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On some classes of loaded equations and their applications

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Abstract. An algorithm for calculating the concentration distribution of absorbing molecules along the laser beam, when the absorbing layer is a medium with fractal geometry, is offered. The algorithm is based on a loaded partial differential equation of the second order that changes its type at a critical time moment, when the concentration of molecules in absorbing medium reaches its maximum.

Key Words and Phrases: Loaded partial differential equations, laser beam, Bouguer-Lambert-Beer law, Tricomi equation, Lavrent'ev-Bitsadze equation, generalized exponential function, nonlocal boundary condition

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Consider the system of the following three equations:

$$\partial_{0x}^{\alpha} u(\xi,t)+\sigma(t)\bar{v}(t)u(x,t)=0,\ 0<x<r,\ \sigma(t)>0; \tag{1}$$

$$k\partial_{\tau t}^{\beta} v(x,\eta)=-\frac{\partial w_1}{\partial x},\ \ \tau=\text{const}\geq 0,\ \ k=\text{const}>0; \tag{2}$$

$$w_1=-\frac{\partial}{\partial x}(av+b)v,\ \ a=\text{const}\geq 0,\ \ b=\text{const}>0. \tag{3}$$

Here $0<\alpha=\text{const}\leq 1$, $0<\beta=\text{const}\leq 1$, and t denotes dimensionless time,

$$\partial_{0x}^{\alpha} u(\xi,t)=D_{0x}^{\alpha-1}\frac{\partial u(\xi,t)}{\partial \xi}=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^x (x-\xi)^{-\alpha}\frac{\partial u(\xi,t)}{\partial \xi}d\xi;$$

$$\partial_{\tau t}^{\beta} v(x,\eta)=\operatorname{sign}(t-\tau)D_{\tau t}^{\beta-1}\frac{\partial v(x,\eta)}{\partial \eta}=\frac{1}{\Gamma(1-\beta)}\int\limits_{\tau}^{t} |t-\eta|^{-\beta}\frac{\partial v(x,\eta)}{\partial \eta}d\eta,$$

where $\Gamma(z)$ is the Euler gamma-function;

$$\bar{v}(t)=\frac{1}{r}\int\limits_0^r v(x,t)dx.$$

For $\alpha = 1$, equation (1) is a differential form of the Bouguer-Lambert-Beer law for plane waves propagating in an absorbing medium along the ray $x \geq 0$ when the attenuation (absorption) coefficient ω^α, the concentration of absorbing particles $v = v(x,t)$ in the layer $0 \leq x \leq r$ and absorption cross-section $\sigma(t)$ are related by

$$\omega^\alpha = \sigma(t)\bar{v}(t). \tag{4}$$

Equation (3) expresses the second law of Fick when diffusion coefficient depends on the concentration linearly.

Equation (2) may be interpreted as the continuity equation or as a fractal differential form of the mass conservation law, and it realizes the relationship between the concentration of absorbing particles with its density $w_1 = w_1(x,t)$ [1, p.26]. The order β of the time derivative with respect to t may depend on x.

By vitue of (4), equation (1) as may be written in the form follows:

$$\partial_{0x}^\alpha u(\xi,t) + \omega^\alpha u(x,t) = 0,\ 0 < x < r,\ t \geq 0. \tag{5}$$

Equation (5) is a loaded partial differential equation with partial derivative of order $\alpha \in]0,1]$ with respect to the spatial variable x.

The order α and the coefficient ω^α may be functions of time t. This equation generalizes the Bouguer-Lambert-Beer law for the intensity $u(x,t) \equiv I_\nu(x,t)$ of the radiation with given frequency ν at the point x and at time t when the absorber layer, $0 \leq x \leq r$, is the medium with fractal dimension that is less then or equal to α. It is the simplest case of an equation that is referred in [2, p. 242] as the generalized fractional oscillation equation.

Any solution $u = u(x,t)$ of (5) can be represented in the form

$$u(x,t) = u(0,t)\mathrm{Exp}_\alpha(-\omega x), \tag{6}$$

where

$$\mathrm{Exp}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}$$

is the generalized exponential function by terminology of V.A. Nakhusheva [3].

It follows from (6) that

$$u(r,t) = \varphi_0(t)\mathrm{Exp}_\alpha(-\omega r), \tag{7}$$

where $\varphi_0(t)$ is the intensity of radiation at the beginning of the route $x = 0$, and $\omega r = \tau_\nu$ is the optical depth (thickness) of the fractal layer $0 \leq x \leq r$.

