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On the statistical type convergence and fundamentality in metric spaces

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Abstract. The concept of $\mathscr{F}$-fundamentality, generated by some filter $\mathscr{F}$ is introduced in metric spaces. Its equivalence to the concept of $\mathscr{F}$-convergence is proved in metric spaces. This convergence generalizes many kinds of convergence, including the well-known statistical convergence.

Key Words and Phrases: $\mathscr{F}$-convergence, $\mathscr{F}$- fundamentality, statistical convergence

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1. Introduction

The idea of statistical convergence (*stat-convergence*) was first proposed by A.Zigmund [1] in his famous monograph where he talked about "almost convergence". The first definition of it was given by H. Fast [2] and H. Steinhaus [3]. Later, this concept has been generalized in many directions. It is impossible to list all the related papers. More details on this matter and its applications can be found in [4-15, 24]. It should be noted that the methods of non-convergent sequences have long been known and they include e.g. Cesaro method, Abel method, etc. These methods are used in different areas of mathematics. For the applicability of these methods it is very important that the considered space has a linear structure. Therefore, the study of statistical convergence in metric spaces is of special scientific interest. Different aspects of this problem have been studied in [16, 17]. Statistical convergence is currently actively used in many areas of mathematics such as summation theory [7, 8, 19], number theory [11, 13], trigonometric series [1], probability theory [8], measure theory [12], optimization [20], approximation theory [21, 22], fuzzy theory [26], etc.

It should be noted that the concept of statistical fundamentality (*stat-fundamentality*) was first introduced by J.A. Fridy [4] who proved its equivalence to *stat-convergence* with respect to numerical sequences. This issue was raised in [10] concerning uniform space $(X; U)$. It was proved that if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is *stat-convergent*, then it is *stat-fundamental*. The problem of the validity of converse statement was also raised in [10].

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Stat-convergence was generalized by many mathematicians (see [10, 15, 23, 24, 25]). The concepts of *I-convergence* and *I^*-convergence* were introduced in [23]. These kinds of convergences generalizes many previously known convergences, including the well-known *stat-convergence*. In present paper we introduce the concepts of *$\mathscr{F}$-convergence and $\mathscr{F}$-fundamentality,* generated by some filter $\mathscr{F} \subset 2^{\mathbb{N}}$. Their equivalence is proved. *$\mathscr{F}$-convergence* generalizes many kinds of convergence, related to such concepts as statistical density, logarithmic density, uniform density, etc. More details on these concepts can be found in [23].

2. Needful information

We will use the standard notation. $\mathbb{N}$ will be a set of all positive integers; $\mathbb{R}$ will be a set of real numbers; $\chi_M(\cdot)$ is the characteristic function of M; $(X;\rho)$ is a metric space. $O_\varepsilon(a)$ is an open ball centered at a with radius ε, i.e. $O_\varepsilon(a) \equiv \{x \in X : \rho(x;a) < \varepsilon\}$. 2^M will be a set of all subsets M; $\bar{M}$ will stand for the closure of M; $|A| = card\, A$ is the number of elements of A. $M^C = N \backslash M$. $\wedge$ will be a quantifier which means "and".

Let us recall the definition of asymptotic (statistical) density of $A \subset \mathbb{N}$. Assume

$$\delta_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k),$$

and let $\delta_*(A) = \liminf_{n\to\infty} \delta_n(A)$, $\delta^*(A) = \limsup_{n\to\infty} \delta_n(A)$. $\delta_*(A)$ and $\delta^*(A)$ are called lower and upper asymptotic density of the set A, respectively. If $\delta_*(A) = \delta^*(A) = \delta(A)$, then $\delta(A)$ is called asymptotic (or statistical) density of A. It should be noted that the statistical convergence is determined by means of this concept, namely, the consequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called statistically convergent to x, if $\delta(A_\varepsilon) = 0$, for $\forall\varepsilon > 0$, where $A_\varepsilon \equiv \{n \in \mathbb{N} : \rho(x_n;x) \geq \varepsilon\}$.

Let us also recall the definitions of the ideal and the filter.

A family of sets $I \subset 2^{\mathbb{N}}$ is called an ideal if: α) $\emptyset \in I$; β) $A;B \in I \Rightarrow A \cup B \in I$; γ) $(A \in I \wedge B \subset A) \Rightarrow B \in I$.

