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Integral Representations of Functions From the Spaces $S_p^l W(G)$, $S_{p,\theta}^l B(G)$ and $S_{p,\theta}^l F(G)$

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Abstract. In the paper we construct an integral representation of functions from $S_p^l W(G)$, $S_{p,\theta}^l B(G)$ and $S_{p,\theta}^l F(G)$, defined in n-dimensional domains and satisfying the flexible φ-horn condition.

Key Words and Phrases: integral representations, flexible φ-horn, the spaces type functions with dominant mixed derivatives.
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1. Introduction

Integral representation of functions from the spaces with dominant mixed derivative Sobolev - $S_p^l W(G)$ Besov -$S_{p,\theta}^l B(G)$ Lizorkin-Triebel- $S_{p,\theta}^l F(G)$ in the case when the domain $G \subset R^n$ satisfies the conditions of rectangles, was first studied in the paper of A.J. Jabrailov [3], and then in the papers of R.A. Mashiev [5], M.K. Aliyev [1] and others, in the case when the domain $G \subset R^n$ satisfies the "flexible horn condition", in the papers of A.M. Najafov [5], [6], [7].

In this paper we construct an integral representation of functions from these spaces, defined in n-dimensional domains and satisfying the flexible φ-horn condition. Let vector functions $\varphi(t) = (\varphi_1(t_1), ..., \varphi_n(t_n))$ be differentiable continuous on $[0, T_j]$ $(0 < T_j < \infty)$, $\varphi_j(t_j) > 0\,(t_j > 0)$, $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$, $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) = A_j \leq \infty\,(j = 1, 2, ..., n)$. Suppose that $e_n = \{1, 2, ..., n\}$, $e \subseteq e_n$ and for each $x \in G$ consider the vector-function

$$\rho(\varphi(t), x) = (\rho_1(\varphi_1(t_1), x), ..., \rho_n(\varphi_n(t_n), x)), \quad 0 \leq t_j \leq T_j, \quad j \in e_n,$$

where $\rho_j(0, x) = 0$ for all $j \in e_n$, the functions $\rho_j(\varphi_j(t_j), x)$ are absolutely continuous on $[0, T_j]$ and $|\rho_j(\varphi_j(t_j), x)| \leq 1$ for almost all $t_j \in [0, T_j]$, $\rho'_j(u_j, x) = \frac{\partial}{\partial u_j} e_j(u_j, x)$, $j \in e_n$. Given $\theta[0,1]^n$, each of the sets $V(x, \theta) = \bigcup\limits_{0 < t_j \leq T_j} [\rho(\varphi(t), x) + \varphi(t)\theta I]$ and $x + V(x, \theta) \subset G$, where $I = [-1, 1]^n$, $\varphi(t)\theta I = \{(\varphi_1(t_1)\theta_1 y_1, ..., \varphi_n(t_n)\theta_n y_n) : y \in I\}$, is called a flexible φ-horn and the point x is called the vertex of the flexible φ-horn $x + V(x, \theta)$. In the case $\varphi_j(t_j) = t_j$ the set $x + V(x, \theta)$ is called the flexible horn introduced in [6], [7].

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Let $1^e = (\delta_1^e, ..., \delta_n^e)$, where $\delta_j^e = 1$ for $j \in e$ and $\delta_j^e = 0$ for j$\in e_n \backslash e = e'$. We suppose that $f \in L^{loc}(G)$ has all needed generalized derivatives on G. Introduce the average of f as follows:

$$f_{\varphi(t)}(x) = \prod_{j\in e_n} (\varphi_j(t_j))^{-1} \int_{R^n} f(x+y)\,\Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}\right) dy, \tag{1}$$

where $\Omega(y,z) = \prod_{j\in e_n} \omega_1(y_j, z_j)$, $\frac{y}{\varphi(t)} = \left(\frac{y_1}{\varphi_1(t_1)}, ..., \frac{y_n}{\varphi_n(t_n)}\right)$, in case $\varphi_j(t_j) = t^{\lambda_j}$ is the kernel introduced by O.V. Besov [2]. The average of (1) is constructed from the values of f at the points $x + y \in x + V(x,\theta) \subset G$. Let $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $0 < \varepsilon_j < T_j\ (j \in e_n)$. Then the following equality is valid:

$$f_{\varphi(\varepsilon)}(x) = \sum_{e\subseteq e_n} (-1)^{|1^e|} \int_{\varepsilon^{en}}^{T^e} D_t^{1^e} f_{\varphi\left(t^e + T^{e'}\right)}(x)\, dt^e, \tag{2}$$

where $t^e + T^{e'} = t_j\ j = e$; $t^e + T^{e'} = T_j, j \in e^i$ and $\int_{a^e}^{b^e} f(x)\, dx^e = \left(\prod_{j\in e_n} \int_{a_j}^{b_j} dx_j\right) f(x)$ i.e., integration is carried out only with respect to the variables x_j whose indices belong to e.