From formula (7) with $\alpha = 1$, we the known representation of Bouguer law

$$u(r,t) = \varphi_0(t)\exp(-\omega r),$$

We assume that $u(x,t)$ satisfies the local boundary value condition

$$u(0,t) = \varphi_0(t),\ 0 \leq t \leq T, \tag{8}$$

and the function $v(x,t)$ satisfies non-local boundary condition (4).

Using (8), the spectral absorption

$$A_\alpha = \frac{\varphi_0(t) - u(r,t)}{\varphi_0(t)}$$

can be calculated by the formula $A_\alpha = 1 - \mathrm{Exp}_\alpha(-\omega r)$.

Due (2) and (3), the function $v(x,t)$ must be a solution of the equation.

$$k\partial^\beta_{\tau t} v(x,\eta) = \frac{\partial^2}{\partial x^2}[(av+b)v]. \tag{9}$$

We differentiate both sides of equation (9) with respect to the time variable t and, taking into account the equality

$$\frac{\partial}{\partial t}\partial^\beta_{\tau t} v(x,\eta) = \frac{\partial}{\partial t}\mathrm{sign}(t-\tau) D^{\beta-1}_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = D^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta},$$

we obtain the equation

$$kD^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = \frac{\partial^2}{\partial x^2}\left[(2av+b)\frac{\partial v}{\partial t}\right]. \tag{10}$$

The boundary value condition (4) gives us some justification to approximate equation (10) by the equation

$$kD^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = (2a\bar{v}+b)\frac{\partial^3 v(x,t)}{\partial x^2 \partial t}. \tag{11}$$

In equation (11) we introduce the new dependent variable

$$w(x,t) = \frac{\partial v(x,t)}{\partial t}. \tag{12}$$

Then, for $w = w(x,t)$, we obtain

$$D^\beta_{\tau t} w(x,\eta) = k_\alpha \frac{\partial^2 w(x,t)}{\partial x^2} \tag{13}$$

with coefficient

$$k_\alpha = \frac{2a\bar{v}+b}{k} = \frac{1}{k}\left[\frac{2a\omega^\alpha}{\sigma(t)} + b\right]. \tag{14}$$

Equation (11) can be approximately replaced by the equation

$$D^\beta_{\tau t}\frac{\partial v(x,\eta)}{\partial \eta} = K(t)\frac{\partial^2 v(x,t)}{\partial x^2}, \tag{15}$$

where

$$K(t) = \frac{2a\bar{v}'(t)}{k} = \frac{2a}{k}\frac{d}{dt}\left[\frac{\omega^2}{\sigma(t)}\right]. \tag{16}$$

Let τ be a time moment when the average value of the concentration of molecules $\bar{v}(t)$ in absorbing medium $0 \le x \le r$ reaches its maximum, and let the function (16) can be represented in the form

$$K(t) = |t-\tau|^m \chi(t) \mathrm{sign}(\tau - t), \tag{17}$$

where $m = \mathrm{const} \ge 0$, $\chi(t)$ is a continuous positive function defined on time interval $[0, T]$ with initial, $t = 0$, and estimated, $t = T$, time moments.

Condition (17) means that equation (15) for $\beta = 1$ is a partial differential equation of mixed type. At the model case, when $\chi(t) \equiv 1$, equation (15) takes the following form:

$$D^{\beta}_{\tau t} \frac{\partial v(x,\eta)}{\partial \eta} = \mathrm{sign}(\tau - t)|t-\tau|^m \frac{\partial^2 v(x,t)}{\partial x^2}. \tag{18}$$

Equation (18) with $\beta = 1$ and $m = 1$ coincides with the Tricomi equation of hypersonic flow

$$(t-\tau)\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{19}$$

which is well known from the theory of gas dynamics, and, coincides with the Lavrent'ev-Bitsadze equation

$$\mathrm{sign}(t-\tau)\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{20}$$

when $\beta = 1$ and $m = 0$.

