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Some Results About Common Fixed Point Theorems for Multi-Valued Mappings

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Abstract. V. Popa has proved common fixed point theorems for multi-valued mappings which verify rational inequalities, which contain the Hausdorff metric in their expressions. Recently, A. Petcu in [1, 2, 3] has proved other common fixed point theorems for two or more multi-valued mappings without using the Hausdorff metric. In this paper by providing some different conditions we shall study existence of common fixed points for multi-valued mappings.

Key Words and Phrases: Complete metric space, Common fixed point, Multi-valued mappings.

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1. Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping or multi-valued mapping T from a topological space X into itself, that is, we can find $x \in X$ such that $Tx = x$ (for mapping) or $x \in Tx$ (for multi-valued mapping).
In [4] V. Popa has proved common fixed point theorems for multi-valued mappings which verify rational inequalities, which contain the Hausdorff metric in their expressions.
In [1] A. Petcu has proved other common fixed point theorems for two or more multi-valued mappings without using the Hausdorff metric.

In this paper by providing some different conditions we study existence of common fixed points for multi-valued mappings.
Let X be a nonempty set, $P(X)$ the set of all nonempty subsets of X, T a multifunction of X into $P(X)$, $F(T)$ the fixed points set of T, that is $F(T) = \{x \in X : x \in Tx\}$. Throughout the paper, for a topological space X we denote the set of all nonempty closed subsets of X by $P_{cl}(X)$ and the set of all nonempty closed and bounded subsets of X by $P_{b,cl}(X)$ when X is a metric space.

Let (X,d) be a metric space, for $x \in X$ and $A, B \subseteq X$, set $D(x,A) = \inf_{y\in A} d(x,y)$ and

$$H(A,B) = \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\}.$$

We also denote

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}.$$

It is known that, H is a metric on closed bounded subsets of X which is called the Hausdorff metric.

2. Main results

Let $\mathcal{F}$ be all multi-valued mappings of X in to $P_{b,cl}(X)$.
Define the following equivalence relation for the elements of $\mathcal{F}$:

$$F \sim G \quad if\ and\ only\ if \quad fixF = fixG \qquad (F, G \in \mathcal{F}).$$

Denote the equivalence class of $\mathcal{F}$ by $\tilde{\mathcal{F}}$ and define it as follows:

$$\tilde{\mathcal{F}} = \frac{\mathcal{F}}{\sim} = \{\tilde{F} : F \in \mathcal{F}\}.$$

Also define $\tilde{d}$ *on* $\tilde{\mathcal{F}}$ such that

$$\tilde{d}(\tilde{F}, \tilde{G}) = H(fixF, fixG).$$

It is easy to see that $(\tilde{\mathcal{F}}, \tilde{d})$ is a metric space.

Lemma 1. *Let (X,d) be a metric space and $S, T : X \to P_{b,cl}(X)$ be two multi-valued mappings such that $\forall x \in X, \forall y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) with*

$$d^{3m}(x,y) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(y,z) d(x,y) - \frac{c^3}{8} d^{3m}(y,z) \geq 0, \tag{2.1}$$

where $m \geq 1, c > 1$ and $F(S) \neq \phi$. Then $F(T) \neq \phi$ and $\tilde{S} = \tilde{T}$.

Proof. Let $u \in F(S)$, that is $u \in Su$. Then there exists $z \in Tu$ and (2.1) becomes

$$d^3(u,u) - \frac{3}{4\sqrt[3]{4}} c^2 d^2(u,z) d(u,u) - \frac{c^3}{8} d^3(u,z) \geq 0,$$

from where we get $-\frac{c^3}{8} d^3(u,z) \geq 0, c > 1$, that is $d(u,z) = 0$. Then $z = u$ and therefore $u \in Tu$ which implies $F(S) \subset F(T)$.

Analogously we prove that $F(T) \subset F(S)$, therefore $F(S) = F(T)$ and hence $\tilde{S} = \tilde{T}$.

Let $V : X \to P_{b,cl}(X)$ with (X,d) a metric space. The following property will be used further:

$$for\ any\ convergent\ sequence\ (x_n)_{n\geq 0}\ \ from\ \ X\ \ with\ \ \lim_{n\to\infty} x_n = x,$$

$$x_{2n-1} \in Vx_{2n-2} (or\ x_{2n} \in Vx_{2n-1}), it\ results\ \ x \in Vx. \qquad (a)$$

Theorem 1. *Let (X,d) be a complete metric space and $S,T : X \to p_{b,cl}(X)$ be two multi-valued mappings such that $\forall x \in X, \forall y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) with inequality (2.1) holding, where $m \geq 1$, $c > 1$.*
If one of the multi-valued mappings S,T verifies condition (a), then $\tilde{S} = \tilde{T}$.

Proof. Let $x_0 \in X$ be arbitrarily fixed and $x_1 \in Sx_0$. Then there exists $x_2 \in Tx_1$ such that

$$d^{3m}(x_0,x_1) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(x_1,x_2) d(x_0,x_1) - \frac{c^3}{8} d^{3m}(x_1,x_2) \geq 0.$$

Then there exists $x_3 \in Sx_2$ such that

$$d^{3m} d(x_1,x_2) - \frac{3}{4\sqrt[3]{4}} c^2 d^{2m}(x_2,x_3) d(x_1,x_2) - \frac{c^3}{8} d^{3m}(x_2,x_3) \geq 0.$$