Differentiating with respect to $t_j\ (j \in e)$, and using [2], we obtain

$$D^{1^e} f_{\varphi\left(t^e + T^{e'}\right)}(x) = \prod_{j\in e} \frac{\partial}{\partial t_j} f_{\varphi\left(t^e + T^{e'}\right)} = (-1)^{|1^e|} \prod_{j\in e'} (\varphi_j(T_j))^{-1} \times$$

$$\times \int_{R^n} K_e^{(k^e + 1^e)} \left(\frac{y}{\varphi(t^e + T^{\varepsilon'})}, \frac{\rho\left(\varphi\left(t^e + T^{e'}\right), x\right)}{\varphi(t^e + T^{e'})}, \rho'\left(\varphi\left(t^e + T^{e'}\right), x\right)\right) \times$$

$$\times = (-1)^{|1^e|} \prod_{j\in e'} (\varphi_j(T_j))^{-1} \times (\varphi_j(t_j))^{-1} \prod_{j\in e} \frac{1}{\varphi_j(t_j)} \prod_{j\in e} \frac{\partial}{\partial t_j} \varphi_j(t_j)\, dy, \tag{3}$$

where $k = (k_1, ..., k_n)$, k_j -number in the kernel Ω can be chosen arbitrarily large,

$$K_e(x,y,z) = \prod_{j\in e'} \omega_j(x_j, y_j) \prod_{j\in e} \rho_j(x_j, y, z_j) \in C_0^\infty(R^n \times R^n \times R^n),$$

ρ_j -is defined in [7] and

$$K_e^{(\alpha)}(x,y,z) = D_x^{(\alpha)} K_e(x,y,z), \int_{R^n} K_e^{(\alpha)}(x,y,z)\, dx = 0 \ \text{ for all } y, z,$$

and α such that $|\alpha| > 0$.

By (2) from (3) we derive

$$f_{\varphi(\varepsilon)}(x)=\sum_{e\subseteq e_n}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\int_{\varepsilon^e}^{T^e}\int_{R^n}K_e^{(k^e+1^e)}\times$$

$$\times\left(\frac{y}{\varphi(t^e+T^{\varepsilon'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times f(x+y)\prod_{j\in e}(\varphi_j(t_j))^{-2}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy. \tag{4}$$

Then, in view of the Remark on Lemma 5.2 of [2], we have the following: if $f\in L^{loc}(G)$ and $1\le p<\infty$, then $f_{\varphi(\varepsilon)}\to f(x)$ as $\varepsilon_j\to 0\,(j\in e_n)$, moreover, for $p>1$ we have $f_{\varphi(\varepsilon)}(x)\to f(x)$ for almost all $x\in G$ by the Remark on Theorem 1.7 of [2]. Then it follows from (4) that

$$f(x)=\sum_{e\subseteq e_n}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\times$$

$$\times\int_{\varepsilon^e}^{T^e}\int_{R^n}K_e^{(k^e+1^e)}\left(\frac{y}{\varphi(t^e+T^{\varepsilon'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times f(x+y)\prod_{j\in e}(\varphi_j(t_j))^{-2}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy. \tag{5}$$

Let $l=(l_1,...,l_n)$, $l_j\in N$, $l^e=(l_1^e,...,l_n^e)$, $l_j^e=l_j$ for $j\in e$, $l_j^e=0$ for $j\in e'$, and let functions f having on G the generalized mixed derivatives $D^{l^e}f\in L^{loc}(G)$ and suppose that $l_j\le k_j$ for $j\in e$

$$f_{\varphi(\varepsilon)}(x)=\sum_{e\subseteq e_n}(-1)^{|l^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\times$$

$$\times\int_{\varepsilon^e}^{T^e}\int_{R^n}M_e\left(\frac{y}{\varphi(t^e+T^{\varepsilon'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)\times$$

