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Approximations of holomorphic functions by generalized Zygmund sums

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Abstract. We determine the asymptotic equality for the upper bounds of deviations of generalized Zygmund sums $Z_{n,\psi}(f)(z) = \widehat{f}_0 + \sum_{k=1}^{n-1}(1 - \psi_n/\psi_k)\widehat{f}_k z^k$ on the functional classes $H_p^{\psi\phi}$ that are convolution of unit ball of the Hardy space H_p with kernels $\sum_{k=0}^{\infty}\psi_{k+1}\phi_{k+1}z^k$ in case when $\psi = \{\psi_k\}_{k=1}^{\infty}$ are the moment sequence. We give necessary and sufficient conditions on the sequence $\phi = \{\phi_k\}_{k=1}^{\infty}$ under which the sums $Z_{n,\psi}(f)$ approximate the class $H_p^{\psi\phi}$ with minimal possible error $|\psi_n|$.

Key Words and Phrases: Zygmund sums, Hardy space, Kolmogorov-Nikol'skii problem, Functions with positive real part.

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1. Introduction

Let $\mathcal{H}$ be a set of functions holomorphic in the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The Hardy space H_p, $1 \le p \le \infty$, is a set of all functions $f \in \mathcal{H}$ for which $\|f\|_p < \infty$, where

$$\|f\|_p := \begin{cases} \sup\limits_{0\le\varrho<1} \left(\int_0^{2\pi} \left| f(\varrho e^{it}) \right|^p \dfrac{dt}{2\pi} \right)^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{z\in\mathbb{D}} |f(z)|, & p = \infty. \end{cases}$$

We will denote by UH_p the unit ball of H_p.

Let $\psi = \{\psi_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that $|\psi_k| > 0$. We define generalized Zygmund sums for functions $f \in \mathcal{H}$ by

$$Z_{n,\psi}(f)(z) := \widehat{f}_0 + \sum_{k=1}^{n-1} \left(1 - \frac{\psi_n}{\psi_k}\right) \widehat{f}_k z^k, \ n \in \mathbb{N},$$

where $\widehat{f}_k := f^{(k)}(0)/k!$. (throughout this paper, we set empty sums equal to zero.)

In the 2π–periodic case the generalized Zygmund sums was introduced for the first time by Aljančić [1], [2]. They coincide with a classical Zygmund sums [15] in the case when $\psi_k = k^{-r}$, $r > 0$, and with the Fejér sums [4] when $\psi_k = k^{-1}$.

Denote by D^ψ the operator defined on $\mathcal{H}$ by the rule

$$D^\psi(f)(z) := \sum_{k=1}^{\infty} \frac{\widehat{f}_k}{\psi_k} z^{k-1}, \quad z \in \mathbb{D}.$$

If $\phi = \{\phi_k\}_{k=1}^\infty$ be another sequence of complex numbers such that $|\phi_k| > 0$, than

$$D^{\psi\phi}(f)(z) := D^\psi\left(zD^\phi(f)\right)(z) = \sum_{k=1}^{\infty} \frac{\widehat{f}_k}{\psi_k\phi_k} z^{k-1}, \quad z \in \mathbb{D}.$$

We assume in what follows that both sequences ψ and ϕ is such that sums of power series $\sum_{k=0}^\infty \psi_{k+1} z^k$ and $\sum_{k=0}^\infty \phi_{k+1} z^k$ defines a functions from $\mathcal{H}$.

By the class $H_p^{\psi\phi}$ we denote the set of functions $f \in \mathcal{H}$, for which $\|D^{\psi\phi}(f)\|_p \leq 1$. In particular, if $\phi_k = 1$ for all $k \in \mathbb{N}$, then

$$H_p^{\psi\mathbf{1}} = H_p^\psi := \left\{ f \in \mathcal{H} : \|D^\psi(f)\|_p \leq 1 \right\}.$$

The aim of the present work is to solve Kolmogorov–Nikolskii problem (K–N problem) for generalized Zygmund sums, that consists in finding the asymptotic formula for the quantity

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) := \sup\{\|f - Z_{n,\psi}(f)\|_p : f \in H^{\psi\phi}\}.$$

More precisely, we find a pair (μ, ν) of functions of natural argument such that $\nu(n) = o(\mu(n))$, $n \to \infty$, and

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = \mu(n) + O(\nu(n)), \quad n \to \infty.$$

Generally speaking, the K–N problem with respect to component ν is not uniquely solved. So finding the solution $(\mu, 0)$, that is computation of the exact value of $\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right)$, is the most desirable.

