus, combining (26) and (27), one obtains the estimate

$$\|u - u^*\|_{W_2^1(Q_T)} \leqslant C\|q\|_\infty. \quad (28)$$

For the problem II the following similar statement holds.

Assertion 4. *Let $T \leqslant 2l$. Then the solution from $\widehat{W}_2^1(Q_T)$ to the problem II where $\varphi_1(x) \equiv 0$ for $x \in [0,l]$, $\psi_1(x) = 0$ a.e. on $[0,l]$ and an arbitrary function $\mu(t) \in W_2^1[0,T]$ such that $\mu(T) = 0$, is defined uniquely; moreover, $u(x,t) \equiv 0$ in the domain $\{(x,t) \mid T-l \leqslant t \leqslant T,\ T-t \leqslant x \leqslant l\} \bigcap Q_T$.*

Let us proceed with the proof of uniqueness for the solution to the boundary control problem III.

Assertion 5. *For any $T \in (0,2l]$, the boundary control problem III has at most one solution from $\widehat{W}_2^1(Q_T)$.*

Proof. Let us consider only[‡] the case $T = 2l$. Suppose that in this case the problem III has two solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ from the class $\widehat{W}_2^1(Q_{2l})$. Then their difference $u(x,t) = u^{(2)}(x,t) - u^{(1)}(x,t)$ gives a solution from the same class to the problem III with zero initial and terminal data. Let $\mu(t) = u(0,t)$. It follows from the definition of the class $\widehat{W}_2^1(Q_{2l})$ that $\mu(t) \in W_2^1[0,2l]$ and $\mu(0) = \mu(2l) = 0$.

[‡]As it can be easily obtained from what follows, in the case $T < 2l$ it is essential that the domains where the solutions considered in Assertions 2-4 vanish should have common points.

The function $u(x,t)$ is the solution from $\widehat{W}_2^1(Q_{2l})$ both to the problem I with zero initial data and to the problem II with zero terminal data coupled with the boundary condition $\mu(t) = u(0,t)$. It follows from Assertions 2-4 that $u(x,t)$ vanishes in the domain $\Delta_0 = \{(x,t) \mid 0 \leqslant t \leqslant 2l,\ l - |l-t| \leqslant x \leqslant l\}$. Let us show that $u(x,t)$ vanishes also in the remaining domain $Q_{2l} \setminus \Delta_0$.

Let t_1 be an arbitrary value in $[0,2l]$. The characteristic line $t - x = t_1$ that starts at the point $(0,t_1)$ intersects the characteristic line $t + x = 2l$ at the point $(l - t_1/2, l + t_1/2) \in \Delta_0$ where $u(x,t) = 0$. Thus Eqs. (6) and (7) yield

$$\mu(t_1) = -\frac{1}{2} \iint\limits_{D_0'(t_1)} q(\xi,\tau) u(\xi,\tau)\, d\xi d\tau, \tag{29}$$

where $D_0'(t_1) = \{(\xi,\tau) \mid t_1/2 \leqslant \tau \leqslant t_1/2 + l,\ \ |\tau - t_1| \leqslant \xi \leqslant l - |l-\tau|\}$.

Consider an arbitrary point $(x,t) \in Q_{2l} \setminus \Delta_0$. It follows from (6) and (7) that $u(x,t) = \mu(t-x) + \frac{1}{2}\iint_{D_0(x,t)} q(\xi,\tau)u(\xi,\tau)\, d\xi d\tau$, hence, by Eq. (29) with $t_1 = t - x$, one obtains the relation

$$u(x,t) = -\frac{1}{2} \iint\limits_{D_0'(t-x)\setminus D_0(x,t)} q(\xi,\tau) u(\xi,\tau)\, d\xi d\tau \equiv [\mathcal{N}_0 u](x,t); \tag{30}$$

here the domain $D_0'(t-x) \setminus D_0(x,t)$ is the triangle $\{(\xi,\tau) \mid (x+t)/2 \leqslant \tau \leqslant l + (t-x)/2,\ x + |t-\tau| \leqslant \xi \leqslant l - |l-\tau|\}$.

Eq. (30) is the homogeneous Volterra-type equation of the second kind since the operator $\mathcal{N}_0$ in its right-hand side is bounded in $L_\infty(Q_{2l} \setminus \Delta_0)$ and satisfies the estimates $\left|[\mathcal{N}_0^k \chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k\, (t-x)^k/k!\ \sup_{(x,t)\in Q_{2l}\setminus\Delta_0} |\chi(x,t)|$. Thus Eq. (30) has only the trivial solution, and it follows from Eq. (29) that $\mu(t_1) = 0$ for all $t_1 \in [0,2l]$. ∎

