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Interpolation Theorems on the Nikolskii-Morrey type Spaces

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Abstract. In the paper was studied a differential and differential-difference properties of functions from intersection of Nikolski-Morrey type spaces $H^{l^\mu}_{p_\mu,\varphi,\beta}(G_\varphi), (\mu = 1, 2, \dots, N)$.

Key Words and Phrases: Nikolskii-Morrey type spaces, integral representation, generalized Hölder condition.

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1. Introduction

In the paper, we study some differential properties functions from spaces type $\bigcup\limits_{\mu=1}^{N} H^{l^\mu}_{p_\mu,\varphi,\beta}(G_\varphi)$, more precisely we prove inequality type Riesz-Torin for functions from spaces type $H^{l^\mu}_{p_\mu,\varphi,\beta}(G_\varphi), (\mu = 1, 2, \dots, N)$, and also we prove that for the functions from intersection this spaces, the generalized mixed derivatives $D^\nu f$ satisfy the Holder condition in the metric $L_q(G)$ and $C(G)$. The space $H^l_{p,\varphi,\beta}(G)$is defined in [1] as a linear normed space of functions f, on G with the finite norm $(m_i > l_i - k_i > 0, i = 1, 2, ..., n)$

$$\|f\|_{H^l_{p,\varphi,\beta}(G)} = \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \sup_{0<h<h_0} \frac{\left\|\Delta_i^{m_i}\left(\varphi_i\left(h\right), G_{\varphi(h)}\right) D_i^{k_i} f\right\|_{p,\varphi,\beta}}{\varphi_i\left(h\right)^{l_i-k_i}}, \quad (1)$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{x\in G, t>0} \left(|\varphi\left([t]_1\right)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)}\right), \quad (2)$$

$\left|\varphi\left([t]_1\right)\right|^{-\beta} = \prod\limits_{j=1}^{n} \left(\varphi_j\left([t]_1\right)\right)^{-\beta_j}$, $\beta_j \in [0, 1]$, $j = 1, 2, ..., n$; $l \in (0, \infty)^n$, $m_i \in \mathbb{N}$, $k_i \in \mathbb{N}_0$, $p \in [1, \infty)$, $[t]_1 = \min\{1, t\}$, and vector-functions $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$ with Lebesgue measurable functions $\varphi_j(t) > 0$, $t > 0$, $\lim\limits_{t\to+0} \varphi_j(t) = 0$, $\lim\limits_{t\to+\infty} \varphi_j(t) = L \le \infty$, $j = 1, 2, ..., n$. Denote by $\mathbb{A}$ the set of vector functions φ.

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For any $x \in R^n$,

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n \right\},$$

Let for any $t > 0$, $|\varphi([t]_1)| \leq C$, where C is some positive constant. Then the embeddings $L_{p,\varphi,\beta}(G) \to L_p(G)$ and $H^l_{p,\varphi,\beta}(G) \to H^l_p(G)$ hold, i.e.

$$\|f\|_{p,G} \leq c\|f\|_{p,\varphi,\beta;G}, \text{ and } \|f\|_{H^l_p(G)} \leq c\,\|f\|_{H^l_{p,\varphi,\beta}(G)}. \tag{3}$$

Note that the spaces $L_{p,\varphi,\beta}(G)$ and $H^l_{p,\varphi,\beta}(G)$ are Banach spaces. The space $H^l_{p,\varphi,\beta}(G)$, in the case $\beta_j = 0\,(j = 1, ..., n)$ it coincides with the Nikolski space $H^l_p(G)$. The spaces of such type with different norms were introduced and studied in [3]-[8].

2. Preliminaries

Assuming that $\varphi_j(t)$ $(j = 1, 2, ..., n)$ are also differentiable on $[0, T]$.