For the classes of 2π-periodical real-valued functions the K–N problem for generalized Zygmund sums was investigated in many works (see review in [14], and [12], [8]). With respect to holomorphic functions such researches held much less. The first case of solving a K–N problem for holomorphic functions should be considered the theorem 1 in [13], from which in our notation for $\psi_k = k^{-1}$, $\phi_k = k^{-s}$, $s \in \mathbb{N}$, follows asymptotic equality

$$\mathcal{Z}_{n,\psi}(H_\infty^{\psi\phi}; H_\infty) = n^{-1} + O(n^{-s-1}), \quad n \to \infty.$$

Actually, it was shown in [10], that the value O in this equation is equal to zero, i.e. the following equality holds

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = n^{-1} \quad \forall\, n \in \mathbb{N},\ 1 \leq p \leq \infty.$$

We generalize these two relations (Corollary 2) for the case when $\psi_k = k^{-r}$, $\phi_k = k^{-s}$, $r, s \geq 0$ and $r + s > 0$, namely, show that

$$\mathcal{Z}_{n,r}\left(H_p^{\psi\phi}; H_p\right) = \begin{cases} n^{-r} + O\left(n^{-r-s}\right), & 0 \leq s < 1, \\ n^{-r}, & s \geq 1, \end{cases} \quad n \in \mathbb{N}. \tag{1}$$

The first ratio in (1) follows from Theorem 1 and Corollary 1 of this paper, which are talking about pointwise approximation of individual functions $H_p^{\psi\phi}$ inside the disk $\mathbb{D}$ and about the solution of K–N problem for a value $\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right)$ in some important cases. The second ratio follows from the Theorem 3, which talks about the exact value of $\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right)$.

It is important to pay attention to the following fact. For any complex sequences ψ and ϕ such that $\psi_1 = \phi_1 = 1$ and $|\psi_k| > 0$, $|\phi_k| > 0$, $k = 2, 3, \ldots$, holds an inequality

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) \geq \|f^* - Z_{n,\psi}(f^*)\|_p = |\psi_n|, \tag{2}$$

where $f^*(z) = z$. Moreover, as it follows from the main result in [2], the ratio $\|f - Z_{n,\psi}(f)\|_p = o(|\psi_n|), n \to \infty$, can not be performed for any function $f \in H_p$ other than constant. Thus the order of $O(|\psi_n|)$ is the maximum order of the smallness of value $\mathcal{Z}_{n,\psi}\left(H_p^{\psi}; H_p\right)$. In connection with this naturally arise the question under what conditions for the sequence ψ the order of smallness is achieved. We show (Theorem 2) that it is sufficient to require for the ψ be a moment sequence in the sense of Hausdorff moment problem and satisfies condition $\psi_k = O(\psi_{2k})$, $k \in \mathbb{N}$.

In Theorem 3 we give a description of all sequences ϕ such that for a given sequence ψ holds an equality

$$\mathcal{Z}_{n,\psi}(H_\infty^{\psi\phi}; H_\infty) = |\psi_n|, \tag{3}$$

i.e. when generalized Zygmund sums $Z_{n\psi}$ approach the class $H_\infty^{\psi\phi}$ with minimum possible error.

2. The main results

Theorem 1. *Let* $1 \leq p \leq \infty$,

$$\psi_k = \int_0^1 \rho^{k-1} d\lambda(\rho), \quad k = 1, 2, \ldots, \tag{4}$$

where λ *be real-valued a bounded nondecreasing function on* $[0, 1]$ *such that* $\int_0^1 d\lambda = 1$, *and* ϕ *be a sequence of complex numbers, such that for all natural* n *beginning with some number* n_0

$$K_{n,\phi}(z) := \frac{1}{2} + \mathrm{Re} \sum_{k=1}^{\infty} \frac{\phi_{k+n}}{\phi_n} z^k \geq 0 \quad \forall\ z \in \mathbb{D}. \tag{5}$$

Then for every function $f \in H_p^{\psi\phi}$ the following equality holds for any natural $n \geq n_0$:

$$f(z) - Z_{n,\psi}(f)(z) = \psi_n z D^{\psi}(f)(z) + \varepsilon_n(z,f) \quad \forall\ z \in \mathbb{D}, \tag{6}$$

where

$$\|\varepsilon_n(\rho\ \cdot, f)\|_p \leq \rho^n |\phi_n| \left(\psi_n + \psi_{[\frac{n+1}{2}]}\right)\ \forall\ n \geq n_0,\ \rho \in [0,1],$$

and $[\cdot]$ is the integer part of number.