A family $\mathscr{F} \subset 2^{\mathbb{N}}$ is called a filter on X, if :

i) $\emptyset \notin \mathscr{F}$;

ii) from $A;B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$;

iii) from $A \in \mathscr{F} \wedge (A \subset B) \Rightarrow B \in \mathscr{F}$.

Filter, satisfying the condition

iv) if $A_1 \supset A_2 \supset \ldots \wedge A_n \in \mathscr{F}, \forall n \in \mathbb{N} \Rightarrow \exists \{n_m\}_{m\in\mathbb{N}} \subset \mathbb{N}; n_1 < n_2 < \ldots :$ $\cup_{m=1}^{\infty}((n_m, n_{m+1}] \cap A_m) \in \mathscr{F}$ is called a monotone closed filter.

Filter $\mathscr{F}$ satisfying the following condition is called a right filter:

v) $F^C \in \mathscr{F}$, for any finite subset $F \subset \mathbb{N}$.

An ideal I is called non-trivial if $I \neq \emptyset \wedge I \neq X$. $I \subset 2^{\mathbb{N}}$ is a non-trivial ideal if and only if $\mathscr{F} = \mathscr{F}(I) = \{X \backslash A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^{\mathbb{N}}$ is called admissible if and only if $I \supset \{\{x\} : x \in X\}$.

In the sequel, we assume that $(X;\rho)$ is a metric space with metric ρ, and $I\subset 2^{\mathbb{N}}$ is some non-trivial ideal.

Definition 1 [23]. *The sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is called I-convergent to $x\in X$ ($I\text{-}\lim\limits_{n\to\infty} x_n=x$),* if $A_\varepsilon\in I$, $\forall\varepsilon>0$, *where* $A_\varepsilon=\{n\in\mathbb{N}:\rho(x_n;x)\geq\varepsilon\}$.

Let $I_d\equiv\{A\subset\mathbb{N}:d(A)=0\}$. I_d is an ideal on $\mathbb{N}$. I_d*-convergence* means the statistical convergence.

It should be noted that if I is an admissible ideal, then the usual convergence in X implies I*-convergence* in X.

Definition 2. *The sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is called I^*-convergent to $x\in X$, if $\exists M\in\mathscr{F}(I)$ (i.e. $\mathbb{N}\backslash M\in I$), $M=\{m_1<m_2<...<m_k<...\}$: $\lim\limits_{k\to\infty}\rho(x_{m_k};x)=0$.*

In the following discussion we will need the following interesting results of [23].

Theorem 1 [23]. *Let I be an admissible ideal. If $I^*\text{-}\lim\limits_{n\to\infty}x_n=x\Rightarrow I\text{-}\lim\limits_{n\to\infty}x_n=x$.*

The converse is not always true, it depends on the structure of space $(X;\rho)$, namely, we have

Theorem 2 [23]. *Let $(X;\rho)$ be a metric space. (i) If X has no accumulation point, then I-convergence and I^*-convergence coincide for each admissible ideal $I\subset 2^{\mathbb{N}}$; (ii) If X has an accumulation point ξ, then there exists an admissible ideal $I\subset 2^{\mathbb{N}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}\subset X$: $I\text{-}\lim\limits_{n\to\infty}y_n=\xi$, but $I^*\text{-}\lim\limits_{n\to\infty}y_n$ does not exist.*

3. Main results

Let $(X;\rho)$ be some complete metric space and $\mathscr{F}\subset 2^{\mathbb{N}}$ be some filter. Accept the following

Definition 3. *Let $\mathscr{F}\subset 2^{\mathbb{N}}$ be some filter. The sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is called $\mathscr{F}$-convergent to $x\in X$ ($\mathscr{F}\text{-}\lim\limits_{n\to\infty}x_n=x$), if $A_\varepsilon\in\mathscr{F}$, $\forall\varepsilon>0$, where $A_\varepsilon\equiv\{n\in\mathbb{N}: x_n\in O_\varepsilon(x)\}$.*

Let us introduce the concept of $\mathscr{F}$*-fundamentality.*

Definition 4. *The sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ is called $\mathscr{F}$-fundamental, if $\forall\varepsilon>0,\exists n_\varepsilon\in\mathbb{N}:\Delta_{n_\varepsilon}\in\mathscr{F}$, where $\Delta_{n_\varepsilon}\equiv\{n\in\mathbb{N}:x_n\in O_\varepsilon(x_{n_\varepsilon})\}$.*