Let $\lambda_\mu \geq 0$ $(\mu = 1, 2, ..., N)$ and $\sum\limits_{\mu=1}^{N} \lambda_\mu = 1$, $\frac{1}{p} = \sum\limits_{\mu=1}^{N} \frac{\lambda_\mu}{p_\mu}$, $\frac{1}{q} = \sum\limits_{\mu=1}^{N} \frac{\lambda_\mu}{q_\mu}$, $l = \sum\limits_{\mu=1}^{N} l^\mu \lambda_\mu$, and $\Omega(\cdot, y)$, $M_i(\cdot, y) \in C_0^\infty(R^n)$ be such that

$$S(M_i) = \text{supp} M_i \subset I_{\varphi(t)} = \left\{ y : |y_j| < \frac{1}{2}, \quad j = 1, 2, ..., n \right\}.$$

Assume that for any $0 < T \leq 1$ is a fixed number:

$$V = \bigcup_{0<t\leq T} \left\{ y : \frac{y}{\varphi(t)} \in S(M_i) \right\}.$$

It is clear that $V \subset I_{\varphi(t)}$ and suppose that $U + V \subset G$. Assume $\varphi(t)(j = 1, 2, \ldots, n)$ are also differentiable on $[0, T]$.

Lemma 1. *Let $1 \leq p_\mu \leq q_\mu \leq r_\mu \leq \infty$; $0 < \eta$, $t < T \leq 1$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_j \geq 0$ are integers, $j = 1, 2, ..., n$; $\Delta_i^{m_i}(\varphi_i(t)) \in L_{p,\varphi,\beta}(G)$ and let*

$$Q_T^i = \int\limits_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1 - \sum\limits_{\mu=1}^{N} l_i^\mu \lambda_\mu}} dt < \infty,$$

$$A(x) = \prod_{j=1}^{n} \int\limits_{R^n} \int\limits_{R^n} f(x + y + z) \Omega^\nu \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)} \right)$$

$$\times \Omega \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)} \right) f(x + y + z)\, dydz. \tag{4}$$

$$A_{\eta}^{i}(x)=\int_{0}^{\eta}L_i(x,t)\prod_{j=1}^{n}(\varphi_j(t))^{\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{5}$$

$$A_{\eta T}^{i}(x)=\int_{\eta}^{T}L_i(x,t)\prod_{j=1}^{n}(\varphi_j(t))^{\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{6}$$

where

$$L_i(x,t)=\int_{R^n}\int_{-\infty}^{+\infty}M_i\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)}\right)$$
$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{2\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi_i(t),x)\right)\Delta_i^{m_i}(\varphi_i(\delta)u)f(x+y+ue_i)\,dudy \tag{7}$$

Then for any $\overline{x}\in U$ *the following inequalities*

$$\sup_{\overline{x}\in U}\|A\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_1\prod_{\mu=1}^{N}\{\|f\|_{p_\mu,\varphi,\beta;G}\}^{\lambda_\mu}\times$$
$$\times\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^{n}(\psi_j[\xi]_1)^{\beta_j\frac{p}{q}}, \tag{8}$$

$$\sup_{\overline{x}\in U}\left\|A_{\eta}^{i}\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_2\prod_{\mu=1}^{N}\left\{\left\|(\varphi_i(t))^{-l_i^{\mu}}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu}$$
$$\times\left|Q_{\eta}^{i}\right|\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}, \tag{9}$$

$$\sup_{\overline{x}\in U}\left\|A_{\eta T}^{i}\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_2\prod_{\mu=1}^{N}\left\{\left\|(\varphi_i(t))^{-l_i^{\mu}}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p,\varphi,\beta;G}\right\}^{\lambda_\mu}$$
$$\times\left|Q_{\eta T}^{i}\right|\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}, \tag{10}$$

where $U_{\psi(\xi)}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j(\xi),j=1,2,...,n\right\}$ *and* $\psi\in A$, C_1, C_2 *are the constants independent of* φ, ξ, η *and* T.

Proof. Using the Minkowsky inequality for any $\overline{x}\in U$

$$\left\|A_{\eta}^{i}\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq\int_{0}^{\eta}\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})}\prod_{j=1}^{n}(\varphi_j(t))^{\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt, \tag{11}$$

and we get

$$\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \le C_1 \left(\int\limits_{U_{\psi(\xi)}(\overline{x})} \prod_{\mu=1}^{N} \{|L_i(\cdot,t)|\}^{\lambda_\mu q} dx \right)^{\frac{1}{q}}.$$