Corollary 1. *Let the conditions of Theorem 1 be satisfied and let the condition (5) hold for any $n \in \mathbb{N}$, $\psi_n = O(\psi_{2n})$, $\phi_1 = 1$ and $\phi_n = o(1)$. Then*

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = \psi_n + O\left(|\phi_n|\psi_n\right), \quad n \to \infty. \tag{7}$$

Relation (7) is a solution of the K–N problem in these cases.

Theorem 2. *Let $1 \leq p \leq \infty$ and ψ be a sequence such as in the Theorem 1. Then*

$$\psi_n \leq \mathcal{Z}_{n,\psi}\left(H_p^{\psi}; H_p\right) \leq \psi_{[\frac{n+1}{2}]} \quad \forall\ n \in \mathbb{N}. \tag{8}$$

In the next statement we describe the set of all sequences ϕ such that

$$\mathcal{Z}_{n,\psi}(H_\infty^{\psi\phi}; H_\infty) = |\psi_n|. \tag{9}$$

Theorem 3. *Suppose $n \in \mathbb{N}$, $\psi = \{\psi_k\}_{k=1}^\infty$ and $\phi = \{\phi_k\}_{k=1}^\infty$ are sequences of complex numbers such that $\psi_1 = \phi_1 = 1$ and $|\psi_k| > 0$, $|\phi_k| > 0$. Equality (9) holds true if and only if*

$$M_{n,\psi,\phi}(z) := \frac{1}{2} + \mathrm{Re}\left(\sum_{k=1}^{n-1} \phi_{k+1} z^k + \sum_{k=n}^{\infty} \frac{\psi_{k+1}\phi_{k+1}}{\psi_n} z^k\right) \geq 0 \quad \forall\ z \in \mathbb{D}. \tag{10}$$

If inequality (10) is true, then the following relation holds for all $p \in [1, \infty)$:

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi}; H_p\right) = |\psi_n|. \tag{11}$$

Denote by H_p^{r+s} the class $H_p^{\psi\phi}$ when $\psi_k = k^{-r}$ and $\phi_k = k^{-s}$ and let $\mathcal{Z}_{n,r} := \mathcal{Z}_{n,\psi}$. Note that in such a case

$$D^{\psi\phi}(f)(z) = \sum_{k=0}^{\infty} (k+1)^{r+s} \widehat{f}_{k+1} z^k.$$

Corollary 2. *Let $1 \leq p \leq \infty$, $r, s \geq 0$ and $r + s > 0$. Then*

$$\mathcal{Z}_{n,r}\left(H_p^{r+s}; H_p\right) = \begin{cases} n^{-r} + O\left(n^{-(r+s)}\right), & 0 \leq s < 1, \\ n^{-r}, & s \geq 1, \end{cases} \quad n \in \mathbb{N}.$$

3. Appendix

In this section we will show that relations (6) and (7), generally speaking, can not be the corollary of Theorem 3. Also we will formulate a simple condition under which the relation (10) are holds.

Proposition 1. *The sequence $\phi = \mathbf{1} := \{1\}_{k=1}^{\infty}$ satisfies the condition (5) for all $n \in \mathbb{N}$, but doesn't satisfy the condition (10) simultaneously for all $n \in \mathbb{N}$ whatever be the sequence ψ except the case $\psi = \mathbf{1} = \{1\}_{k=1}^{\infty}$.*

Proof. Indeed, for any $n \in \mathbb{N}$

$$K_{n,\mathbf{1}}(z) = M_{n,\mathbf{1},\mathbf{1}}(z) = \frac{1}{2} + \operatorname{Re}\sum_{k=1}^{\infty} z^k = \frac{1}{2}\operatorname{Re}\frac{1+z}{1-z} = \frac{1}{2}\frac{1-|z|^2}{|1-z|^2} \geq 0 \quad \forall\, z \in \mathbb{D}. \tag{12}$$

Suppose that the condition (10) holds for all natural n. Take an arbitrary function $g \in UH_\infty$ and with fixing any $n \in \mathbb{N}$ construct the sequence of functions $\{g_N\}_{N=0}^{\infty}$ by the rule

$$g_0 = g, \quad g_N(z) = \frac{1}{\pi}\int_0^{2\pi} g_{N-1}(e^{it}) M_{n,\psi,\mathbf{1}}(ze^{-it})dt,\ N = 1, 2, \ldots.$$

Clear that as a result of (10) $g_N \in UH_\infty$, $N = 0, 1, 2, \ldots$.