Assume that $\exists\mathscr{F}\text{-}\lim\limits_{n\to\infty}x_n=x$. Let $\varepsilon>0$ be an arbitrary number. Consequently, $A_{\varepsilon/2}\in\mathscr{F}$. From the condition i) in the definition of filter above it follows that $A_{\varepsilon/2}\neq\emptyset$. Take $\forall n_\varepsilon\in A_{\varepsilon/2}:\ \rho(x_{n_\varepsilon};x)<\frac{\varepsilon}{2}$. From the relation

$$\rho(x_n;x_{n_\varepsilon})\leq\rho(x_n;x)+\rho(x;x_{n_\varepsilon})<\frac{\varepsilon}{2}+\rho(x_n;x),$$

it directly follows that

$$\left\{n\in\mathbb{N}:\rho(x_n;x)<\frac{\varepsilon}{2}\right\}\subset\{n\in\mathbb{N}:\rho(x_n;x_{n_\varepsilon})<\varepsilon\}.$$

Hence $\Delta_{n_\varepsilon}\in\mathscr{F}$, i.e. the sequence $\{x_n\}_{n\in\mathbb{N}}$ *is $\mathscr{F}$-fundamental* in X.

Now, vice versa, let $\{x_n\}_{n\in\mathbb{N}}\subset X$ be *$\mathscr{F}$-fundamental.* From *$\mathscr{F}$- fundamentality* it follows that $\exists n_j\in\mathbb{N}:K_j\in\mathscr{F}$, where $K_j\equiv\{n\in\mathbb{N}:\rho\left(x_n;x_{n_j}\right)\leq 2^{1-j}\}$, $j=1,2.$

By the definition of filter we obtain $K_1 \cap K_2 \in \mathscr{F}$. Put $M_1 \equiv \overline{O_1(x_{n_1})} \cap \overline{O_{2^{-1}}(x_{n_2})}$. It is obvious that $x_n \in M_1$, $\forall n \in (K_1 \cap K_2) \equiv K_{(1)}$. Thus, $\exists n_3 \in \mathbb{N} : K_3 \in \mathscr{F}$, where $K_3 \equiv \left\{n : \rho(x_n; x_{n_3}) \le 2^{-2}\right\}$. Let $K_{(2)} = K_{(1)} \cap K_3$. It is clear that $K_{(2)} \in \mathscr{F}$. Now let $M_2 \equiv M_1 \cap \overline{O_{2^{-2}}(x_{n_3})}$. Denote by $d_\rho(M)$ the diameter of the set M, i.e.

$$d_\rho(M) = \sup_{x,y \in M} \rho(x;y).$$

Continuing in the same way, we obtain the nested sequence of closed sets $\{M_n\}_{n\in\mathbb{N}} \subset X$: $M_1 \supset M_2 \supset ...$; whose diameters tend to zero: i.e. $d_\rho(M_n) \le 2^{-n+1} \to 0$, $n \to \infty$. Moreover, $K_{(n)} \in \mathscr{F}$, where $K_{(n)} \equiv \{n \in \mathbb{N} : x_n \in M_n\}$. Take $\forall \tilde{x}_n \in M_n$, $\forall n \in \mathbb{N}$. We have

$$\rho(\tilde{x}_n; \tilde{x}_{n+p}) \le d_\rho(M_n) \to 0, \, n \to \infty, \, \forall p \in \mathbb{N}.$$

Hence, the sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is fundamental in X. Let $\lim\limits_{n\to\infty} \tilde{x}_n = x$. It is absolutely clear that $x \in \cap_n M_n$, i.e. $\cap_n M_n$ is non-empty. From $d_\rho(M_n) \to 0$, $n \to \infty$, it directly follows that $\cap_n M_n \equiv \{x\}$, i.e. $\cap_n M_n$ consists of one element. Let us show that $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$. Take $\forall \varepsilon > 0$. Take $n_\varepsilon \in \mathbb{N} : d_\rho(M_{n_\varepsilon}) < \varepsilon$. Let $y \in M_{n_\varepsilon}$ be an arbitrary element. So

$$\rho(y, x) \le d_\rho(M_{n_\varepsilon}) < \varepsilon.$$

Consequently, $M_{n_\varepsilon} \subset O_\varepsilon(x)$. We have $K_{(n_\varepsilon)} \in \mathscr{F}$, where $K_{(n_\varepsilon)} = \{n \in \mathbb{N} : x_n \in M_{n_\varepsilon}\}$. So, $K_{(n_\varepsilon)} \subset \{n \in \mathbb{N} : x_n \in O_\varepsilon(x)\}$, it is clear that $\{n \in \mathbb{N} : x_n \in O_\varepsilon(x)\} \in \mathscr{F} \Rightarrow \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$. Thus, we have proved the following theorem.

