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A Class of Small Deviation Theorems for Nonhomogeneous Markov Chain Fields on a Homogeneous Tree

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Abstract. In this paper, a class of small deviation theorems for the arbitrary bivariate function are established by introducing the sample relative entropy rate as a measure of deviation between the arbitrary random field and Markov chains field on the homogeneous tree. As corollaries, a class of small deviation theorems for the frequencies of states ordered couples and harmonic mean of transitional probability of Markov chains field on the homogeneous tree are obtained.

Key Words and Phrases: harmonic mean, the homogeneous tree, arbitrary random field, Markov chains field, sample relative entropy density.
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1. Introduction

Let T be a homogeneous tree on which each vertex has $N+1$ neighboring vertices. We first fix any vertex as the "root" and label it by 0. Let σ, τ be vertices of a tree. Write $\tau \leq \sigma$ if τ is on the unique path connecting 0 to σ, $|\sigma|$ for the number of edges on this path. For any two vertices σ, τ, denote $\sigma \wedge \tau$ the vertex farthest from 0 satisfying

$$\sigma \wedge \tau \leq \sigma, \quad and \quad \sigma \wedge \tau \leq \tau.$$

If $\sigma \neq 0$, then we let $\bar{\sigma}$ stand for the vertex satisfying $\bar{\sigma} \leq \sigma$ and $|\bar{\sigma}| = |\sigma| - 1$ (we refer to σ as a son of $\bar{\sigma}$). It is easy to see that the root has $N+1$ sons and all other vertices have N sons. The homogeneous tree T is also called Bethe tree $T_{B,N}$. For example, we give the following Fig 1 $T_{B,2}$.

Definition 1(see[7]). Let T be a homogeneous tree, $S = \{s_0, s_1, s_2 \cdots, s_{M-1}\}$ be a finite state space, $\{X_\sigma, \sigma \in T\}$ be a collection of $S-$valued random variables defined on the measurable space $\{\Omega, \mathcal{F}\}$. Let

$$q = \{q(x), x \in S\} \tag{1}$$

be a distribution on S, and

$$Q_n = (Q_n(y|x)), \qquad x, y \in S \tag{2}$$

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be a series of strictly positive stochastic matrices on S^2. If for any vertices σ, τ,

$$Q(X_\sigma = y|X_{\bar{\sigma}} = x, \ \ and \ \ X_\tau \ for \ \ \sigma \wedge \tau \le \bar{\sigma}) \tag{3}$$
$$= Q(X_\sigma = y|X_{\bar{\sigma}} = x) = Q_n(y|x) \quad \forall x, y \in S, \quad n \ge 1.$$

and

$$Q(X_0 = x) = q(x), \qquad \forall x \in S. \tag{4}$$

$\{X_\sigma, \sigma \in T\}$ will be called $S-$valued Markov chains indexed by a homogeneous tree with the initial distribution (1) and transition matrix (2).

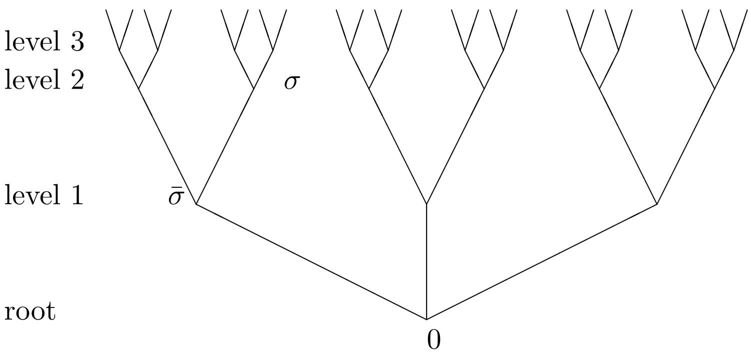

Fig 1. Bethe tree $T_{B,2}$

Two special finite tree-indexed Markov chains are introduced in Kemeny et al.(1976[13]), Spitzer (1975[6]), and there the finite transition matrix is assumed to be positive and reversible to its stationary distribution, and this tree-indexed Markov chains ensure that the cylinder probabilities are independent of the direction we travel along a path. In this paper, we have no such assumption.