On the other hand, by direct computation easily convinced that

$$g_N(z) = \sum_{k=0}^{n-1} \widehat{g}_k z^k + \sum_{k=n}^{\infty}\left(\frac{\psi_{k+1}}{\psi_n}\right)^N \widehat{g}_k z^k \quad \forall\, z \in \mathbb{D}. \tag{13}$$

In particular, putting $g(z) = z^m$, $m \geq n$, we obtain the inequality

$$\left|\frac{\psi_{m+1}}{\psi_n}\right|^N = \|g_N\|_\infty \leq 1 \quad \forall\, N \in \mathbb{Z}_+\ \forall\, m \geq n.$$

Because of the arbitrariness n this relation implies that

$$1 \geq \left|\frac{\psi_{n+1}}{\psi_n}\right| \geq \left|\frac{\psi_{n+2}}{\psi_n}\right| \geq \ldots \quad \forall\, n \in \mathbb{N}.$$

For a given n equate to the number 1 can be achieved only in a finite number of the first row correspondences, or at all at once. We need to consider only the first of this two cases. Therefore without losing generality we consider that

$$1 \geq |\psi_2| = \ldots = |\psi_n| > |\psi_{n+1}| > \ldots. \tag{14}$$

Based on the expansion (13) and inequalities (14) we obtain the ratio

$$\left|\sum_{k=0}^{n-1}\widehat{g}_k z^k\right| \leq |g_N(z)| + \left|\sum_{k=n}^{\infty}\left(\frac{\psi_{k+1}}{\psi_n}\right)^N \widehat{g}_k z^k\right| \leq 1 + \left|\frac{\psi_{n+1}}{\psi_n}\right|^N \frac{|z|^n}{1-|z|} \quad \forall\, z \in \mathbb{D}.$$

Hence when $N \to \infty$ it follows that

$$G_n := \sup\left\{\left|\sum_{k=0}^{n-1} \widehat{g}_k\right| : g \in UH_\infty\right\} \le 1.$$

But as it was shown by E. Landau (see, for example, [7, p. 442]),

$$G_n = 1 + \sum_{k=1}^{n-1}\left(\frac{(2k-1)!!}{(2k)!!}\right)^2 > 1, \quad n \ge 2.$$

We have the contradiction. Hence, our assumption is incorrect, which proves the proposition 1.

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Proposition 2. *Let $n \in \mathbb{N}$, $\psi = \{\psi_k\}_{k=1}^\infty$ be any sequence of positive numbers decreasing to zero as $k \to \infty$ and $\phi = \{\phi_k\}_{k=1}^\infty$ be a sequence of complex numbers such that $|\phi_k| > 0$. Then for the condition (10) is sufficient to require that*

$$P_{n,\phi}(z) := \frac{1}{2} + \operatorname{Re}\sum_{k=1}^{n-1} \phi_{k+1} z^k \ge 0 \quad \forall\ z \in \partial\mathbb{D}\ \forall\ n \in \mathbb{N}. \tag{15}$$

Proof. Applying to the second sum in (10) the Abel transformation for series (this is correctly because of $\psi_n \left|\sum_{j=0}^n \phi_{j+1} z^j\right| \to 0, n \to \infty\ \forall\ z \in \mathbb{D}$), we get

$$\sum_{k=n}^\infty \frac{\psi_{k+1}\phi_{k+1}}{\psi_n} z^k = -\frac{\psi_{n+1}}{\psi_n}\sum_{k=0}^{n-1}\phi_{k+1}z^k + \frac{1}{\psi_n}\sum_{k=n}^\infty(\psi_{k+1}-\psi_{k+2})\sum_{j=0}^k \phi_{j+1}z^j.$$

