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A Generalized Contraction Mapping Principle

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Abstract. We introduce a generalized contraction mapping principle in fuzzy metric spaces with the help of two real functions. The methodology is different from other similar existing results.

Key Words and Phrases: Fixed point, complete fuzzy metric space, contraction mapping, Cauchy sequence.

1. Introduction

In this paper we introduce a new contraction mapping in fuzzy metric spaces which entails the spirit of generalization by Geraghty of the Banach's contraction mapping principle [7]. For this purpose we use two functions one of which has been recently considered by Shen et al in [17] proving a contraction mapping theorem in fuzzy metric spaces and the other by Geraghty [7]. We obtain our result in the fuzzy metric space defined by George and Veeramani [6]. The fuzzy fixed point theory has developed largely based on this space. One of the reasons for such successful development of fuzzy fixed point theory is that the space has a Hausdorff topology, a feature which has been widely utilized in this domain of study. Some recent references on the aforesaid topic are [4, 5, 9, 13, 14]. There are also other definitions of fuzzy metric spaces as, for instance, Kaleva and Seikkala defined a fuzzy metric with the help of fuzzy numbers [11]. A recent fixed point result on this space is deduced in [18].

In the following we state some concepts essential for the discussion in the rest of the paper.

Definition 1.1[10, 16] A binary operation $*: [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm if the following properties are satisfied:

(i) $*$ is associative and commutative,

(ii) $a * 1 = a$ for all $a \in [0,1]$,

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$

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(iv) $*$ is continuous.

Some examples of continuous t-norm are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$ and $a *_4 b = \max\{a + b - 1, 0\}$.
George and Veeramani in their paper [6] introduced the following definition of fuzzy metric space. We will be concerned only with this definition of fuzzy metric space.

Definition 1.2[6] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology and is called the topology on X induced by the fuzzy metric M. This topology is metrizable as we indicated above.

Example 1.3[6] Let X be the set of all real numbers and d be any metric on X. Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. For each $t > 0$, $x, y \in X$, let

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Example 1.4 Let (X, d) be a metric space and ψ be an increasing and continuous function of $(0, \infty)$ into $(0, 1)$ such that $\lim_{t \to \infty} \psi(t) = 1$. Three typical examples of these functions are $\psi(t) = \dfrac{t}{t+1}$, $\psi(t) = \sin(\dfrac{\pi t}{2t+1})$ and $\psi(t) = 1 - e^{-t}$. Let $*$ be any continuous t-norm. For each $t > 0$, $x, y \in X$, let

$$M(x, y, t) = \psi(t)^{d(x,y)}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.5[6] Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

The following lemma was proved by Grabiec [8] for fuzzy metric spaces defined by Kramosil et al [12]. The proof is also applicable to the fuzzy metric space given in Definition 1.2.

Lemma 1.6[8] Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.

Lemma 1.7[15] M is a continuous function on $X^2 \times (0, \infty)$.

Definition 1.8 [17] Let $\psi : [0, 1] \to [0, 1]$ be a function that satisfies the following conditions:

$(P1)$ ψ is strictly decreasing left continuous,

$(P2)$ $\psi(\lambda) = 0$ if and only if $\lambda = 1$.

Obviously, we obtain that $\lim_{\lambda\to 1} \psi(\lambda) = \psi(1) = 0$.

Definition 1.9 [1, 2, 3, 7] Let S be the class of functions $\beta : R^+ \to [0, 1)$ with

(i) $R^+ = \{t \in R/t > 0\}$,

(ii) $\beta(t_n) \to 1$ implies $t_n \to 0$. (1.1)

2. Main Result

Theorem 2.1 Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \to X$ be a mapping, $\psi : [0, 1] \to [0, 1]$ is as in Definition 1.8 and β satisfies the Definition 1.9. If the mapping T satisfies the condition

$$\psi(M(Tx, Ty, t)) \leq \beta(\psi((M(x, y, t))).\psi((M(x, y, t)), \tag{2.1}$$

for all $x, y \in X$, $t > 0$ and $x \neq y$, then T has a unique fixed point in X.

Proof. Starting with x_0 in X, we define the sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = Tx_n. \tag{2.2}$$

Let for all $t > 0$, $n \geq 0$,

$$\delta_n(t) = M(x_n, x_{n+1}, t). \tag{2.3}$$

Now from (2.1), for every $t > 0$, we have

$$\psi(\delta_n(t)) = \psi(M(x_n, x_{n+1}, t))$$

$$\begin{aligned} &= \psi(M(Tx_{n-1}, Tx_n, t)) \\ &\leq \beta(\psi((M(x_{n-1}, x_n, t))).\psi(M(x_{n-1}, x_n, t)) \\ &= \beta(\psi((M(x_{n-1}, x_n, t))).\psi(\delta_{n-1}(t)) \qquad (2.4) \\ &< \psi(\delta_{n-1}(t)) \\ \psi(\delta_n(t)) &< \psi(\delta_{n-1}(t)). \end{aligned}$$

Since $\{\psi(\delta_n(t))\}$ is strictly decreasing for every t, there exists $\delta(t) \geq 0$ such that $\lim_{n\to\infty} \psi(\delta_n(t)) = \delta(t)$.

$$\text{Let } \delta(t) > 0 \text{ for some } t. \qquad (2.5)$$

From (2.4), we have

$$\frac{\psi(\delta_n(t))}{\psi(\delta_{n-1}(t))} \leq \beta(\psi((M(x_{n-1}, x_n, t))) < 1.$$

Taking $n \to \infty$ in the above inequality and using (2.5), we have

$$\lim_{n\to\infty} \beta(\psi((M(x_{n-1}, x_n, t))) = 1.$$

Using the property of (1.1), we have

$$\delta(t) = \lim_{n\to\infty} \psi(\delta_n(t)) = \lim_{n\to\infty} \psi((M(x_{n-1}, x_n, t)) = 0.$$

So we arrive at a contradiction.