If $|\sigma| = n$, it is said to be on the nth level on a tree T. We denote by $T^{(n)}$ the subtree of T containing the vertices from level 0 (the root) to level n, and L_n the set of all vertices on the level n. Let B be a subgraph of T. Denote $X^B = \{X_\sigma, \sigma \in B\}$, and denote by $|B|$ the number of vertices of B. Let $S(\sigma)$ be the set of all sons of vertices σ. It is easy to see that $|S(0)| = N + 1$ and $|S(\sigma)| = N$, where $\sigma \neq 0$.

Let $\Omega = S^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on T and taking values in S, and $\mathcal{F}$ be the smallest Borel field containing all cylinder sets in Ω, μ be the probability measure on $(\Omega, \mathcal{F})$. Let $X = \{X_\sigma, \sigma \in T\}$ be the coordinate stochastic process defined on the measurable space $(\Omega, \mathcal{F})$; that is, for any $\omega = \{\omega(t), t \in T\}$, define

$$X_t(\omega) = \omega(t), \qquad\qquad t \in T^{(n)}$$

$$X^{T^{(n)}} \triangleq \{X_t, t \in T^{(n)}\}, \qquad \mu(X^{T^{(n)}} = x^{T^{(n)}}) = \mu(x^{T^{(n)}}). \quad i = 1, 2. \tag{5}$$

Now we give a definition of Markov chain fields on the tree T by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see[4]).

Definition 2. Let $Q_n = Q_n(j|i)$ and $q = (q(s_0), q(s_1) \cdots, q(s_{M-1}))$ be defined as before, μ_Q be another probability measure on $(\Omega, \mathcal{F})$. If

$$\mu_Q(x_0) = q(x_0) \tag{6}$$

$$\mu_Q(x^{T^{(n)}}) = q(x_0) \prod_{k=0}^{n-1} \prod_{\sigma \in L_k} \prod_{\tau \in S(\sigma)} Q_{k+1}(x_\tau | x_\sigma), \quad n \geq 1. \tag{7}$$

then μ_Q will be called a Markov chain field on the homogeneous tree T determined by the stochastic matrix Q and the distribution q.

There have been some works on limit theorems for tree-indexed stochastic processes. Benjamini and Peres have given the notion of the tree-indexed homogeneous Markov chains and studied the recurrence and ray-recurrence for them (see[1]). Berger and Ye have studied the existence of entropy rate for some stationary random fields on a homogeneous tree (see[2]). Pemantle proved a mixing property and a weak law of large numbers for a PPG-invariant and ergodic random field on a homogeneous tree (see[5]). Ye and Berger, by using Pemantle's result and a combinatorial approach, have studied the asymptotic equipartition property (AEP) in the sense of convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree(see[9-10]). Peng and Yang have studied a class of small deviation theorems for functionals of random field and asymptotic equipartition property (AEP) for arbitrary random field on a homogeneous trees (see[8]). Recently, Yang have studied some limit theorems for countable homogeneous Markov chains indexed by a homogeneous tree and strong law of large numbers and the asymptotic equipartition property (AEP) for finite homogeneous Markov chains indexed by a homogeneous tree (see[7] and [11]). But their results only concern the case of strong limit theorems for Markov chains field, they do not discuss the the case of strong deviation theorems (the strong limit theorems which are represented by the inequalities) for arbitrary random fields.

In this paper, our aim is to establish a class of small deviation theorems (also called strong deviation theorems) for the arbitrary bivariate function by introducing the sample relative entropy rate as a measure of deviation between the arbitrary random fields and the Markov chains field on the homogeneous tree. We apply a new type of techniques to study of the small deviation theorems for arbitrary random field on a homogeneous tree. As corollaries, a class of small deviation theorems for the frequencies of states ordered couples and harmonic mean of transitional probability of Markov chains field on the homogeneous tree are obtained.