4. Main results

First of all let us note a certain peculiarity of the boundary control problem III for the critical value $T = 2l$. It follows from [3] that the function

$$\overset{0}{u}(x,t) = \frac{1}{2}\begin{cases} \varphi(x+t)+\varphi(x-t)+\int\limits_{x-t}^{x+t}\psi(\xi)\,d\xi & \text{in } \Delta_1, \\ \varphi(x+t)+\varphi(0)+\int\limits_{0}^{x+t}\psi(\xi)\,d\xi+\varphi_1(t-x)-\varphi_1(0)+\int\limits_{0}^{t-x}\psi_1(\xi)\,d\xi & \text{in } \Delta_2, \\ \varphi(x-t)+\varphi(2l-x-t)+\int\limits_{x-t}^{l}\psi(\xi)\,d\xi+\int\limits_{2l-x-t}^{l}\psi(\xi)\,d\xi & \text{in } \Delta_3, \\ \varphi(2l-x-t)+\varphi(0)+\int\limits_{2l-x-t}^{l}\psi(\xi)\,d\xi+\int\limits_{0}^{l}\psi(\xi)\,d\xi+ & \\ \qquad +\varphi_1(t-x)-\varphi_1(0)+\int\limits_{0}^{t-x}\psi_1(\xi)\,d\xi & \text{in } \Delta_4, \\ \varphi(2l-x-t)+\varphi(0)+\int\limits_{2l-x-t}^{l}\psi(\xi)\,d\xi+\int\limits_{0}^{l}\psi(\xi)\,d\xi+ & \\ \qquad +\varphi_1(2l+x-t)-\varphi_1(0)+\int\limits_{2l+x-t}^{l}\psi_1(\xi)\,d\xi+\int\limits_{0}^{l}\psi_1(\xi)\,d\xi & \text{in } \Delta_5, \\ \varphi_1(x+t-2l)+\varphi_1(t-x)-\int\limits_{x+t-2l}^{l}\psi_1(\xi)\,d\xi-\int\limits_{t-x}^{l}\psi_1(\xi)\,d\xi & \text{in } \Delta_6, \\ \varphi_1(x+t-2l)+\varphi_1(x-t+2l)-\int\limits_{x+t-2l}^{x-t+2l}\psi_1(\xi)\,d\xi & \text{in } \Delta_7 \end{cases} \tag{31}$$

is a unique solution to the boundary control problem III for the homogeneous wave equation if and only if its data satisfy the relation

$$A_0 \equiv \varphi(0)+\int\limits_0^l \psi(\xi)\,d\xi = \varphi_1(0)-\int\limits_0^l \psi_1(\xi)\,d\xi \equiv B_0. \tag{32}$$

Similarly, in the inhomogeneous case one can prove that the function[§]

$$u(x,t) = \overset{0}{u}(x,t)+\frac{1}{2}\int\limits_0^t \int\limits_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\,d\xi d\tau \tag{33}$$

is the unique solution to the boundary control problem III for the forced oscillations (see (8)) if and only if the relation

$$A_0+\int\limits_0^l\int\limits_\tau^l f(\xi,\tau)\,d\xi d\tau = B_0+\int\limits_l^{2l}\int\limits_{2l-\tau}^{l} f(\xi,\tau)\,d\xi d\tau \tag{34}$$

[§]The integrand in (33) is obtained by the extension of the right-hand side $f(x,t)$ outside Q_T similar to (8).

holds where A_0, B_0 are constants in the left-hand and in the right-hand sides of (32).

The boundary control problem III for the telegraph equation (1) is also governed by a similar condition which is necessary for the existence of its solution from $\widehat{W}_2^1(Q_T)$.

Theorem 1. *Let $T=2l$. Then, for the existence of the solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III, it is necessary to observe the following conditions:*

1) $\varphi(x),\ \varphi_1(x)\in W_2^1[0,l],\ \psi(x),\ \psi_1(x)\in L_2[0,l]$,

2) the initial and terminal data satisfy the relation

$$A_0+\int\limits_0^l\int\limits_\tau^{2l-\tau}\widetilde{q}_A^*(\xi,\tau)A(\xi,\tau)\,d\xi d\tau=B_0+\int\limits_l^{2l}\int\limits_{2l-\tau}^{\tau}\widetilde{q}_B^*(\xi,\tau)B(\xi,\tau)\,d\xi d\tau, \tag{35}$$

where A_0, B_0 are the constants in (32), the values $A(\xi,\tau)$, $B(\xi,\tau)$ are computed via the initial and terminal data by the formulas

$$A(\xi,\tau)=\frac{1}{2}\left[\varphi(l-|\xi+\tau-l|)+\varphi(l-|\xi-\tau-l|)+\int\limits_{\xi-\tau}^{\xi+\tau}\psi(l-|\zeta-l|)\,d\zeta\right], \tag{36}$$

$$B(\xi,\tau)=\frac{1}{2}\left[\varphi_1(l-|\xi+\tau-3l|)+\varphi_1(l-|\xi-\tau+l|)-\int\limits_{\xi+\tau-2l}^{\xi-\tau+2l}\psi_1(l-|\zeta-l|)\,d\zeta\right] \tag{37}$$

and the kernels $\widetilde{q}_A^*(\xi,\tau)$ and $\widetilde{q}_B^*(\xi,\tau)$ of the integral operators are connected with the coefficient $q(\xi,\tau)$ in (1) by the relations