Substituting this formula into expression of the function $M_{n,\psi,\phi}$ and taking into account that according to the maximum modulus principle $P_{n,\phi}(z) \ge 0\ \forall\ z \in \mathbb{D}$, we obtain

$$M_{n,\psi,\phi}(z) = -\frac{1}{2} + \frac{1}{\psi_n}\operatorname{Re}\left(\sum_{k=n-1}^\infty(\psi_{k+1}-\psi_{k+2})\sum_{j=0}^k \phi_{j+1}z^j\right) =$$

$$= -\frac{1}{2} + \frac{1}{\psi_n}\sum_{k=n-1}^\infty(\psi_{k+1}-\psi_{k+2})\left(P_{n,\phi}(z) + \frac{1}{2}\right) \ge$$

$$\ge -\frac{1}{2} + \frac{1}{2\psi_n}\sum_{k=n-1}^\infty(\psi_{k+1}-\psi_{k+2}) = 0 \quad \forall\ z \in \mathbb{D}.$$

◄

Let us note that in the case when all numbers ϕ_k are real the condition (15) is equivalent to the following

$$P_{n,\phi}(e^{it}) = \frac{1}{2} + \sum_{k=1}^{n-1} \phi_{k+1} \cos kt \geq 0 \quad \forall\, t \in [0,\pi]\ \forall\, n \in \mathbb{N}.$$

Detailed review of nonnegative trigonometric polynomials can be found in [7, ch. 4]. Currently the most general sufficient conditions for a real-valued sequence ϕ, for which $P_{n,\phi}(e^{it}) \geq 0$, are given in [3].

4. Proof of the results

Proofs of the Theorems 1 and 2 based on the following statement.

Lemma. *Suppose $1 \leq p \leq \infty$ and ψ are sequences of complex numbers defined by formula (4), where λ are a complex-valued function of bounded variation on $[0,1]$ such that $\int_0^1 d\lambda = 1$. Then for any function $f \in H_p^{\psi\phi}$ in every point $z \in \mathbb{D}$ and almost every point $z \in \mathbb{T}$*

$$f(z) - Z_{n,\psi}(f)(z) =$$

$$= \phi_n z^n \int_0^1 \int_0^{2\pi} D^{\psi\phi}(f)(\rho e^{it}) \rho^{n-1} e^{-i(n-1)t} K_{n,\phi}(\rho e^{it} z) \frac{dt}{\pi}\ d\lambda(\rho^2) + \tag{16}$$

$$+ \psi_n \sum_{k=1}^{n-1} \left(1 - |z|^{2(n-k)} \frac{\overline{\phi}_{2n-k}}{\phi_k} e^{2i \arg \phi_n}\right) \frac{\widehat{f}_k}{\psi_k} z^k \quad \forall\, n \in \mathbb{N}.$$

Proof. Consider the inner integral in (16). Denote for convenience $g(z) := D^{\psi\phi}(f)(\rho z)$, $c_k = \rho^k \phi_{k+n}/\phi_n$ and using the well-known identity (see [6, p. 515]), for any $z \in \mathbb{D}$ and $\rho \in [0,1)$ we obtain

$$\frac{1}{\pi} \int_0^{2\pi} D^{\psi\phi}(f)(\rho e^{it}) e^{-i(n-1)t} K_{n,\phi}(\rho e^{it} z) dt =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) e^{-i(n-1)t} \left(1 + 2\,\mathrm{Re} \sum_{k=1}^{\infty} c_k z^k e^{-ikt}\right) dt =$$

$$= \sum_{k=0}^{n-2} \widehat{g}_k \overline{c}_{n-k-1} \overline{z}^{n-k-1} + \sum_{k=n-1}^{\infty} \widehat{g}_k c_{k-n+1} z^{k-n+1} =$$

$$= \sum_{k=0}^{n-2} \frac{\widehat{f}_{k+1} \rho^k}{\psi_{k+1} \phi_{k+1}} \frac{\overline{\phi}_{2n-k-1} \rho^{n-k-1}}{\overline{\phi}_n} \overline{z}^{n-k-1} + \sum_{k=n-1}^{\infty} \frac{\widehat{f}_{k+1} \rho^k}{\psi_{k+1} \phi_{k+1}} \frac{\phi_{k+1}}{\phi_n} \rho^{k-n+1} z^{k-n+1} =$$

$$= \frac{1}{(\rho z)^{n-1} \phi_n} \left(\sum_{k=0}^{n-2} \frac{\widehat{f}_{k+1}}{\psi_{k+1}} \frac{\overline{\phi}_{2n-k-1}}{\phi_{k+1}} \rho^{2(n-1)} |z|^{2(n-k-1)} z^k + \sum_{k=n-1}^{\infty} \frac{\widehat{f}_{k+1}}{\psi_{k+1}} \rho^{2k} z^k\right).$$