Once again using the Hölder's inequality with indication $\alpha_\mu = \frac{q_\mu}{q\lambda_\mu}$, $\mu = 1,2,\ldots,N$ $(\sum\limits_{\mu=1}^{N} \frac{1}{\alpha_\mu} = q \sum \frac{\lambda_\mu}{q_\mu} = 1)$. Then we have

$$\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \le C_2 \prod_{\mu=1}^{N} \{\|L_i(\cdot,t)\|_{q_\mu U_{\psi(\xi)}(\overline{x})}\}^{\lambda_\mu}. \tag{12}$$

Taking Hölder inequality $(q_\mu \le r_\mu)$ we get

$$\|L_i(\cdot,t)\|_{qU_{\psi(\xi)}(\overline{x})} \le \|L_i(\cdot,t)\|_{rU_{\psi(\xi)}(\overline{x})} \prod_{j=1}^{n} (\psi_j(\xi))^{\frac{1}{q_\mu}-\frac{1}{r_\mu}}. \tag{13}$$

Let X be a characteristic function of the set $S(M_i) = supp\, M_i$. Noting that $1 \le p_\mu \le r_\mu \le \infty$, $s_\mu \le r_\mu$ $\left(\frac{1}{s_\mu} = 1 - \frac{1}{p_\mu} + \frac{1}{r_\mu}\right)$, and apply for $|L_i|$ the Hölder inequality $\left(\frac{1}{p_\mu} + \left(\frac{1}{p_\mu} - \frac{1}{r_\mu}\right) + \left(\frac{1}{s_\mu} - \frac{1}{r_\mu}\right) = 1\right)$, and we obtain

$$\|L_i(\cdot,t)\|_{r_\mu, U_{\psi(\xi)}(\overline{x})} \le$$

$$\le \sup_{x\in U_{\psi(\xi)}(\overline{x})} \left(\int\limits_{R^n} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t),x) \right) \right. \right.$$

$$\left. \left. \times \Delta_i^{m_i}(\varphi_i(t))\, f(x+y+ue_i)\, du \right|^{p_\mu} \chi\left(\frac{y}{\varphi(t)}\right) dy \right)^{\frac{1}{p_\mu}-\frac{1}{r_\mu}}$$

$$\times \sup_{y\in V} \left(\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t),x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t),x) \right) \right. \right.$$

$$\left. \left. \times \Delta_i^{m_i}(\varphi_i(t)u)\, f(x+y+ue_i)\, du \right|^{p_\mu} dx \right)^{\frac{1}{p_\mu}}$$

$$\times \left(\int\limits_{R^n} \left| M_i \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \rho'(\varphi(t),x) \right) \right|^{s_\mu} dy \right)^{\frac{1}{s_\mu}} \tag{14}$$

(suppose that $|M_i(x,y,z)| \le C|\widetilde{M}_i(x)|$).

For any $x \in U$ we have

$$\int\limits_{R^n} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \right.$$

$$\times \Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)\,du|^{p_\mu} \chi\left(\frac{y}{\varphi(t)}\right) dy$$

$$\leq \int\limits_{(U+V)_{\varphi(t)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \right.$$

$$\times\, \Delta_i^{m_i}(\varphi_i(\delta)u) f(y+ue_i)\,du|^{p_\mu}\,dy \leq$$

$$\leq \varphi_i(t)^{p_\mu + p_\mu l_i^\mu} \left\| \varphi_i(t)^{-l_i^\mu} \Delta_i^{m_i}\left(\varphi_i(\delta)u, G_{\varphi(t)}\right) \right\|_{p_\mu,\varphi,\beta}^{p_\mu} \prod_{j=1}^{n} (\varphi_j(t))^{\beta_j p_\mu}. \tag{15}$$

For $y \in V$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)\,du \right|^{p_\mu} dx$$

$$\leq \int\limits_{G_{\varphi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \Delta_i^{m_i}(\varphi_i(\delta)u) f(x+ue_i)\,du \right|^{p_\mu} dx$$

$$\leq (\varphi_i(t))^{p_\mu l_i^\mu} \left\| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x) \right) \right.$$

$$\left. \times \varphi_i(t)^{-l_i^\mu} \Delta_i^{m_i}\left(\varphi_i(\delta)u, G_{\varphi(t)}\right) f du \right\|_{p_\mu, G_{\varphi(t)}(\overline{x})}^{p_\mu}$$