Theorem 3. *Let $(X;\rho)$ be complete metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. The sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is $\mathscr{F}$-convergent in X if and only if it is $\mathscr{F}$-fundamental in X.*

It is easy to see that $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n$ is unique if it exists. In fact, let $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n$ has two values $y_1 \ne y_2$. Take $\forall \varepsilon \in \left(0, \frac{1}{2}\rho(y_1; y_2)\right)$. Let $A_k \equiv \{n \in \mathbb{N} : \rho(x_n; y_k) < \varepsilon\}$, $k = 1, 2$. It is clear that $A_k \in \mathscr{F}$, $k = 1, 2 \Rightarrow A_1 \cap A_2 \in \mathscr{F}$. As $A_1 \cap A_2 = \emptyset \notin \mathscr{F}$, the obtained contradiction proves that $y_1 = y_2$.

Let us consider the sequence $\left\{K_{(n)}\right\}_{n\in\mathbb{N}}$, constructed in the proof of Theorem 1. We have $K_{(1)} \supset K_{(2)} \supset ... \wedge K_{(n)} \in \mathscr{F}$, $\forall n \in \mathbb{N}$. Then from the condition iv) in the definition of filter we have

$$\exists \{n_m : n_1 < n_2 < ...\} \; : \cup_{m=1}^{\infty} \left((n_m, n_{m+1}] \cap K_{(m)}\right) \in \mathscr{F}.$$

Assume

$$\mathbb{N}_0 \equiv \left\{k \in \mathbb{N} : \; k \in (n_m, n_{m+1}] \cap K_{(m)}^C, \, m \in \mathbb{N}\right\} \cup [1, n_1],$$

where $M^C \equiv \mathbb{N} \backslash M$. Define

$$y_k = \begin{cases} x, & k \in \mathbb{N}_0; \\ x_k, & otherwise, \end{cases}$$

where $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$. Take $\forall \varepsilon > 0$. If $k \in \mathbb{N}_0$, then $\rho(y_k; x) = \rho(x; x) < \varepsilon$. If $k \notin N_0$, then $\exists m : n_m < k \le n_{m+1} \wedge k \notin K_{(m)}^c \Rightarrow k \in K_{(m)} \Rightarrow x_k \in M_m$ $(M_1 \supset M_2 \supset ...$

is a sequence from Theorem 1) $\Rightarrow \rho(x_k;x) \le d_\rho(M_m) < \varepsilon$ for sufficiently great values of m (as $x \in M_m$, $\forall m \in \mathbb{N}$). Hence, we have $\lim\limits_{k\to\infty} y_k = x$. Let us show that $\tilde{K} \equiv \{k \in \mathbb{N} : x_k = y_k\} \in \mathscr{F}$. In fact, it is clear that

$$\cup_{m=1}^{\infty}\left((n_m, n_{m+1}] \cap K_{(m)}\right) \subset \tilde{K},$$

holds. Then from the condition iii) in the definition of filter we get $\tilde{K} \in \mathscr{F}$. Thus, if $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, then $\exists \tilde{K} \in \mathscr{F}$: $\lim\limits_{n\to\infty} y_n = x$ and $x_n = y_n$, $\forall n \in \tilde{K}$.

Conversely, let $\lim\limits_{n\to\infty} y_n = x$ and $\tilde{K} \equiv \{n : x_n = y_n\} \in \mathscr{F}$. Take $\forall \varepsilon > 0$. Then $\exists n_\varepsilon \in \mathbb{N} : \rho(y_n;x) < \varepsilon$, $\forall n \ge n_\varepsilon$. We have $\{n \in \mathbb{N} : n \ge n_\varepsilon\} \cap \tilde{K} \subset \{n \in \mathbb{N} : \rho(x_n;x) < \varepsilon\}$. It is clear that $\left(\{n \in \mathbb{N} : n \ge n_\varepsilon\} \cap \tilde{K}\right) \in \mathscr{F}$. Then from the condition iii) in the definition of filter it follows $\{n \in \mathbb{N} : \rho(x_n;x) < \varepsilon\} \in \mathscr{F}$. Thus, the following theorem is true.