$$\times D^{l^e}f(x+y)\prod_{j\in e}(\varphi_j(t_j))^{-2+l_j}\,\varphi_e'(t)\,dt^e dy. \tag{6}$$

where $\varphi_e'(t)=\prod_{j\in e}\varepsilon\varphi_j'(t_j)$, $M_e(x,y,z)=D_x^{k^e}+1^e-l^eK_e(x,y,z)$. Suppose that we obtain

$$f_{\varphi(\varepsilon)}^{(\nu)}(x)=\sum_{e\subseteq e_n}(-1)^{|\nu|+||l^e||}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\times$$

$$\times\int\limits_{\varepsilon^e}^{T^e}\int\limits_{R^n} M_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{\varepsilon'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)},\rho'\left(\varphi\left(t^e+T^{e'}\right),x\right)\right)D^{l^e}f\left(x+y\right)\times$$

$$\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-2+l_j-\nu_j}\varphi_e'\left(t\right)dt^e dy. \tag{7}$$

Suppose that $\varphi_j\left(\varphi_j\left(t_j\right),x\right)$ and $\rho_j'\left(\varphi_j\left(t_j\right),x\right)$ as functions of $\left(\varphi_j\left(t_j\right),x\right)$ are locally summable on $(0,T_j]\times U$ $(j=1,...,n)$, where $U\subset G$ is an open set. Let $\nu=(\nu_1,...,\nu_n)\in N_0^n$; moreover, $l_j\le\nu_j+k_j$ for $j\in e$, and $l_j\le k_j$ for $j\in e'$. Applying the differentiation D_x^ν to both sides of (4) (and moving the differentiation onto the kernel in the summands on the right-hand side), we obtain.

Show now that if

$$\mu_j=l_j-\nu_j>0,\quad j\in e_n \tag{8}$$

then the generalized mixed derivative $D^\nu f\in L_p\left(G\right)$ exists on G. First, establish that

$$f_{\varphi(\varepsilon)}^{(\nu)}-f_{\varphi(\eta)}^{(\nu)}\to 0\quad\text{as}\quad 0<\varepsilon_j<\eta_j\to 0,\ \ j\in e_n, \tag{9}$$

in $L^{loc}\left(U\right)$. Let $F\subset U$ be a compact set. Then $F+hI\subset U$ for some $h>0$. Put

$$M^{(\nu)}\left(x\right)=\max_{e\subseteq e_n}\max_{y_1z\in I}\left|M_e^{(\nu)}\left(x,y,z\right)\right|.$$

By Minkowski's inequality, for sufficiently small ε and $T\equiv\eta$ we have

$$\left\|f_{\varphi(\varepsilon)}^{(\nu)}-f_{\varphi(\eta)}^{(\nu)}\right\|_{1,F+hI}\le C\sum_{e\subseteq e_n}\prod_{j\in e}\left(\varphi_j\left(\eta_j\right)\right)^{\mu_j-\nu_j}\left\|D^{\lambda^e}f\right\|_{1,F+hI}.$$

From here and (8) we obtain (9). Suppose that $D^\nu f$ exists on G, i.e.,

$$f_{\varphi(t)}^{(\nu)}\left(x\right)=D^\nu f\left(x\right)$$

for $x+C\varphi\left(t\right)I\subset G$ with some $C=(C_1,...,C_n)>0$. Pass to the limit in (7) as $\varepsilon_j\to 0\,(j\in e_n)$, observing that the limit exists in the sense of $L^{loc}\left(U\right)$ by (8) and almost everywhere on U by the relation $f_{\varphi(t)}^{(\nu)}\to f\left(x\right)$ as $\varphi_j\left(t_j\right)\to 0\,(j\in e_n)$ applied to $D^\nu f$. Then the equality

$$D^\nu f=\sum_{e\subseteq e_n}(-1)^{|\nu|+|l^e|}\prod_{j\in e}\left(\varphi_j\left(T_j\right)\right)^{-1-\nu_j}\int\limits_{o^e}^{T^e}\int\limits_{R^n}M_e^{(\nu)}\left(,,\right)D^{l^e}f\left(x+y\right)\times$$

$$\times\prod_{j\in e}\left(\varphi_j\left(t_j\right)\right)^{-2+l_j-\nu_j}\varphi_e'\left(t\right)dt^e dy \tag{10}$$

holds for almost all $x \in U$. Recall that the flexible φ- horn $x + V(x,\theta)$ is the support of the representation (10) for $x \in U$. We can assume that the kernels M_e and $M_e^{(\nu)}$ satisfy the following relations for all α and β

$$\int D_x^{\alpha} M_e(x,y,z)\,dx = 0, \qquad \int D_x^{\beta} M_e(x,y,z)\,dx = 0.$$