Using (12), we have

$$\bar{w}(t) = \bar{v}'(t) = \frac{d}{dt}\left[\frac{\omega^\alpha}{\sigma(t)}\right]. \tag{21}$$

Equation (13) may be approximated by the equation

$$k_\alpha \frac{\partial^2 w(x,t)}{\partial x^2} = D^{\beta}_{\tau t}\bar{w}(\eta), \tag{22}$$

where the coefficient k_α is uniquely defined by (14). Hence, we find

$$k_\alpha[w_x(x,t) - w_x(0,t)] = x D^{\beta}_{\tau t}\bar{w}(\eta),$$

$$k_\alpha[w(x,t) - w(r,t) - (x-r)w_x(0,t)] = \frac{1}{2}(x^2 - r^2) D^{\beta}_{\tau t}\bar{w}(\eta), \tag{23}$$

$$k_\alpha\left[\bar{w}(t) - w(r,t) + \frac{1}{2} r w_x(0,t)\right] = \frac{1}{3} r^2 D^{\beta}_{\tau t}\bar{w}(\eta), \tag{24}$$

where

$$w_x(x,t) = \frac{\partial w(x,t)}{\partial x}.$$

Consequently, the solution $w = w(x,t)$ of (22) may be determined uniquely, if we add to the nonlocal condition the local conditions on the edge of absorption layer $0 \le x \le r$

$$w_x(0,t) = \psi_1(t),\ w(r,t) = \psi_0(t),\ 0 \le t \le T, \tag{25}$$

where $\psi_1(t)$ and $\psi_0(t)$ are the given functions continuous on $[0,T]$.

It follows from (21), (23), (24) and (25) that

$$k_\alpha[w(x,t)-\psi_0(t)-(x-r)\psi_1(t)]=\frac{1}{2}(x^2-r^2)D^\beta_{\tau t}\frac{d}{d\eta}\left[\frac{\omega^\alpha}{\sigma(\eta)}\right],$$

$$k_\alpha\left\{\frac{d}{dt}\left[\frac{\omega^\alpha}{\sigma(t)}\right]-\psi_0(t)+\frac{r}{2}\psi_1(t)\right\}=\frac{1}{3}r^2D^\beta_{\tau t}\frac{d}{d\eta}\left[\frac{\omega^\alpha}{\sigma(\eta)}\right]. \tag{26}$$

The algorithm of calculation must involve the checkup of condition (26) for the input data (22).

Equation (18) can be approximated by the following equations:

$$D^\beta_{\tau t}\frac{\partial\bar{v}(\eta)}{\partial\eta}=\mathrm{sign}(\tau-t)|t-\tau|^m\frac{\partial^2 v(x,t)}{\partial x^2}, \tag{27}$$

$$D^\beta_{\tau t}\frac{\partial}{\partial\eta}\frac{1}{h_i}\det\left\|\begin{matrix} v(x_i,\eta) & x_i-x \\ v(x_{i+1},\eta) & x_{i+1}-x\end{matrix}\right\|+$$

$$+\mathrm{sign}(t-\tau)|t-\tau|^m\frac{\partial^2 v(x,t)}{\partial x^2}=0, \tag{28}$$

where $x_i<x<x_{i+1}$, $i=0,1,...,n$.

We use the method of reducing the Samarskii problem to the local boundary value problem, which is posed in the paper [4]. We introduce the new dependent variable in equation (27),

$$U(x,t)=\int\limits_0^x v(\xi,t)d\xi. \tag{29}$$

Function (29) is a solution of partial differential equations of the first order

$$\frac{\partial U}{\partial x}=v(x,t),\ 0\le x\le r, \tag{30}$$

and satisfies the local boundary value conditions

$$U(0,t)=0,\ U(r,t)=r\bar{v}(t),\ 0\le t\le T. \tag{31}$$

Therefore, equation (27) takes the following form:

$$D^\beta_{\tau t}\frac{\partial U(r,\eta)}{\partial\eta}=r\ \mathrm{sign}(\tau-t)|t-\tau|^m\frac{\partial^3 U}{\partial x^3}. \tag{32}$$

We add to condition (4) the boundary condition

$$v(0,t)=\psi_0(t),\ 0\le t\le T, \tag{33}$$

where $\psi_0(t)$ is a given function continuous on $[0, T]$. Because of (30), from this condition it follows that

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \psi_0(t),\ 0 \leq t \leq T. \tag{34}$$

So, the problem is reduced to the following: find a solution $U = U(x, t)$ of (32) at any point x of the absorbing layer and at any time t from the initial $t = 0$ to the estimated $t = T$ time moments, which satisfies the boundary value conditions (31) and (34).

In the case of (28), the function $v = v(x, t)$ must satisfy the conditions (4) and (33).

Due to the fact that equations (15), (19) and (20) are loaded ones of mixed type, we can interpret the absorbing medium as a fractal input-output mixed system [5].

To study the boundary value problems for the equation (13), one can successfully use the Green function that is constructed in [6].

In conclusion, it should be mentioned that the work is based on the report that was made on in the International conference "Physics of Extreme States of Matter" [7].

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