Theorem 4. *Let $(X;\rho)$ be a metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be some filter. Then: 1) if $\mathscr{F}$ is a monotone close and $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, then $\exists \{y_n\}_{n\in\mathbb{N}} \subset X : \lim\limits_{n\to\infty} y_n = x \wedge \{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F}$; 2) if $\mathscr{F}$ is a right filter and $\lim\limits_{n\to\infty} y_n = x \wedge (\{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F})$, then $\mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$.*

The Theorems 1;2 imply the following

Corollary 1. *Let $(X;\rho)$ be a complete metric space, $\mathscr{F} \subset 2^{\mathbb{N}}$ be some monotone close and right filter. Then the following statements are equivalent to each other:*
α) $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$; β) $\{x_n\}_{n\in\mathbb{N}}$ is $\mathscr{F}$-fundamental; γ) $\exists \lim\limits_{n\to\infty} y_n = x \wedge (\{n \in \mathbb{N} : x_n = y_n\} \in \mathscr{F})$.

The Theorem 2 immediately implies the following

Corollary 2. *Let $(X;\rho)$ be a metric space and $\mathscr{F} \subset 2^{\mathbb{N}}$ be a right filter. If $\exists \mathscr{F}\text{-}\lim\limits_{n\to\infty} x_n = x$, then $\exists \{n_k : n_1 < n_2 < ...\} \in \mathscr{F}$: $\lim\limits_{k\to\infty} x_{n_k} = x$.*

4. Filters

I. An ordinary convergence. $\mathscr{F} \equiv \{M \subset \mathbb{N} : M^C \equiv \mathbb{N}\backslash M$ *is a finite set*$\}$. *$\mathscr{F}$-convergence,* generated by this filter, coincides with the ordinary convergence.

II. Statistical convergence. Assume $\mathscr{F}_\delta \equiv \{M \subset \mathbb{N} : \delta(M) = 1\}$. $\mathscr{F}_\delta$ is a filter. It is easy to see that $\mathscr{F}_\delta$ is a right filter. Let us show that $\mathscr{F}_\delta$ is a monotone close filter. Let $K_1 \supset K_2 \supset ... \wedge (\delta(K_n) = 1, \forall n \in \mathbb{N})$. It is clear that $\delta(K_n^c) = 0$, $\forall n \in \mathbb{N}$. Therefore $\exists \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$; $n_1 < n_2 < ...$:

$$\frac{1}{n}|I_n \cap K_m^c| < \frac{1}{m}, \ \forall n \ge n_m.$$

Let $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_n$, where $\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m < k \le n_{m+1} \wedge (k \in K_m^c)\}$. It is obvious that $\delta(\mathbb{N}_0) = \delta\left(\tilde{\mathbb{N}}_0\right)$. Take $\forall n \in \mathbb{N}$. Then $\exists m \in \mathbb{N} : n_m < n \le n_{m+1}$. Without loss of generality, we may suppose that $n > n_1$. Let us show that

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c). \tag{1}$$

Let $k \in \left(I_n \cap \tilde{\mathbb{N}}_0\right) \Rightarrow \exists m_0 \leq m : n_{m_0} < k \leq n_{m_0+1} \wedge \left(k \in K_{m_0}^c\right) \Rightarrow k \in K_m^c$. So, the inclusion (1) is true. Consequently

$$\frac{1}{n}\left|I_n \cap \tilde{\mathbb{N}}_0\right| \leq \frac{1}{n}\left|I_n \cap K_m^c\right| < \frac{1}{m}. \tag{2}$$

From (2) it directly follows that $\delta\left(\tilde{\mathbb{N}}_0\right) = 0$. As a result, $\delta(\mathbb{N}_0) = 0 \Rightarrow \delta(\mathbb{N}_0^c) = 1 \Rightarrow \mathbb{N}_0^c \in \mathscr{F}_\delta$. In the sequel, it should be pointed out that $\mathbb{N}_0^c \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\}$. Thus, $\mathscr{F}_\delta$ is a monotone close filter. Fulfilment of condition v) by $\mathscr{F}_\delta$ is obvious. Then, the statement of Corollary 1 is true with respect to $\mathscr{F}_\delta$*-convergence.* So,we get the validity of