Now we construct an integral representation for studying the properties of functions from $S^l_{p,\theta} B(G\varphi)$ defined in n-dimensional domains and satisfying the flexible φ-horn condition. Introduce the average of f as follows:

$$\overline{f}_{\varphi(t)}(x) = \left(f_{\varphi(t)}\right)_{\varphi(t)}(x) = \prod_{j\in e_n} (\varphi_j(t_j))^{-2} \int\int \Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)}\right) \times$$

$$\times \Omega\left(\frac{z}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)}\right) f(x+y+z)\,dydz. \tag{11}$$

Obviously, $\Omega\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{2\varphi\left(t^e+T^{e'}\right)}\right) \neq 0$ is possible only for $\left|y_j - \frac{1}{2}\rho_j(\varphi_j(t_j),x)\right| < \sigma_j\left[1 + m_j + \frac{1}{2}m_j\right]\varphi_j(t_j)$, here σ_j, m_j are integers in formula of ω_j determined in [2]. Hence, it follows that double averaging was constructed by contraction of f on $x + \rho\left(\varphi\left(t^e + T^{e'}\right),x\right) + m\varphi\left(\sigma\left(t^e + T^{e'}\right)\right) I$ and was defined for $0 < \sigma < \frac{\eta}{m_0}$, $m_0 = \max(2 + 3m_j)$. Let

$$\overline{f}^{(\nu)}_{\varphi(t)}(x) = (-1)^{|\nu|} \prod_{j\in e_n} (\varphi_j(t_j))^{-2-\nu_j} \int_{R^n}\int_{R^n} \Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)}\right) \times$$

$$\times \Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{2\varphi(t)}\right) f(x+y+z)\,dydz. \tag{12}$$

Note that if there exists $D^{\nu} f \in L^{loc}(G)$, then by $G(2)$ [2] $\overline{f}^{(\nu)}_{\varphi(t)}(x) = (D^{\nu} f)^{(x)}_{\varphi(t)}$ for $x \in U$, $0 < t_j \le T_j$ $(j \in e_n)$. Applying the equality

$$g(z_j) = \int_{-\infty}^{\infty} \omega_j\left(\frac{y}{\varphi_j(t_j)}, \frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)}\right) f(x+y+z)\,dy_j$$

we have

$$D_t^{1^e} \overline{f}_{\varphi\left(t^e+T^{e'}\right)}(x) = (-1)^{|1^e|} \prod_{j\in e_n} (\varphi_j(T_j))^{-1} \prod_{j\in e} A_j^{-1} \prod_{j\in e} (\varphi_j(t_j))^{-3} \prod_{j\in e} \varphi'_j(t_j) \times$$

$$\times \int_{R^n}\int_{-\infty}^{\infty} \psi_e\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)}, \frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)}\right) \times$$

$$\times\prod_{j\in e} S_j\left(\frac{u_j}{\varphi_j(t_j)}-\frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)},\frac{1}{2}\rho_j'(\varphi_j(t_j),x)\right)\Delta^{m^e}(\varphi(\delta u))\,f(x+y+u^e)\,du^e dy=$$

$$=(-1)^{|1^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1}\prod_{j\in e}(\varphi_j(t_j))^{-3}\prod_{j\in e}\varphi_j'(t_j)\times$$

$$\times\int_{R^n}\int_{-\infty}^{\infty}\psi_e\left(\frac{y}{\varphi(t^e+T^{e'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})}\right)\times$$

$$\times S_e\left(\frac{u}{\varphi(t)}-\frac{\rho(\varphi(t),x)}{2\varphi(t)}\right)\Delta^{m^e}(\varphi(\delta u))\,f(x+y+u^e)\,dy du^e, \tag{13}$$

where $\Delta^{m^e}(t)f=\prod\limits_{j\in e}\Delta_j^{m_j}(t_j)f$,

$$\psi_e\left(\frac{y}{\varphi(t^e+T^{e'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})}\right)=$$