2. Main results and its proof

Lemma 1.(see[3]) Let μ_1 and μ_2 be two probability measure on $(\Omega, \mathcal{F})$, $D \in \mathcal{F}$, $\{\tau_n, n \geq 0\}$ be a positive-value stochastic sequence such that

$$\liminf_n \frac{\tau_n}{|T^{(n)}|} > 0. \qquad \mu_1 - a.s. \;\; D \tag{8}$$

then

$$\limsup_{n\to\infty} \frac{1}{\tau_n} \log \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \le 0. \qquad \mu_1 - a.s. \;\; D \tag{9}$$

Particularly, let $\tau_n = |T^{(n)}|$, then

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \le 0. \qquad \mu_1 - a.s. \tag{10}$$

Proof. See reference [3].

Let

$$\varphi(\mu|\mu_Q) = \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})}. \tag{11}$$

$\varphi(\mu|\mu_Q)$ is called the sample relative entropy rate with respect to μ and μ_Q. $\varphi(\mu|\mu_Q)$ is also called asymptotic logarithmic likelihood ratio. By (10) and (11),

$$\varphi(\mu|\mu_Q) \ge \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \log \frac{\mu(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})} \ge 0. \quad \mu - a.s. \tag{12}$$

Hence $\varphi(\mu|\mu_Q)$ can be look on as a type of a measure of the deviation between the arbitrary random field and the Markov chains field on the homogeneous tree.

Although $\varphi(\mu|\mu_Q)$ is not a proper metric between two probability measures, we nevertheless think of it as a measure of "dissimilarity" between their joint distribution μ and Markov distribution μ_Q. Obviously, $\varphi(\mu|\mu_Q) = 0$ if and only if $\mu = \mu_Q$. It has been shown in (12) that $\varphi(\mu|\mu_Q) \ge 0$, a.s. in any case. Hence, $\varphi(\mu|\mu_Q)$ can be used as a random measure of the deviation between the true joint distribution $\mu(x^{T^{(n)}})$ and the Markov distribution $\mu_Q(x^{T^{(n)}})$. Roughly speaking, this deviation may be regarded as the one between coordinate stochastic process $x^{T^{(n)}}$ and the Markov case. The smaller $\varphi(\mu|\mu_Q)$ is, the smaller the deviation is.

Theorem 1. Let $X = \{X_\sigma, \sigma \in T\}$ be an arbitrary random field on the homogeneous tree T. $\varphi(\mu|\mu_Q)$ is defined as (11). Let $f(x,y)$ be an arbitrary real bivariate function defined on S^2, $c > 0$. Denote

$$D(c) = \{\omega : \varphi(\mu|\mu_Q) \le c\}, \tag{13}$$

then

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \{f(X_\sigma, X_\tau) - E_Q[f(X_\sigma, X_\tau)|X_\sigma]\} \le (2\sqrt{c}+c) \sum_{i\in S}\sum_{j\in S} |f(i,j)|,$$

$$\mu - a.s. \;\; \omega \in D(c) \tag{14}$$

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \{f(X_\sigma, X_\tau) - E_Q[f(X_\sigma, X_\tau)|X_\sigma]\} \ge -2\sqrt{c} \sum_{i\in S}\sum_{j\in S} |f(i,j)|.$$

$$\mu - a.s. \;\; \omega \in D(c) \tag{15}$$

Where $E_Q(\cdot)$ represents the expectation with respect to the Markov measure μ_Q.

Proof. Consider the probability space $(\Omega, \mathcal{F}, \mu)$, let λ be an arbitrary real number, $\delta_i(j)$ be Kronecker function. We construct the following product distribution:

$$\mu_Q(x^{T^{(n)}};\lambda) = \lambda^{\sum\limits_{k=0}^{n-1}\sum\limits_{\sigma\in L_k}\sum\limits_{\tau\in S(\sigma)}\delta_i(x_\sigma)\delta_j(x_\tau)} \prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)}[\frac{1}{1+(\lambda-1)Q_{k+1}(j|i)}]^{\delta_i(x_\sigma)}$$

$$\cdot q(x_0)\prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)} Q_{k+1}(x_\tau|x_\sigma). \tag{16}$$