Statement 1. *Filter $\mathscr{F}_\delta$, generated by statistical density, is a monotone close and right filter.*

III. Logarithmic convergence. Let $M \subset \mathbb{N}$. Assume

$$l_n(M) = \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_M(k)}{k},$$

where $s_n = \sum_{k=1}^n \frac{1}{k}$. If $\exists \lim\limits_{n\to\infty} l_n(M) = l(M)$, then $l(M)$ is called a logarithmic density of the set M. Let $\mathscr{F}_l \equiv \{M \subset \mathbb{N} : l(M) = 1\}$. The following lemma is true.

Lemma 1. *If* $l(M_k) = 1,\ k = 1,2 \Rightarrow l(M_1 \cap M_2) = 1.$

Proof. We have

$$M_1 \cap M_2 = (M_1 \cup M_2) \setminus [(M_2 \backslash M_1) \cup (M_1 \backslash M_2)].$$

Consequently

$$M_1 \cap M_2 \cap I_n = [(M_1 \cup M_2) \cap I_n] \setminus [((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) \cap I_n]. \tag{3}$$

From

$$((M_2 \backslash M_1) \cap I_n) \subset (M_1^c \cap I_n),$$

we get

$$\frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_2\backslash M_1}(k) \leq \frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_1^c}(k). \tag{4}$$

It is absolutely clear that, if $l(M) = 1$, then $l(M^c) = 0$. Then from (4) we obtain $l(M_2 \backslash M_1) = 0$. Similarly, we have $l(M_1 \backslash M_2) = 0$. So

$$((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) \cap I_n = ((M_2 \backslash M_1) \cap I_n) \cup ((M_1 \backslash M_2) \cap I_n).$$

It is clear that

$$l((M_2 \backslash M_1) \cup (M_1 \backslash M_2)) = 0. \tag{5}$$

It is easy to see that $l(M_1 \cup M_2) = 1$. From (3) we have

$$\frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_1\cap M_2}(k) = \frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{M_1\cup M_2}(k) - \frac{1}{s_n}\sum_{k=1}^{n}\frac{1}{k}\chi_{(M_2\backslash M_1)\cup(M_1\backslash M_2)}(k).$$

Taking into account (5) we get $l(M_1 \cap M_2) = 1$. Lemma is proved.

This lemma implies that $\mathscr{F}_l$ is a filter. If $M \subset \mathbb{N}$ is a finite set, then it is clear that $M^C \in \mathscr{F}_l$, i.e. $\mathscr{F}_l$ satisfies the condition v). Then it is absolutely clear that $l(M) = 0$. Let us show that $\mathscr{F}_l$ is a monotone close filter. Let $K_1 \supset K_2 \supset ... \wedge (l(K_n) = 1, \forall n \in \mathbb{N}) \Rightarrow l(K_n^c) = 0, \forall n \in \mathbb{N}$. Therefore

$$\exists \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, n_1 < n_2 < ... : \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_{K_m^c}(k)}{k} < \frac{1}{m}, \ \forall n \geq n_m.$$

Similarly to the previous example, let $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_n$, where

$$\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m \leq k \leq n_{m+1} \wedge (k \in K_m^c)\}.$$

It is clear that $l(\mathbb{N}_0) = l\left(\tilde{\mathbb{N}}_0\right)$. Let $n \in \mathbb{N} \Rightarrow \exists m \in \mathbb{N} : n_m < n \leq n_{m+1}$. As before, we assume that $n > n_1$. It is clear that (1) is true, i.e. .

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c).$$

Hence

$$\frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_{\tilde{N}_0}(k)}{k} \leq \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_{K_m^c}(k)}{k} < \frac{1}{m}, \ \forall n \geq n_m.$$

Consequently, $l\left(\tilde{\mathbb{N}}_0\right) = 0 \Rightarrow l(\mathbb{N}_0) = 0 \Rightarrow l(\mathbb{N}_0^c) = 1 \Rightarrow \mathbb{N}_0^c \in \mathscr{F}_l$. It is obvious that

$$\mathbb{N}_0^c \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\}.$$

It directly follows that $\mathscr{F}_l$ is a right filter. Thus, we have proved

Statement 2. *Filter $\mathscr{F}_l$, generated by logarithmic density, is a monotone close and right filter.*

Note that, if $\exists\delta(M) \Rightarrow \exists l(M) \wedge l(M) = \delta(M)$. The converse is not generally true.