$$=2^{|1^e|}\prod_{j\in e'}A_j^{-1}\prod_{j\in e'}(\varphi_j(t_j))^{-1}\left\{\int\prod_{j\in e'}\omega_j\left(\frac{y_j}{\varphi_j(T_j)},\frac{\rho_j(\varphi_j(T_j),x)}{2\varphi_j(T_j)}\right)\times\right.$$

$$\left.\times\omega_j\left(\frac{z}{\varphi_j(T_j)},\frac{\rho_j(\varphi_j(T_j),x)}{2\varphi_j(T_j)}\right)dz^{e'}\right\}\times$$

$$\times\prod_{j\in e}\omega_j\left(\frac{y_j}{\varphi_j(t_j)},\frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)}\right)\frac{\partial}{\partial y_j}\omega_j\left(\frac{z_j}{\varphi_j(t_j)},\frac{\rho_j(\varphi_j(t_j),x)}{2\varphi_j(t_j)}\right),$$

S_j, A_j are defined in [7, p. 88].

The equality (13) is valid in some vicinity of $x^{(0)}\in U$ also for the vector function $\rho\left(\varphi(t),x^{(0)}\right)$ instead of $\rho(\varphi(t),x)$. In this case, differentiating it with respect to x in the vicinity of the point $x^{(0)}$, taking into account possibility of carrying over the differentiation operation on the kernel, we have

$$D_t^{1^e}\overline{f}^{(\nu)}_{\varphi(t^e+T)}(x)=(-1)^{|\nu|+|1^e|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\prod_{j\in e}(\varphi_j(t_j))^{-3-\nu_j}\prod_{j\in e}\varphi_j'(t_j)\times$$

$$\times\int_{R^n}\int_{-\infty}^{\infty}\psi_e^{(\nu)}\left(\frac{y}{\varphi(t^e+T^{e'})},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi(t^e+T^{e'})}\right)\times$$

$$\times S_e\left(\frac{u}{\varphi(t)}-\frac{\rho(\varphi(t),x)}{2\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\Delta^{m^e}(\varphi(\delta u))\,f(x+y+u^e)\,dy du^e. \tag{14}$$

Hence we get

$$D^{\nu}f(x)=\sum_{e\subseteq e_n}(-1)^{|\nu|}\int\limits_{0^e}^{T^e}D_t^{1e}\overline{f}^{(\nu)}_{\varphi\left(t^e+T^{e'}\right)}(x)\,dt^e=$$

$$=\sum_{e\subseteq e_n}(-1)^{|\nu|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\times$$

$$\times\int\limits_{0^e}^{T^e}\int\limits_{R^n}\int\limits_{-\infty^e}^{\infty^e}\psi_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{\varphi\left(t^e+T^{e'}\right)}\right)\times$$

$$\times S_e\left(\frac{u}{\varphi(t)}-\frac{\rho(\varphi(t),x)}{2\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\Delta^{m^e}(\varphi(\delta u))f(x+y+u^e)\times$$

$$\times\prod_{j\in e}(\varphi_j(T_j))^{-3-\nu_j}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy du^e. \qquad (15)$$

Note that $\psi_e(y,z)\in C^{\infty}(R^n\times R^n)$, i.e. is infinitely differentiable with respect to all variables, and $\psi_e(\cdot,z)$ is uniformly finite with respect to z from the arbitrary compact. The equality (15) is valid almost everywhere on V, the set $x+V(x,\theta)$ is a support of this representation.

Show that if the function f satisfies the conditions

$$\left(\int\limits_{0^e}^{\infty^e}\prod_{j\in e_n}(\varphi_j(t_j))^{-1-\theta_e l_j}\left\|\Delta^{m^e}(\varphi(t),E)f\right\|_p^{\theta_\varepsilon}dt^e\right)^{\frac{1}{\theta_e}}\le A_e(E),\quad e\subseteq e_n,$$

where $A_e(E)$ are the constants independent of E and the vector $\nu=(\nu_1,...,\nu_n)$,$\nu_j\ge 0$ are entire $(j\in e_n)$ satisfy the conditions $\varepsilon_j=l_j-\nu_j>0\,(j\in e_n)$, then there exists the derivative $D^{\nu}f\in L_p^{loc}(G)$ and identity (15) is valid.