$$\begin{aligned}
&\widetilde{q}_A^*(\xi,\tau)=q(l-|\xi-l|,\tau)\sum_{k=0}^{\infty}\widetilde{q}_A^{(k)}(l,l;\xi,\tau), \qquad \widetilde{q}_A^{(0)}(x,t;\xi,\tau)\equiv 1/2;\\
&\widetilde{q}_A^{(k+1)}(x,t;\xi,\tau)=\tfrac12\int_\tau^t\int_{\max(x-t+\tau_1,\xi+\tau-\tau_1)}^{\min(x+t-\tau_1,\xi-\tau+\tau_1)}q(l-|\xi_1-l|,\tau_1)\widetilde{q}_A^{(k)}(\xi_1,\tau_1;\xi,\tau)\,d\xi_1d\tau_1,\\
&\widetilde{q}_B^*(\xi,\tau)=q(l-|\xi-l|,\tau)\sum_{k=0}^{\infty}\widetilde{q}_B^{(k)}(l,l;\xi,\tau), \qquad \widetilde{q}_B^{(0)}(x,t;\xi,\tau)\equiv 1/2;\\
&\widetilde{q}_B^{(k+1)}(x,t;\xi,\tau)=\tfrac12\int_t^\tau\int_{\max(x+t-\tau_1,\xi-\tau+\tau_1)}^{\min(x-t+\tau_1,\xi+\tau-\tau_1)}q(l-|\xi_1-l|,\tau_1)\widetilde{q}_B^{(k)}(\xi_1,\tau_1;\xi,\tau)\,d\xi_1d\tau_1.
\end{aligned} \tag{38}$$

Proof. Let the function $u(x,t)$ be the solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III. Then it is also the solution to the problem I in the triangles Δ_1, Δ_3 and the solution to the problem II in the triangles Δ_6, Δ_7. Mimicking the proof of Assertion 2, we construct the integral relations for the function $u(x,t)$ in the domains Δ_1, Δ_3, Δ_6 and Δ_7.

Let $(x,t)\in\Delta_1\bigcup\Delta_3$. Denoting by $\overset{0}{u}_1(x,t)$ and $\overset{0}{u}_3(x,t)$ the solutions (31) to the problem I for the homogeneous wave equation in Δ_1 and Δ_3 respectively, one obtains the equations

$$u(x,t)=\overset{0}{u}_1(x,t)+\frac{1}{2}\iint\limits_{\Omega_1}q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau, \quad (x,t)\in\Delta_1, \tag{39}$$

$$u(x,t) = \overset{0}{u}_3(x,t) + \frac{1}{2}\left(\iint\limits_{\Omega_3'} + \iint\limits_{\Omega_3''}\right) q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau, \quad (x,t)\in\Delta_3, \tag{40}$$

where $\Omega_1(x,t) = D_1(x,t)$, $\Omega_3'(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant t,\ x-t+\tau \leqslant \xi \leqslant (l+x+t-\tau)/2 - |(\tau-x-t+l)/2|\}$ and $\Omega_3''(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant x+t-l,\ 2l-x-t+\tau \leqslant \xi \leqslant l\}$.

Let the functions $u(x,t)$, $q(x,t)$ in (40) and the initial data $\varphi(x)$, $\psi(x)$ in $\overset{0}{u}_3(x,t)$ be continued evenly over $x = l$ to the domain $Q_{2l}' = [l \leqslant x \leqslant 2l] \times [0 \leqslant t \leqslant 2l]$. Denote these new functions by $\widetilde{u}(x,t)$, $\widetilde{q}(x,t)$, $\widetilde{\varphi}(x)$, $\widetilde{\psi}(x)$ respectively. Thus $\overset{0}{u}_3(x,t) = \frac{1}{2}[\widetilde{\varphi}(x+t) + \widetilde{\varphi}(x-t) + \int\limits_{x-t}^{x+t} \widetilde{\psi}(\xi)d\xi] \equiv \overset{0}{\widetilde{u}}_1(x,t)$ and using this continuation one can rewrite Eq. (40) in the following form

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_1(x,t) + \frac{1}{2}\iint\limits_{\Omega_1} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau, \quad (x,t)\in\Delta_3. \tag{41}$$

One can easily see that Eq. (41) transforms into Eq. (39) for $(x,t) \in \Delta_1$ and, by the symmetry, keeps its form if the point (x,t) belongs to the triangle which is mirror symmetric to the triangle $\Delta_1 \bigcup \Delta_3$ with respect to $x = l$. Thus the continued solution $\widetilde{u}(x,t)$ satisfies Eq. (41) for all $(x,t) \in \widetilde{\Delta}_1 = \{(x,t) \mid 0 \leqslant t \leqslant l,\ t \leqslant x \leqslant 2l-t\}$.