IV. Uniform convergence. Let $M \subset \mathbb{N} \wedge (t \in \mathbb{Z}_+; s \in \mathbb{N})$. Assume

$$M(t+1; t+s) = |n \in M : t+1 \leq n \leq t+s|.$$

Put

$$\beta_s(M) = \liminf_{t\to\infty} M(t+1; t+s),$$

$$\beta^s(M) = \limsup_{t\to\infty} M(t+1; t+s).$$

If $\lim\limits_{s\to\infty} \frac{\beta_s(M)}{s} = \lim\limits_{s\to\infty} \frac{\beta^s(M)}{s} = \beta(M)$, then the quantity $\beta(M)$ is called the uniform density of the set M. Let $\mathscr{F}_\beta \equiv \{M \subset \mathbb{N} : \beta(M) = 1\}$. Let us show that $\mathscr{F}_\beta$ is a filter. It is clear that

$$M(t+1;t+s) + M^c(t+1;t+s) = |\,[t+1,t+s]\,| = s.$$

Hence it directly follows that $\beta(M) = 1 \Leftrightarrow \beta(M^c) = 0$. $I_\beta \equiv \{M \subset \mathbb{N} : \beta(M) = 0\}$ is a non-trivial ideal [23]. Therefore, $\mathscr{F}_\beta$ is a filter. It is clear that $\mathscr{F}_\beta$ satisfies the condition v). Let us show that $\mathscr{F}_\beta$ is a monotone close filter. Let $K_1 \supset K_2 \supset \ldots \wedge (\beta(K_n) = 1, \forall n \in \mathbb{N}) \Rightarrow \beta(K_n^c) = 0, \forall n \in \mathbb{N} \Rightarrow \exists \{n_k\}_{k\in\mathbb{N}} \subset N,\ n_1 < n_2 < \ldots :$

$$\frac{\beta^s(K_m^c)}{s} < \frac{1}{m},\ \forall s \geq n_m.$$

As before, we set $\mathbb{N}_0 = \tilde{\mathbb{N}}_0 \cup I_{n_1}$, where $\tilde{\mathbb{N}}_0 \equiv \{k \in \mathbb{N} : n_m \leq k \leq n_{m+1} \wedge (k \in K_m^c)\}$. It is clear that $\beta(\mathbb{N}_0) = \beta\left(\tilde{\mathbb{N}}_0\right)$. Let $n > n_1$ be an arbitrary integer. Then $\exists m \in \mathbb{N} : n_m < n \leq n_{m+1}$. It is obvious that the inclusion

$$\left(I_n \cap \tilde{\mathbb{N}}_0\right) \subset (I_n \cap K_m^c),$$

is true in this case, too. From the arbitrariness of n we have

$$\left(\tilde{\mathbb{N}}_0 \cap [t+1;t+s]\right) \subset (K_m^c \cap [t+1;t+s]).$$

Consequently

$$\tilde{\mathbb{N}}_0(t+1;t+s) \leq K_m^c(t+1;t+s),$$

and as a result

$$\beta^s\left(\tilde{\mathbb{N}}_0\right) \leq \beta^s(K_m^c).$$

Thus

$$\frac{\beta^s\left(\tilde{\mathbb{N}}_0\right)}{s} \leq \frac{\beta^s(K_m^c)}{s} < \frac{1}{m},\ \ \forall s \geq n_m.$$

From this relation it directly follows

$$\beta\left(\tilde{\mathbb{N}}_0\right) = 0 \Rightarrow \beta(\mathbb{N}_0) = 0 \Rightarrow \beta(\mathbb{N}_0^c) = 1 \Rightarrow \mathbb{N}_0^c \in \mathscr{F}_\beta,$$

where

$$\mathbb{N}_0^c \equiv \{k \in \mathbb{N} : n_m < k \leq n_{m+1} \wedge (k \in K_m)\},$$

i.e. $\mathscr{F}_\beta$ is a monotone close filter. As a result, we obtain the validity of the following

Statement 3. *Filter $\mathscr{F}_\beta$, generated by the uniform convergence, is a monotone close and right filter.*

Following [23], number of such examples can be extended.

Remark 1. *Similar results can be obtained with respect to concepts of I-convergence and I^*-convergence.*

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