Therefore $\delta(t) > 0$ for all t, and we have

$$\lim_{n\to\infty} M(x_{n-1}, x_n, t) = 1. \qquad (2.6)$$

We now prove that the sequence $\{x_n\}$ is a Cauchy sequence. If not, then there exist $0 < \epsilon < 1$ and two sequences $\{m(k)\}$ and $\{n(k)\}$, where $m(k) > n(k) > k$ for every $n \geq 0$ and $t > 0$, such that

$$M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon.$$
$$\text{and } M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \epsilon. \qquad (2.7)$$

Then, $M(x_{m(k)-1}, x_{n(k-1)}, t) \geq M(x_{m(k)-1}, x_{n(k)}, \frac{t}{2}) * M(x_{n(k)}, x_{n(k)-1}, \frac{t}{2})$

$$\geq (1-\epsilon) * M(x_{n(k)}, x_{n(k)-1}, \tfrac{t}{2}). \qquad (2.8)$$

Since M is continuous we can find $\eta > 0$ such that

$$1 - \epsilon \geq M(x_{m(k)}, x_{n(k)}, t) \geq$$
$$M(x_{m(k)-1}, x_{n(k)}, \frac{\eta}{2}) * M(x_{m(k)-1}, x_{n(k)-1}, t-\eta) * M(x_{n(k)-1}, x_{n(k)}, \frac{\eta}{2}). \qquad (2.9)$$

Taking $k \to \infty$ in the above two inequalities (2.8) and (2.9), using (2.7) and the fact that $*$ is continuous, we have

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t) \geq (1-\epsilon) \geq \lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t-\eta).$$

Since M is continuous and η is arbitrary, we have

$$\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, t) = 1 - \epsilon. \qquad (2.10)$$

Now by (2.1), we have

$$\psi(1-\epsilon) \leq \psi(M(x_{m(k)}, x_{n(k)}, t)) \leq \beta(\psi(M(x_{m(k)-1}, x_{n(k)-1}, t))).\psi(M(x_{m(k)-1}, x_{n(k)-1}, t)).$$

Taking $k \to \infty$, we have

$$\psi(1-\epsilon) \leq \lim_{k\to\infty} \beta(\psi(M(x_{m(k)-1}, x_{n(k)-1}, t))).\psi(1-\epsilon).$$

Using definition 1.7, the last inequality implies $\lim_{k\to\infty}\beta(\psi(M(x_{m(k)-1},x_{n(k)-1},t)))=1$.
Since $\beta\in S$, we have $\lim_{k\to\infty}\psi(M(x_{m(k)-1},x_{n(k)-1},t))=0$,
which implies $\lim_{k\to\infty}M(x_{m(k)-1},x_{n(k)-1},t)=1$.
This fact and (2.10) give us $\epsilon=0$, which is a contradiction.
Hence the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x\in X$ such that $\lim_{n\to\infty}x_n=x$.

$$\begin{aligned}\psi(M(x_{n+1},Tx,t)) &= \psi(M(Tx_n,Tx,t))\\ &\le \beta(\psi(M(x_n,x,t)).\psi(M(x_n,x,t))\\ &< \psi(M(x_n,x,t)).\end{aligned}$$

Taking $n\to\infty$ on the both sides of the above inequality, we have

$$\begin{aligned}\psi(M(x,Tx,t)) &\le \psi(M(x,x,t)),\\ &= \psi(1),\\ &= 0.\end{aligned}$$

Since $\psi(M(x,Tx,,t))=0$, by the property of $(P2)$, we have $M(x,Tx,t)=1$,
that is, $x=Tx$,
that is, x is a fixed point of T.
To show uniqueness, let $y\neq x$ be another fixed point of T.
$\psi(M(x,y,t))=\psi(M(Tx,Ty,t))\le\beta(\psi(M(x,y,t))).\psi(M(x,y,t))<\psi(M(x,y,t))$,
which is a contradiction.
The proof is completed.

Particularly, taking $\beta(t)=k$, $0<k<1$ we obtain the following corollary.

Corollary 2.2 Let $(X,M,*)$ be a complete fuzzy metric space. Let $T:X\to X$ be a mapping, $\psi:[0,1]\to[0,1]$ is as in Definition 1.7 and β satisfies the Definition 1.8. If the mapping T satisfies the condition

$$\psi(M(Tx,Ty,t))\le k.\psi((M(x,y,t)),$$

for all $x,y\in X$, $t>0$ and $x\neq y$, then T has a unique fixed point in X.

Conclusion: The corollary can be viewed as a version of the contraction mapping principle in fuzzy metric spaces. With this consideration, Theorem 2.1 is an extension of the fuzzy contractions mapping principle.

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