Similarly denoting by $\overset{0}{u}_6(x,t)$ and $\overset{0}{u}_7(x,t)$ the solutions (31) to the problem II for the homogeneous wave equation in Δ_6 and Δ_7 respectively, one obtains from the equations

$$u(x,t) = \overset{0}{u}_7(x,t) + \frac{1}{2}\iint\limits_{\Omega_7} q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau,\ (x,t)\in\Delta_7, \tag{42}$$

$$u(x,t) = \overset{0}{u}_6(x,t) + \frac{1}{2}\left(\iint\limits_{\Omega_6'} + \iint\limits_{\Omega_6''}\right) q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau,\ (x,t)\in\Delta_6, \tag{43}$$

where $\Omega_7(x,t) = \{(\xi,\tau) \mid t \leqslant \tau \leqslant 2l,\ x+t-\tau \leqslant \xi \leqslant x-t+\tau\}$, $\Omega_6'(x,t) = \{(\xi,\tau) \mid t \leqslant \tau \leqslant 2l,\ x+t-\tau \leqslant \xi \leqslant (l+x-t+\tau)/2 - |(\tau+x-t-l)/2|\}$, $\Omega_6''(x,t) = \{(\xi,\tau) \mid l-x+t \leqslant \tau \leqslant 2l,\ 2l-x+t-\tau \leqslant \xi \leqslant l\}$, that the evenly extended solution $\widetilde{u}(x,t)$ for all $(x,t) \in \widetilde{\Delta}_7 = \{(x,t) \mid l \leqslant t \leqslant 2l,\ 2l-t \leqslant x \leqslant t\}$ satisfies the relation

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_7(x,t) + \frac{1}{2}\iint\limits_{\Omega_7} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\,d\xi d\tau. \tag{44}$$

Here $\overset{0}{\widetilde{u}}_7(x,t) = \frac{1}{2}[\widetilde{\varphi}_1(x+t-2l) + \widetilde{\varphi}_1(x-t+2l) - \int\limits_{x+t-2l}^{x-t+2l} \widetilde{\psi}_1(\xi)d\xi]$ where $\widetilde{\varphi}_1(x), \widetilde{\psi}_1(x)$ are the terminal data which are extended evenly over $x = l$.

Since the function $u(x,t)$ (and therefore the function $\widetilde{u}(x,t)$) is continuous in Q_{2l} (as it belongs to $\widehat{W}_2^1(Q_{2l})$) the value $u(l,l) = \widetilde{u}(l,l)$ should be the same no matter whether it is computed from (41) or from (44). Therefore

$$\overset{0}{\widetilde{u}}_1(l,l) + \frac{1}{2} \iint\limits_{\Omega_1(l,l)} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\, d\xi d\tau = \overset{0}{\widetilde{u}}_7(l,l) + \frac{1}{2} \iint\limits_{\Omega_7(l,l)} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\, d\xi d\tau. \tag{45}$$

As $\overset{0}{\widetilde{u}}_1(l,l) = \overset{0}{u}_3(l,l) = \varphi(0) + \int_0^l \psi(\xi)d\xi = A_0$, $\overset{0}{\widetilde{u}}_7(l,l) = \overset{0}{u}_6(l,l) = \varphi_1(0) - \int_0^l \psi_1(\xi)d\xi = B_0$ and moreover $\Omega_1(l,l) = \Delta_1'$, $\Omega_7(l,l) = \Delta_7'$, Eq. (45) yields the relation

$$A_0 + \frac{1}{2} \iint\limits_{\Delta_1'} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\, d\xi d\tau = B_0 + \frac{1}{2} \iint\limits_{\Delta_7'} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\, d\xi d\tau. \tag{46}$$

Following an approach introduced in [14], Eq. (46) can be transformed to its final form (35) by expressing $\widetilde{u}(x,t)$ via $\overset{0}{\widetilde{u}}_1(x,t)$ in (41) and $\overset{0}{\widetilde{u}}_7(x,t)$ in (44) using the corresponding Neumann series and substituting the obtained expressions in the left-hand and the right-hand sides of (46).

Introducing the operators $[\mathcal{G}_j\widetilde{u}](x,t) = (1/2)\iint_{\Omega_j} \widetilde{q}(\xi,\tau)\widetilde{u}(\xi,\tau)\, d\xi d\tau$, $j = 1,7$, one obtains

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_1(x,t) + \sum_{k=1}^{\infty} [\mathcal{G}_1^k \overset{0}{\widetilde{u}}_1](x,t) \quad \text{for } (x,t) \in \widetilde{\Delta}_1 \tag{47}$$

and

$$\widetilde{u}(x,t) = \overset{0}{\widetilde{u}}_7(x,t) + \sum_{k=1}^{\infty} [\mathcal{G}_7^k \overset{0}{\widetilde{u}}_7](x,t) \quad \text{for } (x,t) \in \widetilde{\Delta}_7. \tag{48}$$

The series in the right-hand sides of (47) and (48) are absolutely convergent since the operators $\mathcal{G}_1$ and $\mathcal{G}_7$ satisfy the estimates:

$$\Big|[\mathcal{G}_1^k\chi](x,t)\Big| \leqslant (2\|q\|_\infty)^k \frac{t^{2k}}{(2k)!} \sup_{(x,t)\in\widetilde{\Delta}_1} |\chi(x,t)|, \tag{49}$$

$$\Big|[\mathcal{G}_7^k\chi](x,t)\Big| \leqslant (2l\|q\|_\infty)^k \frac{(2l-t)^k}{k!} \sup_{(x,t)\in\widetilde{\Delta}_7} |\chi(x,t)|. \tag{50}$$