$$\leq \varphi_i(t)^{p+pl_i^\mu} \left\| \varphi_i(t)^{-l_i^\mu} \Delta_i^{m_i}\left(\varphi_i(\delta), G_{\varphi(t)}\right) \right\|_{p_\mu,\varphi,\beta}^{p_\mu} \prod_{j=1}^{n} (\psi_j([\xi]_1))^{\beta_j p_\mu} \tag{16}$$

$$\left(\int\limits_{R^n} \left| \widetilde{M}_i\left(\frac{y}{\varphi(t)}\right) \right|^{s_\mu} dy \right)^{\frac{1}{s_\mu}} = \left\| \widetilde{M}_i \right\|_{s_\mu}^{s_\mu} \cdot \prod_{j=1} \varphi_j(t). \tag{17}$$

From inequalities (11)-(17) for $(r_\mu = q_\mu)$ and for any $\overline{x} \in U$ reduce to the estimation

$$\left\| A_\eta^i \right\|_{qU_{\psi(\xi)}(\overline{x})} \leq C_1 \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}(\varphi_i(\delta)u) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}$$

$$\times \left|Q^i_\eta\right| \prod_{j=1}^{n} \left(\psi_j([\xi]_1)\right)^{\beta_j \frac{p}{q}} \quad \left(Q^i_\eta < \infty\right). \tag{18}$$

In the case $Q^i_{\eta,T} < \infty$ inequality (10) and (8) is proved in the same way.

From last inequalities it follows that

$$\left\|A^i_\eta\right\|_{q,\psi,\beta^1;U} \leq C^1 \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l^\mu_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}, \tag{19}$$

$$\left\|A^i_{\eta T}\right\|_{q,\psi,\beta^1;U} \leq C^2 \prod_{\mu=1}^{N} \left\{ \left\| (\varphi_i(t))^{-l^\mu_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}. \tag{20}$$

C'_1 and C'_2 are the constants independent of φ.

3. Main results

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn[1], $1 \leq p_\mu \leq q_\mu \leq \infty, \mu = 1, 2, \ldots, N$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \geq 0$ be entire $j = 1, 2, ..., n$, $Q^i_T < \infty$ $(i = 1, 2, ..., n)$ and let $f \in \bigcup_{\mu=1}^{N} H^{l^\mu}_{p_\mu,\varphi,\beta}(G_\varphi)$. Then the following embedding hold*

$$D^\nu : \bigcup_{\mu=1}^{N} H^{l^\mu}_{p_\mu,\varphi,\beta}(G_\varphi) \to L_{q,\psi,\beta^1}(G)$$

more precisely, for $f \in \bigcup_{\mu=1}^{N} H^{l^\mu}_{p_\mu,\varphi,\beta}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are valid:

$$\|D^\nu f\|_{q,G} \leq C_1 H(t) \prod_{\mu=1} \left\{ \|f\|_{H^{l^\mu}_{pmu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu}, \tag{21}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \leq C_2 \prod_{\mu=1} \left\{ \|f\|_{H^{l^\mu}_{pmu,\varphi,\beta}(G_\varphi)} \right\}^{\lambda_\mu}, \; p \leq q < \infty. \tag{22}$$

In particular, if

$$Q^i_{T,0} = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\frac{1}{p}} \times$$

$$\times \frac{\varphi'_i(t)}{(\varphi_i(t))^{1-\sum_{\mu=1}^{N} l^\mu \lambda_\mu}} dt < \infty, (i = 1, 2, \ldots, n),$$

then the function $D^{\nu} f(x)$ *is continuous on* G, *and*

$$\sup_{x \in G} |D^{\nu} f(x)| \leq C_1 H_0(t) \prod_{\mu=1} \left\{ \|f\|_{H^{l^{\mu}}_{p^{\mu},\varphi,\beta}(G_{\varphi})} \right\}^{\lambda_{\mu}} \tag{23}$$

where $H(T) = \sum\limits_{i=0}^{n} |Q^i_T|$, $H_0(T) = \sum\limits_{i=0}^{n} |Q^i_{T,0}|$,

$$Q^0_T = \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)(\frac{1}{p} - \frac{1}{q})}$$