By integrating the last equality with respect to the measure $d\lambda(\rho^2)$, and then reordering the change of integration and summation, we obtain

$$\frac{\phi_n z^n}{\pi}\int_0^1\int_0^{2\pi} D^{\psi\phi}(f)(\rho e^{it})\rho^{n-1}e^{-i(n-1)t}K_{n,\phi}(\rho e^{it}z)dt\ d\lambda(\rho^2)=$$

$$=z\left(\sum_{k=0}^{n-2}\frac{\psi_n}{\psi_{k+1}}\widehat{f}_{k+1}\frac{\overline{\phi}_{2n-k-1}}{\phi_{k+1}}|z|^{2(n-k-1)}z^k+\sum_{k=n-1}^{\infty}\widehat{f}_{k+1}z^k\right)=$$

$$=f(z)-Z_{n,\psi}(f)(z)-\sum_{k=1}^{n-1}\frac{\psi_n}{\psi_k}\widehat{f}_k z^k+\sum_{k=1}^{n-1}\frac{\psi_n}{\psi_k}\widehat{f}_k\frac{\overline{\phi}_{2n-k}}{\phi_k}|z|^{2(n-k)}z^k,$$

which proves the equality (16).

◀

Proof of Theorem 1. Set $g(z)=zD^{\psi}(f)(z)$,

$$U_{n,\phi}(g)(z):=\sum_{k=0}^{n-1}\left(1-|z|^{2(n-k)}\frac{\overline{\phi}_{2n-k}}{\phi_k}e^{2i\arg\phi_n}\right)\widehat{g}_k z^k,$$

and denote by $I_{n,\psi,\phi}(f)(z)$ the integral in (16). Then the formula (16) can be rewritten in the following form:

$$f(z)-Z_{n,\psi}(f)(z)=\psi_n g(z)+\phi_n z^n I_{n,\psi,\phi}(f)(z)+\psi_n\left(U_{n,\phi}(g)(z)-g(z)\right)\qquad(17)$$

Evaluate the second and third summands in a righthand side of (17). According to the condition (5) $\int_0^{2\pi}K_{n,\phi}(\rho e^{it}z)dt=\pi$, by Holder inequality we have

$$|I_{n,\psi,\phi}(f)(z)|^p\le\int_0^1\int_0^{2\pi}\left|D^{\psi}(f)(\rho e^{it})\right|^p\rho^{n-1}K_{n,\phi}(\rho e^{it}z)\frac{dt}{\pi}\ d\lambda(\rho^2)\times$$

$$\times\left(\int_0^1\int_0^{2\pi}\rho^{n-1}K_{n,\phi}(\rho e^{it}z)\frac{dt}{\pi}\ d\lambda(\rho^2)\right)^{p/q}=$$

$$=\psi^{p/q}_{[\frac{n-1}{2}]}\int_0^1\int_0^{2\pi}\left|D^{\psi}(f)(\rho e^{it})\right|^p\rho^{n-1}K_{n,\phi}(\rho e^{it}z)\frac{dt}{\pi}\ d\lambda(\rho^2),\quad \frac{1}{p}+\frac{1}{q}=1.$$

Therefore

$$\|I_{n,\psi,\phi}(f)(\rho\cdot)\|_p\le\psi^{1/p}_{[\frac{n-1}{2}]}\psi^{1/q}_{[\frac{n-1}{2}]}=\psi_{[\frac{n+1}{2}]}.\qquad(18)$$

Since $g\in H_p^{\phi}$ and ϕ satisfies the condition (5), then by result from [11, theorem 2],

$$\|U_{n,\phi}(g)(\rho\cdot)-g(\rho\cdot)\|_p\le\rho^n|\phi_n|\|g(\rho\cdot)\|_p\le\rho^n|\phi_n|.\qquad(19)$$

Combining (18), (19) and equality (17) the result follows.

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Proof of Corollary 1. By Cauchy formula we have the equality

$$D^{\psi}(f)(z)=\frac{1}{\pi}\int_0^{2\pi} D^{\psi\phi}(f)(e^{i\theta})K_{1,\phi}(ze^{i\theta})d\theta \quad \forall\, z\in\mathbb{D},$$

where $D^{\psi\phi}(f)(e^{i\theta})$ means nontangential boundary values of function $D^{\psi\phi}$ on the circle $\mathbb{T}:=\{z:|z|=1\}$, due to the fact that $D^{\psi\phi}\in H_p$.