Applying (47) and (48) in (46), changing the order of integration and taking into account that $A(x,t) = \overset{0}{\widetilde{u}}_1(x,t), B(x,t) = \overset{0}{\widetilde{u}}_7(x,t)$, $\widetilde{\varphi}(x) = \varphi(l-|x-l|)$, $\widetilde{\psi}(x) = \psi(l-|x-l|)$, $\widetilde{\varphi}_1(x) = \varphi_1(l-|x-l|)$, $\widetilde{\psi}_1(x) = \psi_1(l-|x-l|)$, $\widetilde{q}(x,t) = q(l-|x-l|,t)$, one obtains (35)–(38). ■

Let us show that the necessary condition (35) in Theorem 1 is also sufficient for the existence of the solution to the boundary control problem III.

Theorem 2. *Let $T = 2l$ and the condition 1 in Theorem 1 be satisfied. Then the relation (35) is sufficient for the existence of the unique solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III.*

Proof. Let us define the function $u(x,t)$ in the triangles Δ_1 and Δ_3 as the solution to Eq. (41) constructed in the proof of Theorem 1 and in the triangles Δ_6 and Δ_7 – as the solution to Eq. (44). Due to the estimates (49) and (50) these equations have bounded solutions which are given by the series (47) and (48). As the right-hand sides of (41) and (44) are continuous these solutions are also continuous in $\Delta_1 \bigcup \Delta_3$ and $\Delta_6 \bigcup \Delta_7$. If the relation (35) holds true then, as it follows from the proof of Theorem 1, the relation (45) is satisfied and thus the function $u(x,t)$ is continuous in the union of domains $\Delta_1 \bigcup \Delta_3 \bigcup \Delta_6 \bigcup \Delta_7$.

Let $u_j(x,t)$ stand for the obtained solution $u(x,t)$ in Δ_j for $j = 1, 3, 6, 7$ respectively and consider the remaining parts of the rectangle Q_{2l}.

For $(x,t) \in \Delta_4$ the following relation holds:

$$u(x,t) = \overset{0}{u}_4(x,t) +$$

$$+\frac{1}{2}\left[\iint\limits_{\Omega'_{41}} + \iint\limits_{\Omega''_{41}} + \iint\limits_{\Omega'_{43}} + \iint\limits_{\Omega''_{43}} - \iint\limits_{\Omega'_{46}} - \iint\limits_{\Omega''_{46}} - \iint\limits_{\Omega_{47}} - \iint\limits_{\Omega_4}\right] q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{51}$$

where $\overset{0}{u}_4(x,t)$ is the solution to the boundary control problem III for the homogeneous wave equation which is defined by (31) in the triangle Δ_4, and the integration domains are given by the inequalities: $\Omega'_{41}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant l/2,\ \tau \leqslant \xi \leqslant l-\tau\}$, $\Omega''_{41}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant (t+x-l)/2,\ 2l-x-t+\tau \leqslant \xi \leqslant l-\tau\}$, $\Omega'_{43}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant (x+t)/2,\ l/2+|\tau-l/2| \leqslant \xi \leqslant (l+x+t-\tau)/2 - |(l-x-t+\tau)/2|\}$, $\Omega''_{43}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant x+t-l,\ (3l-x-t)/2 + |\tau-(x+t-l)/2| \leqslant \xi \leqslant l\}$, $\Omega'_{46}(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant l-x+t,\ (2l+x-t)/2+|\tau-(2l-x+t)/2| \leqslant \xi \leqslant l\}$, $\Omega''_{46}(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant (3l-x+t)/2,\ l/2+|\tau-(3l)/2| \leqslant \xi \leqslant (3l-x+t-\tau)/2-|(\tau-l+x-t)/2|\}$, $\Omega_{47}(x,t) = \{(\xi,\tau) \mid (3l)/2 \leqslant \tau \leqslant 2l,\ 2l-\tau \leqslant \xi \leqslant (l-x+t)/2 - |\tau-(3l-x+t)/2|\}$, $\Omega_4(x,t) = \{(\xi,\tau) \mid (x+t)/2 \leqslant \tau \leqslant (2l-x+t)/2,\ x+|\tau-t| \leqslant \xi \leqslant l-|\tau-l|\}$.

Since $\Omega'_{41} \cup \Omega''_{41} \subset \Delta_1$, $\Omega'_{43} \cup \Omega''_{43} \subset \Delta_3$, $\Omega'_{46} \cup \Omega''_{46} \subset \Delta_6$, $\Omega_{47} \subset \Delta_7$, $\Omega_4 \subset \Delta_4$, all the integral terms in the right-hand side of (51), except the last one, are known and therefore the relation (1) can be treated as the integral equation for $u(x,t)$ in the domain Δ_4:

$$u(x,t) = F_4(x,t) - [\mathcal{G}_4 u](x,t) \tag{52}$$

where $[\mathcal{G}_4\chi](x,t) = (1/2)\iint_{\Omega_4} q(\xi,\tau)\chi(\xi,\tau)\,d\xi d\tau$ while the function $F_4(x,t)$ is already known. The operator $\mathcal{G}_4$ is bounded in $L_\infty(\Delta_4)$ and satisfies the estimate

$$\left|[\mathcal{G}_4^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k\,\frac{(t-x)^k}{k!}\sup_{(x,t)\in\Delta_4}|\chi(x,t)|. \tag{53}$$

This estimate yields that Eq. (52) has a bounded in Δ_4 solution. Let us denote it by $u_4(x,t)$. It follows from Eq. (51) that the function $u_4(x,t)$ is continuous in Δ_4.