By (16) we have

$$\begin{aligned}
&\sum_{x^{L_n}\in S}\mu_Q(x^{T^{(n)}};\lambda)\\
= &\sum_{x^{L_n}\in S} q(x_0)\prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)}\lambda^{\delta_i(x_\sigma)\delta_j(x_\tau)}[\frac{1}{1+(\lambda-1)Q_{k+1}(j|i)}]^{\delta_i(x_\sigma)}\cdot Q_{k+1}(x_\tau|x_\sigma)\\
= &\mu_Q(x^{T^{(n-1)}};\lambda)\sum_{x^{L_n}\in S}\prod_{\sigma\in L_{n-1}}\prod_{\tau\in S(\sigma)}\lambda^{\delta_i(x_\sigma)\delta_j(x_\tau)}[\frac{1}{1+(\lambda-1)Q_n(j|i)}]^{\delta_i(x_\sigma)}Q_n(x_\tau|x_\sigma)\\
= &\mu_Q(x^{T^{(n-1)}};\lambda)\prod_{\sigma\in L_{n-1}}\prod_{\tau\in S(\sigma)}\sum_{x_\tau\in S}\lambda^{\delta_i(x_\sigma)\delta_j(x_\tau)}[\frac{1}{1+(\lambda-1)Q_n(j|i)}]^{\delta_i(x_\sigma)}Q_n(x_\tau|x_\sigma)
\end{aligned}$$

$$\mu_Q(x^{T^{(n-1)}};\lambda)\prod_{\sigma\in L_{n-1}}\prod_{\tau\in S(\sigma)}\frac{\lambda^{\delta_i(x_\sigma)}Q_n(j|x_\sigma)+1-Q_n(j|x_\sigma)}{[1+(\lambda-1)Q_n(j|i)]^{\delta_i(x_\sigma)}}, \tag{17}$$

when $x_\sigma = i$, we obtain from (17) that

$$\sum_{x^{L_n}\in S}\mu_Q(x^{T^{(n)}};\lambda)$$

$$\mu_Q(x^{T^{(n-1)}};\lambda)\prod_{\sigma\in L_{n-1}}\prod_{\tau\in S(\sigma)}\frac{1+(\lambda-1)Q_n(j|i)}{1+(\lambda-1)Q_n(j|i)} = \mu_Q(x^{T^{(n-1)}};\lambda), \tag{18}$$

when $x_\sigma \neq i$, we acquire from (17) that

$$\sum_{x^{L_n}\in S}\mu_Q(x^{T^{(n)}};\lambda) =$$

$$= \mu_Q(x^{T^{(n-1)}};\lambda)\prod_{\sigma\in L_{n-1}}\prod_{\tau\in S(\sigma)}(Q_n(j|x_\sigma)+1-Q_n(j|x_\sigma)) = \mu_Q(x^{T^{(n-1)}};\lambda). \tag{19}$$

Therefore $\mu_Q(x^{T^{(n)}};\lambda)$, $n=1,2,\cdots$ are a family of consistent distribution functions on $S^{T^{(n)}}$. Let

$$U_n(\lambda,\omega)=\frac{\mu_Q(X^{T^{(n)}};\lambda)}{\mu(X^{T^{(n)}})}. \tag{20}$$

By (16) and (20), we have

$$U_n(\lambda,\omega)=\lambda^{\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\delta_j(X_\tau)}\prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)}[\frac{1}{1+(\lambda-1)Q_{k+1}(j|i)}]^{\delta_i(X_\sigma)}$$

$$\cdot q(X_0)\prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)}Q_{k+1}(X_\tau|X_\sigma)\Big/\mu(X^{T^{(n)}}). \tag{21}$$

Since μ and μ_Q are two probability measures, we easily see that $U_n(\lambda,\omega)$ is a nonnegative sup-martingale from Doob's martingale convergence theorem(see[12]) . Moreover,

$$\lim_{n\to\infty}U_n(\lambda,\omega)=U_\infty(\lambda,\omega)<\infty. \qquad \mu-a.s. \tag{22}$$

By (10) and (20) we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\log U_n(\lambda,\omega)\le 0. \qquad \mu-a.s. \tag{23}$$