Hence by Minkowski's inequality we have

$$\|D^{\psi}(f)\|_p\le\|D^{\psi\phi}(f)\|_p\le 1 \quad \forall\, f\in H_p^{\psi\phi}.$$

Thus, from the relation (6) and the last inequality follows an upper bound

$$\mathcal{Z}_{n,\psi}\left(H_p^{\psi\phi};H_p\right)\le\psi_n+O(|\phi_n|\psi_n),$$

which together with a lower bound (2) proved the Corollary 1.

◄

Proof of Theorem 2. Put in (16) $\phi_k=1$, $k=1,2,\ldots$. By Proposition 1 the condition (5) is satisfied. Thus estimate (18) takes place.

Therefore, according to (18)

$$\|f_\rho-Z_{n,\psi}(f_\rho)\|_{L_p}\le\rho^n\|I_{n,\psi,\phi}(f_\rho)\|_p+\psi_n\sum_{k=1}^{n-1}\left(1-\rho^{2(n-k)}\right)\frac{|\widehat{f}_k|}{\psi_k}\rho^k\le$$

$$\le\psi_{[\frac{n+1}{2}]}+\psi_n\|D^{\psi}(f)\|_p\sum_{k=1}^{n-1}\left(1-\rho^{2(n-k)}\right)\rho^k.$$

Taken in these correspondences the limit when $\rho\to 1-$ and taking into account that for any function $f\in H_p$ $\|f\|_p=\lim_{\rho\to1-}\|f_\rho\|_{L_p}$ (see, for example, [5, p. 55]), we obtain

$$\|f-Z_{n,\psi}(f)\|_p\le\psi_{[\frac{n+1}{2}]},$$

that together with (2) proves Theorem 2.

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Proof theorem 3. Let $D^{\psi\phi}(f)(e^{it})$, as before, denote the nontangential boundary value in a point e^{it} of function $D^{\psi\phi}(f)$.

Applying the Cauchy formula, easy to show that for any function $f\in H_p^{\psi\phi}$, $1\le p\le\infty$,

$$f(z)-Z_{n,\psi}(f)(z)=\frac{z\psi_n}{\pi}\int_0^{2\pi}D^{\psi\phi}(f)(e^{i(\theta+t)})M_{n,\psi,\phi}(\rho e^{it})dt \quad \forall\, z\in\mathbb{D}, z=\rho e^{i\theta}. \tag{20}$$

Based on this formula, taking into account correspondence (2), it is easy to verify that the condition (10) is sufficient. Also provided (10) applying to the evaluation of integral in the right part the Minkowski's inequality, we obtain the equality (11).

To prove the necessity of conditions (10), we proceed as follows.

From the formula (20) considering the invariance of class $H^{\psi\phi}_{\infty}$ with respect to rotation of argument ($f \in H^{\psi\phi} \Rightarrow f(e^{i\theta}\cdot) \in H^{\psi\phi}_{\infty}\ \forall\ \theta \in [0, 2\pi]$), and also by the principle of maximum modulus, for any $z \in \mathbb{D}$ we obtain an inequality

$$|\psi_n||z|\mathcal{M}_n(|z|) = \sup\left\{|f(z) - Z_{n,\psi}(f)(z)| : f \in H^{\psi\phi}_{\infty}\right\} \leq |\psi_n|,$$

where

$$\mathcal{M}_n(\rho) := \sup\left\{\left|\frac{1}{\pi}\int_0^{2\pi} F(e^{it})M_{n,\psi,\phi}(\rho e^{it})dt\right| : F \in UH_\infty\right\}.$$

So $\mathcal{M}_n(\rho) \leq 1/\rho\ \forall\ \rho \in [0, 1)$.

On the other hand, according to the relations of duality for holomorphic functions (see, for example, [5, p. 129]) holds the equality

$$\mathcal{M}_n(\rho) = \min\{\|2M_{n,\psi,\phi}(\rho\cdot) - g_n(\rho,\cdot)\|_1 : g_n(\rho,\cdot) \in H^0_1\}, \quad \rho \in [0, 1), \tag{21}$$

where minimum is achieved for a unique function $w \mapsto g^*_n(\rho, w)$, $w \in \mathbb{D}$, from the space $H^0_1 := \{f \in H_1 : f(0) = 0\}$.