On the common boarder of Δ_3 and Δ_4, i.e. for $x=t$, $l/2\leqslant t\leqslant l$, Eq. (51) transforms into the relation

$$u_4(t,t)=\overset{0}{u}_4(t,t)+\frac{1}{2}\sum_{k=1}^{2}\left[\iint\limits_{\Omega'_{4,2k-1}(t,t)}+\iint\limits_{\Omega''_{4,2k-1}(t,t)}\right]q(\xi,\tau)u_{2k-1}(\xi,\tau)\,d\xi d\tau. \tag{54}$$

Since $\overset{0}{u}_4(t,t)=\overset{0}{u}_3(t,t)$, $\Omega'_{41}(t,t)=\Omega'_3(t,t)\bigcap\Delta_1=\Delta_1$, $\Omega''_{41}(t,t)=\Omega''_3(t,t)\bigcap\Delta_1$, $\Omega'_{43}(t,t)=\Omega'_3(t,t)\bigcap\Delta_3$, $\Omega''_{43}(t,t)=\Omega''_3(t,t)\bigcap\Delta_3$, Eqs. (40) and (54) yield $u_4(t,t)=u_3(t,t)$.

On the common boarder of Δ_4 and Δ_6, i.e. for $x=2l-t$, $l\leqslant t\leqslant(3l)/2$, Eq. (51) transforms into the relation

$$u_4(2l-t,t)=\overset{0}{u}_4(2l-t,t)+\iint\limits_{\Delta_1}q(\xi,\tau)u_1(\xi,\tau)\,d\xi d\tau+\iint\limits_{\Delta_3}q(\xi,\tau)u_3(\xi,\tau)\,d\xi d\tau-$$
$$-\frac{1}{2}\left[\iint\limits_{\Omega'_{46}(2l-t,t)}+\iint\limits_{\Omega''_{46}(2l-t,t)}\right]q(\xi,\tau)u_6(\xi,\tau)\,d\xi d\tau-\frac{1}{2}\iint\limits_{\Omega_{47}(2l-t,t)}q(\xi,\tau)u_7(\xi,\tau)\,d\xi d\tau. \tag{55}$$

As $\overset{0}{u}_4(2l-t,t)=\overset{0}{u}_6(2l-t,t)+A_0-B_0$, Eq. (46) (which is equivalent to (35)) holds, and due to the relation (43), one comes to the equality $u_4(2l-t,t)=u_6(2l-t,t)$.

Similarly, for $(x,t)\in\Delta_2$ one obtains the equation

$$u(x,t)=\overset{0}{u}_2(x,t)+\frac{1}{2}\left[\iint\limits_{\Omega_{21}}-\iint\limits_{\Omega_{24}}-\iint\limits_{\Omega'_{26}}-\iint\limits_{\Omega''_{26}}-\iint\limits_{\Omega_{27}}-\iint\limits_{\Omega_{2}}\right]q(\xi,\tau)u(\xi,\tau)\,d\xi d\tau \tag{56}$$

where $\overset{0}{u}_2(x,t)$ is the solution to the boundary control problem III for the homogeneous wave equation in the triangle Δ_2 (see (31)), and the integration domains are given by the inequalities $\Omega_{21}(x,t)=\{(\xi,\tau)\mid 0\leqslant\tau\leqslant(t+x)/2,\ \tau\leqslant\xi\leqslant x+t-\tau\}$, $\Omega_{24}(x,t)=\{(\xi,\tau)\mid l/2\leqslant\tau\leqslant(2l-x+t)/2,\ (l+x-t)/2+|\tau-(l-x+t)/2|\leqslant\xi\leqslant l-|l-\tau|\}$, $\Omega'_{26}(x,t)=\{(\xi,\tau)\mid l\leqslant\tau\leqslant(3l-x+t)/2,\ l/2+|\tau-(3l)/2|\leqslant\xi\leqslant(3l-x+t-\tau)/2-|(l-x+t-\tau)/2|\}$, $\Omega''_{26}(x,t)=\{(\xi,\tau)\mid l\leqslant\tau\leqslant l-x+t,\ (2l+x-t)/2+|\tau-(2l-x+t)/2|\leqslant\xi\leqslant l\}$, $\Omega_{27}(x,t)=\{(\xi,\tau)\mid(3l)/2\leqslant\tau\leqslant 2l,\ 2l-\tau\leqslant\xi\leqslant(l-x+t)/2-|\tau-(3l-x+t)/2|\}$, $\Omega_2(x,t)=\{(\xi,\tau)\mid(x+t)/2\leqslant\tau\leqslant(l-x+t)/2,\ x+|\tau-t|\leqslant\xi\leqslant l/2-|(\tau-l/2|\}$.