Let $\rho_j(\varphi_j(t_j),x)=0$, $0\le t_j\le T_j\,(j\in e_n)$ and the compact $F\subset G$. Then for all rather small $h=(h_1,...,h_n)$, $h_j>0\,(j\in e_n)\,F+hI$ is contained in some compact $E\subset G$. Based on (15), Minskovsky-Young and Holder generalized inequalities, we successively get

$$\left\|\overline{f}^{(\nu)}_{\varphi(\varepsilon)}-\overline{f}^{(\nu)}_{\varphi(T)}\right\|_{p,F}\le$$

$$\le\sum_{e\subseteq e_n}C_e\left\|\psi_e^{(\nu)}\right\|_1\|S_e\|_{\theta_e}A_e(E)\prod_{j\in e}(\varphi_j(T_j))^{l_j-\nu_j}$$

hence it follows that $\left\|\overline{f}^{(\nu)}_{\varphi(\varepsilon)}-\overline{f}^{(\nu)}_{\varphi(T)}\right\|_{p,F}\to 0$ for $0<\varepsilon_j<T_j\to 0$, $j\in e_n$. Then

$$\overline{f}_{\varphi(\varepsilon)}(x)\to f(x)$$

as $\varepsilon_j \to 0\,(j \in e_n)$ in the sense of convergence $L^{loc}(G)$. Based on lemma 6.2 [2] we deduce that there exists $D^\nu f \in L_p(G)$.

Now, for studying the space $S^l_{p,\theta}F(G)$ we construct integral representation of functions being some modification of the representation (15). Introduce the following averaging that differ from previous ones:

$$\widetilde{f}_{\varphi(t)}(x) = \left(\left(f^{(x)}_{\varphi(t)}\right)_{\varphi(t)}\right)_{\varphi(t)}(x) =$$

$$= \prod_{j\in e_n} (\varphi_j(t_j))^{-1} \int_{R^n} \Omega\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{3\varphi(t)}\right) f(x+y)\,dy, \tag{16}$$

and assume that $m\varphi(\sigma t)\,I \subset (m\varphi(\eta t)\,I)_{\varphi(\eta t)} \subset (\varphi(\eta t)\,I)_{\varphi(\sigma t)}$ σ_j, m_j are the numbers contained in the formula of ω_j that were determined in [2]. In other words, $\widetilde{f}_{\varphi(t)}(x)$ was constructed by contraction of f on $G_{\varphi(\sigma t)}$. Differentiating the equality (16) with respect to x in the neighborhood of the point $x^{(0)}$ taking into account possibility to transfer the differential operation on the kernel, we get

$$\widetilde{f}^{(\nu)}_{\varphi(t)}(x) = (-1)^{|\nu|} \prod_{j\in e_n} (\varphi_j(t_j))^{-1-\nu_j} \int_{R^n} \Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{3\varphi(t)}\right) f(x+y)\,dy, \tag{17}$$

Differentiating with respect to t_j and under the condition $0 < \varepsilon_j < T_j$, $j \in e_n$ from (17) we get

$$\widetilde{f}^{(\nu)}_{\varphi(\varepsilon)}(x) = \sum_{e\subseteq e_n} (-1)^{|1^\varepsilon|} \int_{\varepsilon^e}^{T^\varepsilon} D^{1^e}_t f_{\varphi\left(t^e+T^{e'}\right)}(x)\,dt^e =$$

$$= \sum_{e\subseteq e_n} (-1)^{|\nu|+|1^\varepsilon|} \prod_{j\in e'} (\varphi_j(T_j))^{-1-\nu_j} \times$$

$$\times \int_{\varepsilon^e}^{T^\varepsilon} \int_{R^n} M^{(\nu)}_e \left(\frac{y}{\varphi(t^e+T^{e'})}, \frac{\rho\left(\varphi\left(t^e+T^{e'}\right), x\right)}{3\varphi(t^e+T^{e'})}\right) f_e(x+y,t) \times$$

$$\times \prod_{j\in e} (\varphi_j(t_j))^{-2-\nu_j} \prod_{j\in e} \varphi'_j(t_j)\,dt^e dy, \tag{18}$$

where $M^{(\nu)}_e(y,a) = D^{(\nu)}_y M_e(y,a)$; $y, a \in R^n$,

$$M^{(\nu)}_e(y,a) = 3\prod_{j\in e} A_j^{-1} \left\{\prod_{j\in e'} \omega_j\left(y_j - z_j - u_j, \frac{a_j}{3}\right) \omega_j\left(z_j, \frac{a_j}{3}\right) \omega_j, \frac{a_j}{3}\right\} \times$$