Thus

$$1 = \frac{1}{2\pi}\int_0^{2\pi} 2M_{n,\psi,\phi}(\rho e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi}\left(2M_{n,\psi,\phi}(\rho e^{it}) - g^*_n(\rho, e^{it})\right)dt \leq \mathcal{M}_n(\rho).$$

Therefore,

$$1 \leq \mathcal{M}_n(\rho) \leq \frac{1}{\rho} \quad \forall\ \rho \in (0, 1). \tag{22}$$

Now show that the function $\rho \mapsto \mathcal{M}_n(\rho)$ is not decreasing on $[0, 1)$.

Let $0 \leq \rho_1 < \rho_2 < 1$. By the Poisson's formula applying to the function $z \mapsto 2M_{n,\psi,\phi}(\rho_2 z) - g^*_n(\rho_2, z)$, we obtain

$$2M_{n,\psi,\phi}(\rho_1 e^{it}) - g^*_n\left(\rho_2, \frac{\rho_1}{\rho_2}e^{it}\right) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(2M_{n,\psi,\phi}(\rho_2 e^{i\theta}) - g^*_n(\rho_2, e^{i\theta})\right)\frac{\rho_2^2 - \rho_1^2}{|\rho_2 - \rho_1 e^{i(t-\theta)}|^2}dt.$$

Hence

$$\mathcal{M}_n(\rho_1) \leq \left\|2M_{n,\psi,\phi}(\rho_1\cdot) - g^*_n\left(\rho_2, \frac{\rho_1}{\rho_2}\,\cdot\right)\right\|_1 \leq \mathcal{M}_n(\rho_2).$$

Therefore, $\mathcal{M}_n(\rho) \nearrow$. Combining this fact with the equation $\lim_{\rho\to 1-}\mathcal{M}_n(\rho) = 1$, which follows from (22), we see that $\mathcal{M}_n(\rho) = 1$ for any $\rho \in [0, 1)$. It follows that the value in a righthand side of equality (21) also equals to 1. According to the theorem 2 in [9] it is possible if and only if the condition (10).

◀

Proof of Corollary 2. Suffices to show that sequences $\psi = \{k^{-r}\}_{k=1}^{\infty}$ and $\phi = \{k^{-s}\}_{k=1}^{\infty}$ satisfy conditions of Corollary 1 and Theorem 3 under the proper conditions on the parameter s.

Indeed, since

$$\psi_k = k^{-r} = \frac{1}{\Gamma(r)} \int_0^1 \rho^{k-1} \left(\ln \frac{1}{\rho}\right)^{r-1} d\rho,$$

and moreover $\psi_k = 2^r \psi_{2k}$, then ψ satisfies the conditions of Corollary 1.

For the sequence ϕ we have

$$K_{n,\phi}(z) = \frac{1}{2} + \mathrm{Re} \sum_{k=1}^{\infty} \frac{n^s}{(k+n)^s} z^k = n^s \left(\frac{a_0(z)}{2} + \sum_{k=1}^{\infty} a_k(z) \cos kx \right), \tag{23}$$

where $a_k(z) = |z|^k (k+n)^{-s}$, $x = \arg z$.

Since for each $z \in \mathbb{D}$ the sequence $\{a_k(z)\}_{k=1}^{\infty}$ is convex, and clearly, $a_k(z) \downarrow 0$, according to the well-known statement (see, for example, [16, p. 183]) a sum of series in righthand side of (23) is a nonnegative.

So $K_{n,\phi}$ satisfies the condition (5) for all $s \geq 0$. Hence holds true (7) for all $r \geq 0$ and $s \geq 0$.

If $s \geq 1$, then

$$P_{n,\phi}(z) = \frac{b_0(z)}{2} + \sum_{k=1}^{n-1} \frac{b_k(z) \cos kx}{k+1},$$

where $b_k(z) = |z|^k (k+1)^{1-s}$ and $x = \arg z$.

Since for each $z \in \overline{\mathbb{D}}$ coefficients $b_k(z)$, $k = 1, 2, \ldots$, are nonnegative and not increasing, then by the theorem of Rogosinskii–Szego (see, for example, [7, p. 330]), $P_{n,\phi}$ satisfies the condition (15) for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Therefore, according to Proposition 2 the condition (10) is satisfied. Hence the condition (11) is also satisfied.

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