Continuing this reasoning we obtain a sequence
$x_0, x_1, x_3, \ldots, x_{n-1}, x_n \ldots$ with $x_{2n-1} \in Sx_{2n-2}$, $x_{2n} \in Tx_{2n-1}$ which verifies the inequality

$$d^{3m}(x_n,x_{n-1}) - \frac{3c^2}{4\sqrt[3]{4}} d^{2m}(x_n,x_{n+1}) d(x_n,x_{n-1}) - \frac{c^3}{8} d^{3m}(x_n,x_{n+1}) \geq 0, \qquad (2.2)$$

for all $n \geq 1$. The first member in the inequality (2.2) is a third degree trinomial in the variable $d^m(x_n,x_{n-1})$ with the discriminant

$$\Delta = 4\left(\frac{-3}{4\sqrt[3]{4}} c^2 d^{2m}(x_n,x_{n+1})\right)^3 + 27\left(\frac{-c^3}{8} d^{3m}(x_n,x_{n+1})\right)^2 = 0.$$

Inequality (2.2) holds if

$$d^m(x_n,x_{n-1}) \geq -2\sqrt[3]{\frac{c^3}{8} d^{3m}(x_n,x_{n+1})} = c d^m(x_n,x_{n+1}).$$

We denote $k^m = \frac{1}{c}$. Then we have $k < 1$ and

$$0 \leq d^m(x_n,x_{n+1}) < k^m d^m(x_n,x_{n-1}),$$

that is

$$d(x_n,x_{n+1}) \leq k d(x_{n-1},x_n), \quad \forall n \geq 1,$$

from where we deduce

$$d(x_n,x_{n+1}) \leq k^n d(x_0,x_1), \quad \forall n \geq 1.$$

A routine calculation leads to

$$d(x_n, x_{n+p}) \le \frac{k^n}{1-k} d(x_0, x_1), \quad n, p \in \mathbb{N},$$

which shows that $(x_n)_{n\ge 0}$ is a Cauchy sequence and since the space X is complete it results that $(x_n)_{n\ge 0}$ is convergent. Let $u = \lim_{n\to\infty} x_n,\ u \in X$. We have
$x_{2n-1} \in Sx_{2n-2}$ and assuming that S verifies (a) it results that $u \in Su$. With lemma (1) we deduce that $u \in Tu$ and $F(S) = F(T)$ and so $\tilde{S} = \tilde{T}$.

Lemma 2. *[5] If $A, B \in B(X)$ and $k \in \mathbb{R}, k > 1$, then for any $a \in A$ there exists $b \in B$ such that $d(a, b) \le kH(A, B)$.*

According to the above lemma the following lemma is true.

Lemma 3. *Let $k > 1$ and the multi-valued mappings $S, T : X \to P_{cl,b}(X)$ be given. Then for any $y \in Sx$(or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sy$) such that*

$$d(y, z) \le kH(Sx, Ty).$$

Theorem 2. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{b,cl}(X)$ be two multi-valued mappings such that*

$$H^m(T_1x, T_2y) \le \frac{8d^{3m}(x, T_1x)}{c^2\delta^{2m}(y, T_2y) + 6c\delta^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^{2m}(x, T_1x)}, \tag{2.3}$$

and for any x, y from X

$$c^2\delta^{2m}(y, T_2y) + 6c\delta^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^{2m}(x, T_1x) \ne 0,$$

where $m \ge 1, c > 1$.Then $\tilde{T_1} = \tilde{T_2}$.

Proof. Eliminating the denominator, (2.3) becomes

$$\begin{aligned} H^m(T_1x, T_2y)(c^2\delta^{2m}(y, T_2y) + 6c\delta^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^{2m}(x, T_1x)) \\ \le 8d^{3m}(x, T_1x). \end{aligned} \tag{2.4}$$

Inequality (2.4) is valid for any x, y from X and in particular for $y \in T_1x$.
Let $1 < c < k^m$. For $x \in X,\ y \in T_1x$ with lemma (3) it results that there exists $z \in T_2y$ such that $d(y, z) \le kH(T_1x, T_2y)$, and from here we have

$$cd^m(y, z)(c^2d^{2m}(y, z) + \frac{6c}{\sqrt[3]{4}}d^m(y, z)d^m(x, y) \le 8d^{3m}(x, y).$$

Consequently, $\forall x \in X, \forall y \in T_1x$, there exists $z \in T_2y$ such that

$$d^{3m}(x, y) - \frac{3}{4\sqrt[3]{4}}cd^m(y, z)d^m(x, y) - \frac{c^3}{8}d^{3m}(y, z) \ge 0,$$

where $m \geq 1, 1 < c < k^m$, condition which has the form of inequality (2.1). We prove now that T_1 verifies condition (a). Let $(x_n)_{n\geq 0}$ be a convergent sequence from X with $\lim_{n\to\infty} x_n = x \in X$ and $x_{2n-1} \in T_1 x_{2n-2},\ x_{2n} \in T_2 x_{2n-1}$.
We have

$$d(T_1 x, x_{2n}) \leq H(T_1 x, T_2 x_{2n-1}),$$

from where with (2.4) we obtain

$$cd^m(T_1x, x_{2n})(c^2 d^{2m}(x_{2n-1}, x_{2n}) + 6cd^m(x_{2n-1}, x_{2n})d^m(x_{2n}, T_1x) + 8d(x_{2n}, T_1x)) \leq 8d^{3m}(x_{2n}, T_1x),$$

from where, for $n \to \infty$, it results

$$d(x, T_1x) \leq \frac{1}{c} d(x, T_1x),$$

that is $d(T_1x, x) = 0$. Because T_1x is a closed set we deduce $x \in T_1x$ and by previous theorem and lemma we obtain $F(T_1) = F(T_2)$, therefore $\tilde{T_1} = \tilde{T_2}$.

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