$0 < T \leq \min\{1, T_0\}$, T_0 *is a fixed number;* C_1, C_2 *are the constants independent of* f, *also* C_1 *is independent from* T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu} f$ on G. Indeed, from the condition $Q^i_T < \infty$ for all $(i = 1, 2, ..., n)$ it follows that for $f \in H^{l^{\mu}}_{p_{\mu},\varphi,\beta}(G) \to H^{l^{\mu}}_{p_{\mu}}(G)$, there exists $D^{\nu} f \in L_{p_{\mu}}(G)$ and for almost every point of $x \in G$ integral representation is valid.

$$D^{\nu} f(x) = f^{(\nu)}_{\varphi(t)}(x) + (-1)^{|\nu|} \sum_{i=1}^{n} \int_0^T \int_{-\infty}^{+\infty} \int_{R^n} K^{(\nu)}_i \left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)} \right)$$

$$\times \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{2\varphi_i(t)}, \frac{1}{2} \rho'_i(\varphi_i(t), x) \right)$$

$$\times \Delta^{m_i}_i (\varphi_i(\delta) u) f(x + y + u e_i) \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - 2} \frac{\varphi'_i(t)}{\varphi_i(t)} dt du dy, \tag{24}$$

$$f^{(\nu)}_{\varphi(T)}(x) = \prod_{j=1}^{n} (\varphi_j(T))^{-2-\nu_j} \times$$

$$\times \int_{R^n} \int_{R^n} \Omega^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)} \right) \Omega \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(T), x)}{2\varphi(T)} \right) f(x + y + z) dy dz, \tag{25}$$

Applying the Minkowsky inequality we have

$$\|D^{\nu} f\|_{q,G} \leq \left\| f^{(\nu)}_{\varphi(T)} \right\|_{q,G} + \sum_{i=1}^{n} \|A^i_T\|_{q,G}. \tag{26}$$

By means of inequality (8) and (9) for $M_i = K^{(\nu)}_i$, $\eta = T$ we get inequality (21). By means of inequality (19) for $M_i = K^{(\nu)}_i$, $\eta = T$ we get inequality (22).

Now let conditions $Q^i_T < \infty (i = 1, 2, \ldots, n)$, then take into account (24),and (25), from inequality (26) we get

$$\left\| D^{\nu} f - f^{(\nu)}_{\varphi(T)} \right\|_{\infty,G} \leq$$

$$\leq C\sum_{i=1}^{n}\left|Q_T^i\right|\prod_{\mu=1}\left\{\sup_{0<t<t_0}\left\|\frac{\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f}{\left(\varphi_i(t)\right)^{l_i^\mu}}\right\|_{p,\varphi,\beta;G}\right\}^{\lambda_\mu}.$$

As $T\to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}^{(\nu)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the absolutely convergence. Consequently, the derivative function is continuous G.

Let γ be an n-dimensional vector.

Theorem 2. *Let all the conditions of Theorem 1 be fulfilled. Then for $Q_T^i<\infty$ $(i=1,2,...,n)$ the derivative $D^\nu f$ satisfies on G the Hölder generalized condition, i.e. the following inequality is valid:*

$$\left\|\Delta\left(\gamma,G\right)D^\nu f\right\|_{q,G}\leq$$

$$\leq C\prod_{\mu=1}\left\{\|f\|_{H^{l^\mu}_{p^\mu,\varphi,\beta}(G_\varphi)}\right\}^{\lambda_\mu}\cdot\left|R\left(|\gamma|,\varphi;T\right)\right|, \tag{27}$$

in particular, if $Q_{T,0}^i<\infty$, $(i=1,2,\ldots,n)$, then

$$\sup_{x\in G}\left|\Delta\left(\gamma,G\right)D^\nu f\left(x\right)\right|\leq C\prod_{\mu=1}\left\{\|f\|_{H^{l^\mu}_{p^\mu,\varphi,\beta}(G_\varphi)}\right\}^{\lambda_\mu}\cdot\left|R_0\left(|\gamma|,\varphi,T\right)\right|. \tag{28}$$

where $R\left(|\gamma|,\varphi,T\right)=\max\limits_i\left\{|\gamma|,Q^i_{|\gamma|},Q^i_{|\gamma|,T}\right\}$ $\left(h_0\left(|\gamma|,\varphi,T\right)\ =\max\limits_i\left\{|\gamma|,Q^i_{|\gamma|,0},Q^i_{|\gamma|,T,0}\right\}\right)$.