$$\times \prod_{j\in e} \frac{\partial}{\partial y_j} \omega_j\left(y_j, \frac{a_j}{3}\right),$$

$$f_e(x,t)=\prod_{j\in e}(\varphi_j(t_j))^{-2}\int\int\prod_{j\in e}\omega_j\left(\frac{z_j}{\varphi_j(t_j)},\frac{\rho(\varphi(t^e),x)}{3\varphi_j(t_j)}\right)\times$$
$$\times\rho_j\left(\frac{u_j}{\varphi_j(t_j)}-\frac{\rho_j(\varphi_j(t_j),x)}{3\varphi_j(t_j)},\frac{1}{3}\rho_j(\varphi_j(t_j),x)\right)\times$$
$$\times\Delta^{m^e}\left(\varphi(\delta u),G_{\varphi(\eta t)}\right)f(x+z^e+u^e)\,du^e dz^e,$$

furthermore, we can show that

$$|f_e(x,t)|\le C\int_{-1^e}^{1^e}\delta^{m^e}(\varphi(\delta t))\,f(x+v\varphi(t))\,dv^e.$$

Note that under the condition $l_j-\nu_j>0\,(j\in n)$ there exists a generalized derivative $D^\nu f\in L^{loc}(G)$. Then $\widetilde{f}^{(\nu)}_{\varphi(\varepsilon)}(x)=\left(D^\nu\widetilde{f}\right)_{\varphi(\varepsilon)}(x)$ and as $\varepsilon_j\to 0\,(j\in n)\,\widetilde{f}_{\varphi(\varepsilon)}\to D^\nu f$ almost everywhere on G and b in the sense of $L^{loc}(G)$. Passing to the limit $\varepsilon_j\to 0$, $j\in e_n$ for $f\in S^l_{p,\theta}F(G)$

$$D^\nu f(x)=\sum_{e\subseteq e_n}(-1)^{|\nu|}\prod_{j\in e'}(\varphi_j(T_j))^{-1-\nu_j}\times$$
$$\times\int_{0^e}^{T^e}\int_{R^n}M_e^{(\nu)}\left(\frac{y}{\varphi\left(t^e+T^{e'}\right)},\frac{\rho\left(\varphi\left(t^e+T^{e'}\right),x\right)}{3\varphi\left(t^e+T^{e'}\right)}\right)f_e(x+y,t)\times$$
$$\times\prod_{j\in e'}(\varphi_j(t_j))^{-2-\nu_j}\prod_{j\in e}\varphi_j'(t_j)\,dt^e dy, \tag{19}$$

where the equality is fulfilled almost everywhere on G in the sense $L^{loc}(G)$ and the set $X+\bigcup_{0<t_j\le T_j}[\rho(\varphi(t),x)+m\varphi(\sigma t)I]$ is the support of the representation (19). It should be noted that $M_e\in C_0^\infty(R^n x R^n)$ and $\int M_e^{(\nu)}(y,a)=0$; $ty,a\in R^n$, $\nu\in N_0^n$.

References

[1] M.K. Aliyev, *Estimations of the norms of functions reduced to polynomials determined in n-dimensional domains*, Phd thesis, Baku, 1996, 140 p.

[2] O.V. Besov, V.P. Il'in, S.M. Nikolsky, *Integral representation of functions and imbedding theorems*, Nauka, Moscow, 1996, 480 p.

[3] A.D. Jabrailov, *Studying differential properties of functions determined on n-dimensional domain*, Doctor's dissertation, Moscow, 1971, 355 p.

[4] R.A. Matiyev, *On an integral representation of functions and imbedding theorems of some functional spaces*, Phd dissertation, Baku, 1988, 135 p.

[5] A.M. Najafov, *Embedding theorems in the Sobolev-Morrey type spaces* $S^{l}_{p,a,\varkappa,\tau}W(G)$ *with domain mixed derivatives*, Sib. M. Jour., **47(3)**, 2006, 505-516.

[6] A.M. Najafov, *On integral representation of functions from the spaces with domainant mixed derivative*, Vestnik BSU, ser. Phys. Math. Sec., **3**, 2005, 31-39.

[7] A.M. Najafov, A.T. Orujova, *Integral representation of functions from the space* $\bigcap\limits_{i=0}^{n} L^{\langle l^i \rangle}_{p^i,\theta}(G)$ *in the domain satisfying the flexible horn condition*, Vestmnik ChGU ser. Mathem. Physics, 2013, 25-29.

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