By (7), (21) and (23) we have

$$\limsup_{n\to\infty}\{\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\delta_j(X_\tau)\log\lambda$$

$$-\quad\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\log[1+(\lambda-1)Q_{k+1}(j|i)]+\frac{1}{|T^{(n)}|}\log\frac{\mu_Q(X^{T^{(n)}})}{\mu(X^{T^{(n)}})}\}$$

$$\le 0. \qquad \mu-a.s. \tag{24}$$

Letting $\lambda=0$ in (24), we have

$$\varphi(\mu|\mu_Q)\ge\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\log\frac{\mu(X^{T^{(n)}})}{\mu_Q(X^{T^{(n)}})}\ge 0.\ \mu-a.s. \tag{25}$$

By (13) and (24) we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\{\delta_j(X_\tau)\log\lambda-\log[1+(\lambda-1)Q_{k+1}(j|i)]\}\le$$

$$\varphi(\mu|\mu_Q)\le c.\ \ \mu-a.s. \qquad \omega\in D(c) \tag{26}$$

In the case $\lambda > 1$, dividing two sides of (26) by $\log\lambda$, we obtain

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\{\delta_j(X_\tau)-\frac{\log[1+(\lambda-1)Q_{k+1}(j|i)]}{\log\lambda}\}\le\frac{c}{\log\lambda}.$$

$$\mu-a.s. \qquad \omega\in D(c) \tag{27}$$

By (27), the inequalities $1-1/x\le\ln x\le x-1, 0\le\delta_i(X_\sigma)\le 1$ and the properties of superior limit

$$\limsup_{n\to\infty}(a_n-b_n)\le d\Rightarrow\limsup_{n\to\infty}(a_n-c_n)\le\limsup_{n\to\infty}(b_n-c_n)+d,$$

we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{\delta_i(X_\sigma)\delta_j(X_\tau)-\delta_i(X_\sigma)Q_{k+1}(j|i)\}\\
\le\ &\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\{\frac{\log[1+(\lambda-1)Q_{k+1}(j|i)]}{\log\lambda}-Q_{k+1}(j|i)\}+\frac{c}{\log\lambda}\\
\le\ &\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\{\frac{(\lambda-1)Q_{k+1}(j|i)}{\log\lambda}-Q_{k+1}(j|i)\}+\frac{c}{\log\lambda}\\
\le\ &\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}(\lambda-1)Q_{k+1}(j|i)+\frac{c}{\lambda-1}+c\\
\le\ &\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}(\lambda-1)+\frac{c}{\lambda-1}+c\\
=\ &(\lambda-1)+\frac{c}{\lambda-1}+c.
\end{aligned}$$

$$\mu-a.s. \qquad \omega\in D(c) \tag{28}$$

It is easy to show that in the case $c>0$, the function $f(\lambda)=(\lambda-1)+c+\frac{c}{\lambda-1}(\lambda>1)$ attains its smallest value $f(1+\sqrt{c})=2\sqrt{c}+c$ at $\lambda=1+\sqrt{c}$. Hence letting $\lambda=1+\sqrt{c}$ in (28), we have

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{\delta_i(X_\sigma)\delta_j(X_\tau)-\delta_i(X_\sigma)Q_{k+1}(j|i)\}\le 2\sqrt{c}+c.$$

$$\mu-a.s. \qquad \omega\in D(c) \tag{29}$$

In the case $c=0$, (29) also follows from (28) by choosing $\lambda_i\to 1+(i\to\infty)$.

In the case $0 < \lambda < 1$, dividing two sides of (26) by $\log \lambda$, we obtain

$$\begin{aligned} & \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \{\delta_i(X_\sigma)\delta_j(X_\tau) - \delta_i(X_\sigma)Q_{k+1}(j|i)\} \\ \geq & \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \delta_i(X_\sigma)(\frac{\lambda-1}{1-1/\lambda} - 1)Q_{k+1}(j|i) + \frac{c}{\lambda-1} \\ \geq & \liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \delta_i(X_\sigma)(\lambda-1) + \frac{c}{\lambda-1} \\ = & (\lambda-1) + \frac{c}{\lambda-1}. \end{aligned}$$

$$\mu - a.s. \qquad \omega \in D(c) \tag{30}$$

It is obvious to show that in the case $c > 0$, the function $h(\lambda) = (\lambda-1) + \frac{c}{\lambda-1}(0 < \lambda < 1)$ attains its largest value $h(1-\sqrt{c}) = -2\sqrt{c}$ at $\lambda = 1-\sqrt{c}$. Hence letting $\lambda = 1-\sqrt{c}$ in (30), we obtain

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \delta_i(X_\sigma)\{\delta_j(X_\tau) - Q_{k+1}(j|i)\} \geq -2\sqrt{c}.$$

$$\mu - a.s. \qquad \omega \in D(c) \tag{31}$$

In the case $c = 0$, (31) also follows from (30) by choosing $\lambda_i \to 1-\, (i \to \infty)$.