Since $\Omega_{21}\subset\Delta_1$, $\Omega_{24}\subset\Delta_4$, $\Omega'_{26}\cup\Omega''_{26}\subset\Delta_6$, $\Omega_{27}\subset\Delta_7$, $\Omega_2\subset\Delta_2$, all the integral terms on the right-hand side of (56), except the last one, are already known and therefore Eq. (56) is the integral equation of the form

$$u(x,t)=F_2(x,t)-[\mathcal{G}_2u](x,t) \tag{57}$$

for finding $u(x,t)$ in the domain Δ_2. Here the operator $[\mathcal{G}_2\chi](x,t) = (1/2)\iint_{\Omega_2} q(\xi,\tau)\chi(\xi,\tau)\, d\xi d\tau$ is bounded in $L_\infty(\Delta_2)$ and satisfies the estimate

$$\left|[\mathcal{G}_2^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k \frac{(t-x)^k}{k!} \sup_{(x,t)\in\Delta_2} |\chi(x,t)|. \tag{58}$$

Thus Eq. (57) has a bounded and continuous in Δ_2 solution $u(x,t) = u_2(x,t)$.

Eqs. (39), (51), (56) yield that on the boarder between Δ_1, Δ_2: $u_2(t,t) = u_1(t,t)$ and on the boarder between Δ_2, Δ_4: $u_2(l-t,t) = u_4(l-t,t)$.

Finally, for $(x,t) \in \Delta_5$ the following relation holds:

$$u(x,t) = \overset{0}{u}_5(x,t)+$$

$$+\frac{1}{2}\left[\iint\limits_{\Omega'_{51}} + \iint\limits_{\Omega''_{51}} + \iint\limits_{\Omega'_{53}} + \iint\limits_{\Omega''_{53}} - \iint\limits_{\Omega_{54}} -2\iint\limits_{\Omega_{56}} - \iint\limits_{\Omega'_{57}} - \iint\limits_{\Omega''_{57}} - \iint\limits_{\Omega_5}\right] q(\xi,\tau)u(\xi,\tau)\, d\xi d\tau \tag{59}$$

where $\overset{0}{u}_5(x,t)$ is the solution to the boundary control problem III for the homogeneous wave equation in the triangle Δ_5 (see (31)), and the integration domains are given by the inequalities: $\Omega'_{51}(x,t) = \{(\xi,\tau) \bigm| 0 \leqslant \tau \leqslant (x+t-l)/2,\ 2l-x-t+\tau \leqslant \xi \leqslant l-\tau\}$, $\Omega''_{51}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant l/2,\ \tau \leqslant \xi \leqslant l-\tau\}$, $\Omega'_{53}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant (x+t)/2,\ l/2+|\tau-l/2| \leqslant \xi \leqslant (l+x+t-\tau)/2-|(l-x-t+\tau)/2|\}$, $\Omega''_{53}(x,t) = \{(\xi,\tau) \mid 0 \leqslant \tau \leqslant x+t-l,\ (3l-x-t)/2+|\tau-(x+t-l)/2| \leqslant \xi \leqslant l\}$, $\Omega_{54}(x,t) = \{(\xi,\tau) \mid (x+t)/2 \leqslant \tau \leqslant (3l)/2,\ (x+t-l)/2+|\tau-(x+t+l)/2| \leqslant \xi \leqslant l-|\tau-l|\}$, $\Omega_{56}(x,t) = \{(\xi,\tau) \mid l \leqslant \tau \leqslant 2l,\ l/2+|\tau-(3l)/2| \leqslant \xi \leqslant l\}$, $\Omega'_{57}(x,t) = \{(\xi,\tau) \mid (3l)/2 \leqslant \tau \leqslant 2l,\ 2l-\tau \leqslant \xi \leqslant \tau-l\}$, $\Omega''_{57}(x,t) = \{(\xi,\tau) \mid (3l)/2 \leqslant \tau \leqslant 2l,\ (2l+x-t)/2+|\tau-(2l-x+t)/2| \leqslant \xi \leqslant \tau-l\}$, $\Omega_5(x,t) = \{(\xi,\tau) \mid (t+x+l)/2 \leqslant \tau \leqslant (2l-x+t)/2,\ x+|\tau-t| \leqslant \xi \leqslant l/2-|\tau-(3l)/2|\}$.