Proof. According to Lemma 8.6 from [2] there exists a domain

$$G_\omega\subset G\left(\omega=\zeta r\left(x\right),\,\zeta>0\;r\left(x\right)=\rho\left(x,\partial G\right),\;x\in G\right)$$

and assume that $|\gamma|<\omega$, then for any $x\in G_\omega$ the segment connecting the points $x,x+\gamma$ is contained in G. For all the points of this segment,from identies (24), (25) after same transformations we get

$$\left\|\Delta\left(\gamma,G\right)D^\nu f\left(x\right)\right\|\leq\left\|B\left(\cdot,\gamma\right)\right\|_{q,G}+$$

$$+\sum_{i=1}^{n}\left(\left\|B_1\left(\cdot,\gamma\right)\right\|_{q,G}+\left\|B_2\left(\cdot,\gamma\right)\right\|_{q,G}\right), \tag{29}$$

where

$$B\left(x,\gamma\right)=\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-2-\nu_j}$$

$$\times\int\limits_{R^n}\int\limits_{R^n}\left|f\left(x+y+z\right)\right|\left|\Omega^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{2\varphi(t)}\right)\right.$$

$$-\Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(T)}\right)\Bigg| dydz \leq \prod_{j=1}^{n} (\varphi_j(t))^{-2-\nu_j} \times$$

$$\times \int_0^{|\gamma|} d\zeta \int_{R^n}\int_{R^n} |f(x+\zeta e_\zeta + z)| \left| D_j \Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) - \right.$$

$$\left. -\Omega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(T)}\right)\right| dydz,$$

$$B_1(x,\gamma) = \int_0^{|\gamma|}\int_{R^n}\int_{-\infty}^{+\infty} \left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho'(\varphi(t),x)\right)\right| \times$$

$$\times \left| K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)}\right)\right| |\Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+ue_i)| \, dydudt$$

$$B_2(x,\gamma) = \int_{|\gamma|}^{T}\int_{R^n}\int_{-\infty}^{+\infty} \left| K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)}\right)\right| \times$$

$$\times \left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t),x)\right)\right| \times$$

$$\times \int_0^1 |\Delta_i^{m_i}(\varphi_i(\delta)u) f(x+y+v\gamma)| \, dvdudydt.$$

Here $0 < T \leq \min\{1, T_0\}$. Additionally, we assume that $|\gamma| < T$, then $|\gamma| < \min(\omega, T)$ and for $x \in G \setminus G_\omega$ then

$$\Delta(\gamma, G) D^\nu f(x) = 0.$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, based around the generalized Minkowsky inequality, from inequality (8) for $U = G$, we have

$$\|B(\cdot,\gamma)\|_{q,G_\omega} \leq C_1 |\gamma| \prod_{\mu=1} \left\{\|f\|_{H^{l^\mu}_{p^\mu,\varphi,\beta}(G_\varphi)}\right\}^{\lambda_\mu}. \tag{30}$$

By means of inequality (9),(10) for $U = G$, $M_i = K_i^{(\nu)}$, $\eta = |\gamma|$ we get

$$\|B_1(\cdot,\gamma)\|_{q,G_\omega} \leq C_2 \left|Q^i_{|\gamma|}\right| \times$$

$$\times \prod_{\mu=1} \left\{\left\|(\varphi_i(t))^{-l_i^\mu} \Delta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p_\mu,\varphi,\beta;G}\right\}^{\lambda_\mu} \tag{31}$$

$$\|B_2(\cdot,\gamma)\|_{q,G_\omega} \leq C_3 \left|Q^i_{|\gamma|,T}\right| \times$$

$$\times \prod_{\mu=1} \left\{ \left\| (\varphi_i(t))^{-l_i^{\mu}} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p_\mu,\varphi,\beta;G} \right\}^{\lambda_\mu}. \tag{32}$$

From inequalities (29) –(32) we get the required inequality (27).

Let $|\gamma| \geq \min(\omega, T)$, then

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \leq 2 \|D^\nu f\|_{q,G} \leq C(\omega T) \|D^\nu f\|_{q,G} |R(|\gamma|, \varphi; T)|.$$

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (21), in this case we get estimation (27).

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