It follows from (31) and (29) that for any $f(i,j)$,

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \delta_i(X_\sigma)f(i,j)\{\delta_j(X_\tau) - Q_{k+1}(j|i)\} \leq (2\sqrt{c}+c)|f(i,j)|,$$

$$\mu - a.s. \qquad \omega \in D(c) \tag{32}$$

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \delta_i(X_\sigma)f(i,j)\{\delta_j(X_\tau) - Q_{k+1}(j|i)\} \geq -2\sqrt{c}|f(i,j)|.$$

$$\mu - a.s. \qquad \omega \in D(c) \tag{33}$$

Noticing that

$$\begin{aligned} & f(X_\sigma, X_\tau) - E_Q[f(X_\sigma, X_\tau)|X_\sigma] \\ = & \sum_{i\in S}\sum_{j\in S} \delta_i(X_\sigma)\delta_j(X_\tau)f(i,j) - \sum_{j\in S} f(X_\sigma, j)Q_{k+1}(j|X_\sigma) \\ = & \sum_{i\in S}\sum_{j\in S} \delta_i(X_\sigma)\delta_j(X_\tau)f(i,j) - \sum_{i\in S}\sum_{j\in S} \delta_i(X_\sigma)f(i,j)Q_{k+1}(j|i) \end{aligned}$$

$$\sum_{i\in S}\sum_{j\in S}\delta_i(X_\sigma)f(i,j)\{\delta_j(X_\tau)-Q_{k+1}(j|i)\}. \qquad \mu-a.s. \qquad \omega\in D(c). \tag{34}$$

By virtue of the properties of superior limit and inferior limit, combing (32)-(34), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{f(X_\sigma,X_\tau)-E_Q[f(X_\sigma,X_\tau)|X_\sigma]\}\\ \leq\ &\sum_{i\in S}\sum_{j\in S}\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)f(i,j)\{\delta_j(X_\tau)-Q_{k+1}(j|i)\}\\ &\leq(2\sqrt{c}+c)\sum_{i\in S}\sum_{j\in S}|f(i,j)|.\end{aligned} \tag{35}$$

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{f(X_\sigma,X_\tau)-E_Q[f(X_\sigma,X_\tau)|X_\sigma]\}\\ =\ &\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\sum_{i\in S}\sum_{j\in S}\delta_i(X_\sigma)f(i,j)\{\delta_j(X_\tau)-Q_{k+1}(j|i)\}\\ \geq\ &\sum_{i\in S}\sum_{j\in S}\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)f(i,j)\{\delta_j(X_\tau)-Q_{k+1}(j|i)\}\\ &\geq-2\sqrt{c}\sum_{i\in S}\sum_{j\in S}|f(i,j)|.\end{aligned} \tag{36}$$

(14), (15) follow from (35) and (36), respectively. The proof is finished.

Corollary 1. Let $X=\{X_\sigma,\sigma\in T\}$ be the Markov chains field determined by the measure μ_Q on the homogeneous tree with the initial distribution (6) and joint distribution (7). $f(x,y)$ is defined as above. Then

$$\lim_{n\to\infty}\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{f(X_\sigma,X_\tau)-E_Q[f(X_\sigma,X_\tau)|X_\sigma]\}=0. \qquad \mu_Q-a.s. \tag{37}$$

Proof. In this case, $\mu\equiv\mu_Q$. It is obvious that $\varphi(\mu|\mu_Q)\equiv 0$. Hence letting $c=0$ in Theorem 1, we obtain $D(0)=\Omega$, (37) follows from (14) and (15) immediately.