Since $\Omega'_{51}\cup\Omega''_{51} \subset \Delta_1$, $\Omega'_{53}\cup\Omega''_{53} \subset \Delta_3$, $\Omega_{54} \subset \Delta_4$, $\Omega_{56} \subset \Delta_6$, $\Omega'_{57}\cup\Omega''_{57} \subset \Delta_7$, $\Omega_5 \subset \Delta_5$, all the integral terms on the right-hand side of (59), except the last one, are already known and therefore Eq. (59) is the integral equation of the form

$$u(x,t) = F_5(x,t) - [\mathcal{G}_5 u](x,t), \tag{60}$$

for finding $u(x,t)$ in the domain Δ_5. The operator $[\mathcal{G}_5\chi](x,t) = (1/2)\iint_{\Omega_5} q(\xi,\tau)\chi(\xi,\tau)\, d\xi d\tau$ is bounded in $L_\infty(\Delta_5)$, and as it satisfies the estimate

$$\left|[\mathcal{G}_5^k\chi](x,t)\right| \leqslant (l\|q\|_\infty/2)^k \frac{(2l-t-x)^k}{k!} \sup_{(x,t)\in\Delta_5} |\chi(x,t)|, \tag{61}$$

Eq. (60) has the bounded and continuous in Δ_5 solution $u(x,t) = u_5(x,t)$.

Applying Eqs. (42), (51) and (59) one can easily approve that on the boarder between Δ_5 and Δ_4: $u_5(t-l,t) = u_4(t-l,t)$, and, by virtue of Eq. (35), on the boarder between Δ_5 and Δ_7: $u_5(2l-t,t) = u_7(2l-t,t)$.

Thus the solutions to the integral equations (39), (40), (42), (43), (51), (56) and (59) define the continuous in Q_{2l} function $u(x,t)$ for which $u(x,t) = u_j(x,t)$ if $(x,t) \in \Delta_j$, $j = \overline{1,7}$.

Differentiating both parts of these integral equations with respect to x and t, one can easily show that the function $u(x,t)$ belongs to $\widehat{W}_2^1(Q_{2l})$ and $u_x(l,t) = 0$ for all $t \in [0, 2l]$. The direct substitution of the integral equations for $u(x,t)$ in the identity (5) and smoothness arguments similar to those in the proof of Assertion 2, show that $u(x,t)$ is the acquired generalized solution to the boundary control problem III. ■

Remark 1. Estimates (49), (50), (53), (58), (61) and formulas that define the solutions $u_j(x,t)$, $j = \overline{1,7}$, to the corresponding integral equations in the form of the Neumann series (see, e.g., Eqs. (47), (48) for $j = 1$ and $j = 7$), yield the a priori estimate for the solution to the boundary control problem III

$$\|u(x,t)\|_{W_2^1(Q_{2l})} \leqslant C \left(\|\varphi\|_{W_2^1[0,l]} + \|\varphi_1\|_{W_2^1[0,l]} + \|\psi\|_{L_2[0,l]} + \|\psi_1\|_{L_2[0,l]} \right);$$

it claims that this solution is stable with respect to perturbations of initial and terminal data.

Remark 2. If Eq. (35) holds true then, generally speaking, the function $\overset{0}{u}(x,t)$ defined in (31) is not a solution from $\widehat{W}_2^1(Q_{2l})$ to the boundary control problem III for the homogeneous wave equation. Let us define the constant $\widetilde{C}_0 = \widetilde{C}_0(q)$ by the formula

$$\widetilde{C}_0 = -2 \left(\int\limits_0^l \int\limits_\tau^{2l-\tau} \widetilde{q}_A^*(\xi,\tau) A(\xi,\tau)\, d\xi d\tau + 2 \int\limits_l^{2l} \int\limits_{2l-\tau}^{\tau} \widetilde{q}_B^*(\xi,\tau) B(\xi,\tau)\, d\xi d\tau \right). \tag{62}$$

If one adds this constant $\widetilde{C}_0$ to the expressions that define the function $\overset{0}{u}(x,t)$ in the domains Δ_6 and Δ_7, the new function $\overset{0}{u}_*(x,t)$ becomes the generalized solution to the considered problem for the homogeneous wave equation but with a modified first terminal condition $\overset{0}{u}_*(x,2l) = \varphi_1(x) + \widetilde{C}_0$.

Applying Eqs. (36)–(38), one can easily show that if $\|q\|_\infty \to 0$ then the constant $\widetilde{C}_0$, defined in (62), vanishes while the function $\overset{0}{u}_*(x,t)$ transforms into $\overset{0}{u}(x,t)$.

Moreover, the estimates (49), (50), (53), (58), (61) and integral representations for partial derivatives of the solution $u(x,t)$ show that if $\|q\|_\infty \to 0$ then $\|u - \overset{0}{u}\|_{W_2^1(Q_{2l})} \to 0$ and respectively $\|\mu - \overset{0}{\mu}\|_{W_2^1[0,2l]} \to 0$ where $\overset{0}{\mu}(t) = \overset{0}{u}(0,t)$. In other words, the solution to the boundary control problem III is regular with respect the additive perturbation $q(x,t)u(x,t)$ of the wave operator in (1) with a bounded and measurable coefficient $q(x,t)$.

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Kritskov Leonid Vladimirovich
Moscow State University, Dpt. of Computational Mathematics and Cybernetics, 115172, Russian Federation, Moscow, Novospassky per., 3-2-151
E-mail: kritskov_leo@mail.ru

Abdukarimov Mahmadsalim Faizulloevich
Moscow State University,119991, Russian Federation, Moscow, Lenin Hills Street 1
*E-mail:*mahmadsalim_86@mail.ru