Corollary 2. Let $X=\{X_\sigma,\sigma\in T\}$ be an arbitrary random field on the homogeneous tree T. $\varphi(\mu|\mu_Q)$ is defined as (11). Denote $c>0$. Let $i,j\in S$, $S_n(i,j)$ be the number of couple (i,j) in the couples of random variables

$$(X_\sigma,X_\tau),\ 0\leq k\leq n-1,\ \sigma\in L_k,\ \tau\in S(\sigma),\ n\geq 1.$$

That is

$$S_n(i,j)=\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)\delta_j(X_\tau).$$

Then

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\{S_n(i,j)-\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}N\delta_i(X_\sigma)Q_{k+1}(j|i)\}\leq 2\sqrt{c}+c, \qquad \mu-a.s. \quad \omega\in D(c) \tag{38}$$

$$\liminf_{n\to\infty}\frac{1}{|T^{(n)}|}\{S_n(i,j)-\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}N\delta_i(X_\sigma)Q_{k+1}(j|i)\}\geq -2\sqrt{c}. \qquad \mu-a.s. \quad \omega\in D(c) \tag{39}$$

Proof. Letting $f(x,y)=\delta_i(x)\delta_j(y)$ in Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{f(X_\sigma,X_\tau)-E_Q[f(X_\sigma,X_\tau)|X_\sigma]\}\\
=\;&\frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{\delta_i(X_\sigma)\delta_j(X_\tau)-E_Q[\delta_i(X_\sigma)\delta_j(X_\tau)|X_\sigma]\}\\
=\;&\frac{1}{|T^{(n)}|}\{S_n(i,j)-\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_i(X_\sigma)Q_{k+1}(j|X_\sigma)\}\\
=\;&\frac{1}{|T^{(n)}|}\{S_n(i,j)-\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}N\delta_i(X_\sigma)Q_{k+1}(j|i)\}
\end{aligned} \tag{40}$$

and

$$\sum_{u\in S}\sum_{v\in S}|f(u,v)|=\sum_{u\in S}\sum_{v\in S}\delta_i(u)\delta_j(v)=\sum_{u\in S}\delta_i(u)=1. \tag{41}$$

Therefore, (38) and (39) follows from (14), (15), (40) and (41).

Corollary 3. Let $X=\{X_\sigma,\sigma\in T\}$ be an arbitrary random field on the homogeneous tree T. $\varphi(\mu|\mu_Q)$ is defined as (11). Denote $c>0$. Let $S_n(j)$ be the number of j in the set of random variables $X^{T^{(n)}}=\{X_\sigma,\sigma\in T^{(n)}\}$. That is

$$S_n(j)=\sum_{k=0}^{n}\sum_{\sigma\in L_k}\delta_j(X_\sigma)=\delta_j(X_0)+\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\delta_j(X_\tau).$$

Then

$$\limsup_{n\to\infty}\frac{1}{|T^{(n)}|}\{S_n(j)-\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}NQ_{k+1}(j|X_\sigma)\}\leq(2\sqrt{c}+c)M, \qquad \mu-a.s. \quad \omega\in D(c) \tag{42}$$

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|}\{S_n(j) - \sum_{k=0}^{n-1}\sum_{\sigma\in L_k} NQ_{k+1}(j|X_\sigma)\} \geq -2\sqrt{c}M. \quad \mu - a.s. \quad \omega \in D(c) \tag{43}$$

Proof. Letting $f(x,y) = \delta_j(y)$ in Theorem 1, we have by the definition of $S_n(j)$ that

$$\begin{aligned}
& \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{f(X_\sigma, X_\tau) - E_Q[f(X_\sigma, X_\tau)|X_\sigma]\} \\
= \quad & \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\{\delta_j(X_\tau) - E_Q[\delta_j(X_\tau)|X_\sigma]\} \\
= \quad & \frac{1}{|T^{(n)}|}\{S_n(j) - \delta_j(X_0) - \sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\sum_{x_\tau\in S}\delta_j(x_\tau)Q_{k+1}(x_\tau|X_\sigma)\}
\end{aligned}$$

$$= \frac{1}{|T^{(n)}|}\{S_n(j) - \delta_j(X_0) - \sum_{k=0}^{n-1}\sum_{\sigma\in L_k} NQ_{k+1}(j|X_\sigma)\} \tag{44}$$

and

$$\sum_{u\in S}\sum_{v\in S}|f(u,v)| = \sum_{u\in S}\sum_{v\in S}\delta_j(v) = M. \tag{45}$$

Therefore, (42) and (43) follows from (14), (15), (44) and (45).

3. Strong deviation theorem for harmonic mean of the transitional probability of the homogeneous Markov chain field on a homogeneous tree

In the definition 2, if for all $n \geq 1$,

$$Q_n = Q = (Q(y|x)), \quad \forall x, y \in S.$$

$X = \{X_\sigma, \sigma \in T\}$ will be also called S-valued homogeneous Markov chain indexed by a homogeneous tree. At the moment, we have

$$\mu_Q(x_0) = q(x_0)$$

$$\mu_Q(x^{T^{(n)}}) = q(x_0)\prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)} Q(x_\tau|x_\sigma), \quad n \geq 1. \tag{46}$$

We present a small deviation theorem for harmonic mean of the transitional probability of Markov chain indexed by a homogeneous tree as follows:

Theorem 2. Let $X = \{X_\sigma, \sigma \in T\}$ be an arbitrary random field on the homogeneous tree. $D(c)$ is defined as (13). $c > 0$. Then

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} Q(X_\tau|X_\sigma)^{-1} \le M + (2\sqrt{c}+c) \sum_{i\in S}\sum_{j\in S} Q(j|i)^{-1},$$

$$\mu - a.s. \quad \omega \in D(c) \tag{47}$$

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} Q(X_\tau|X_\sigma)^{-1} \ge M - 2\sqrt{c} \sum_{i\in S}\sum_{j\in S} Q(j|i)^{-1}.$$

$$\mu - a.s. \quad \omega \in D(c) \tag{48}$$

Proof. Letting $f(x,y) = Q(y|x)^{-1}$ in Theorem 1, by (14) we can write

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \{f(X_\sigma, X_\tau) - E_Q[f(X_\sigma, X_\tau)|X_\sigma]\} \\
= &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \{Q(X_\tau|X_\sigma)^{-1} - E_Q(Q(X_\tau|X_\sigma)^{-1}|X_\sigma)\} \\
= &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \{Q(X_\tau|X_\sigma)^{-1} - \sum_{x_\tau\in S} Q(x_\tau|X_\sigma)^{-1} Q(x_\tau|X_\sigma)\} \\
= &\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} Q(X_\tau|X_\sigma)^{-1} - M
\end{aligned}$$

$$\le (2\sqrt{c}+c) \sum_{i\in S}\sum_{j\in S} Q(j|i)^{-1}. \qquad \mu - a.s. \qquad \omega \in D(c). \tag{49}$$

Analogously, from (15) we can get

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} Q(X_\tau|X_\sigma)^{-1} - M \ge -2\sqrt{c} \sum_{i\in S}\sum_{j\in S} Q(j|i)^{-1}. \tag{50}$$

(47), (48) follow from (49) and (50), respectively.

4. Conclusion

In this paper, we mainly investigate a kind of small deviation theorems for the arbitrary bivariate function indexed by a homogeneous tree by introducing the sample relative entropy rate. $\varphi(\mu|\mu_Q)$ is regarded as a measure of "dissimilarity" between their joint distribution μ and Markov distribution μ_Q. When the difference between the joint distribution μ and Markov distribution μ_Q is controlled in a certain range, the difference between the functions $f(X_\sigma, X_\tau)$ and the conditional expectation of $f(X_\sigma, X_\tau)$ under the Markov measure μ_Q can also be controlled in a certain range determined by the bound of $\varphi(\mu|\mu_Q)$. The smaller the bound c of $\varphi(\mu|\mu_Q)$ is, the smaller the deviation of $f(X_\sigma, X_\tau)$ relative to $E_Q[f(X_\sigma, X_\tau)|X_\sigma]$ is. As corollaries, a class of small deviation theorems for the frequencies of states ordered couples and harmonic mean of transitional probability of Markov chains field on the homogeneous tree are obtained.

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