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Legendre Collocation Polynomials Method to Solve Linear Fuzzy Volterra Integral Equations

M. Arab Ameri *, M. Shahrezaee

Abstract. In this paper, we apply the method of Legendre collocation polynomials to approximate the solution of linear fuzzy Volterra integral equation (FVIE). First we present the fuzzy set and properties of Legendre polynomials then verify this properties that apply to reduce the FVIE to two crisp linear Volterra integral equations (VIEs). Also, the convergency of the proposed method is approved. Finally several examples are solved by the mentioned method.

Key Words and Phrases: Fuzzy function, Fuzzy Volterra integral equations, Legendre polynomials.

2010 Mathematics Subject Classifications: 65R20

1. Introduction

The topic of fuzzy integral equations is related to two definitions, fuzzy function and the integral of fuzzy function. For the first time, Chang and Zade introduced the fuzzy mapping function [4]. Dubois and Prade [6] defined the topic of integration of fuzzy function. While Goetchel and Voxman [9] preferred the integral in Riemann type concept, the Lebesgue integral type approach was introduced by Kaleva [10].

The concept of the fuzzy integral equation was first presented by Congxin and Ming [5]. Then, this subject was continued by Mordeson and Newman [13]. Also this topic was followed by Wu and Mu in [11].

There are several research papers about solving FVIE [3, 8, 15, 16, 17]. Most of them have converted a FVIE to a linear crisp system of integral equations using the fuzzy number in parametric form, then have applied a method to obtain the solutions of the resulted system. Also, the subject of the existence and uniqueness of solution to FVIE is given in [15].

In this paper, we approximate the solution of FVIE using Legendre collocation polynomial method. We also use the fuzzy number in parametric form and transform a linear FVIE to a crisp system of two VIEs, then solve the obtained system using Legendre polynomial.

*Corresponding author.

The frame of this paper is as follows: Some prerequistes about fuzzy set theory is reviewed in section 2. In Section 3, we present some elementary properties of Legendre polynomials and drive our method to solve FVIEs. Also in section 3 the convergency of the mentioned method is approved. Section 4, includes numerical solution of three instances by present method for proving the efficiency of method. Eventually the conclusion highlights are given in section 5.

2. Preliminaries

In this section, we gather some well-known definitions from [9, 12, 7] which we will use throughout the paper.

Definition 1. *The parametric form of fuzzy number is represented by an ordered pair of functions* $(\underline{x}(\alpha), \overline{x}(\alpha))$, $0 \leqslant \alpha \leqslant 1$ *which satisfy the following requierments:*

1. $\underline{x}(\alpha)$ *is a bounded left-continuous non-decreasing function over* $[0,1]$ *;*

2. $\overline{x}(\alpha)$ *is a bounded left-continuous non-increasing function over* $[0,1]$ *;*

3. $\underline{x}(\alpha) \leqslant \overline{x}(\alpha)$ $0 \leqslant \alpha \leqslant 1$.

For the crisp number, a, we have $\underline{x}(\alpha) = \overline{x}(\alpha) = a, 0 \leqslant \alpha \leqslant 1$.

For arbitrary $x = (\underline{x}(\alpha), \overline{x}(\alpha))$, $y = (\underline{y}(\alpha), \overline{y}(\alpha))$ *and* $k \in \mathbb{R}$ *we define addition and multiplication by* k *as:*

$$(\underline{x+y})(\alpha) = \underline{x}(\alpha) + \underline{y}(\alpha),$$

$$(\overline{x+y})(\alpha) = \overline{x}(\alpha) + \overline{y}(\alpha),$$

$$\underline{kx}(\alpha) = k\underline{x}(\alpha) \ , \ \overline{kx}(\alpha) = k\overline{x}(\alpha) \qquad if \qquad k \geqslant 0,$$

$$\overline{kx}(\alpha) = k\overline{x}(\alpha) \ , \ \underline{kx}(\alpha) = k\underline{x}(\alpha) \qquad if \qquad k < 0.$$

We represent the set of all fuzzy numbers by $\mathbb{E}^1$.

For fuzzy number $x = (\underline{x}(\alpha), \overline{x}(\alpha))$, $0 \leqslant \alpha \leqslant 1$, let

$$x^c(\alpha) = \frac{\underline{x}(\alpha) + \overline{x}(\alpha)}{2},$$

$$x^d(\alpha) = \frac{\overline{x}(\alpha) - \underline{x}(\alpha)}{2}.$$

Clearly, $x^d(\alpha) \geqslant 0$, $\underline{x}(\alpha) = x^c(\alpha) - x^d(\alpha)$ and $\overline{x}(\alpha) = x^c(\alpha) + x^d(\alpha)$, also a fuzzy number $x \in \mathbb{E}^1$ is said symmetric if $x^c(\alpha)$ is independent of α for all $0 \leqslant \alpha \leqslant 1$.

For $x=(\underline{x}(\alpha),\overline{x}(\alpha)), y=(\underline{y}(\alpha),\overline{y}(\alpha))$ and arbitrary real numbers k, s, let $z=kx+sy$, then

$$z^c(\alpha)=kx^c(\alpha)+sy^c(\alpha),$$

$$z^d(\alpha)=|k|\,x^d(\alpha)+|s|\,y^d(\alpha).$$

Definition 2. *The distance between fuzzy numbers $x=(\underline{x}(\alpha),\overline{x}(\alpha)), y=(\underline{y}(\alpha),\overline{y}(\alpha))$ in Hausdorff metric is defined by*

$$D(x,y)=max\{\sup_{0\leqslant\alpha\leqslant1}|\underline{x}(\alpha)-\underline{y}(\alpha)|,\ \sup_{0\leqslant\alpha\leqslant1}|\overline{x}(\alpha)-\overline{y}(\alpha)|\}.$$

Definition 3. *A fuzzy function $x:\mathbb{R}\to\mathbb{E}^1$ is said to be continuous for arbitrary fixed $t_0\in[a,b]$, if for every one $\varepsilon>0$ there exists $\delta>0$ such that if $|t-t_0|<\delta$ then $D\left(x\left(t\right),x\left(t_0\right)\right)<\varepsilon$.*

Definition 4. *Let $x:[a,b]\to\mathbb{E}^1$, for each partition $\Delta=\{s_0,s_1,...,s_n\}$ of $[a,b]$ and for arbitrary $\xi_p: s_{p-1}<\xi_p<s_p$, $1\leqslant p\leqslant n$, define*

$$R_\Delta=\sum_{p=0}^{n}x(\xi_p)(s_p-s_{p-1})$$

and

$$I=\lim_{h\to0}R_\Delta;\qquad h=\max_{1\leqslant p\leqslant n}|s_p-s_{p-1}|$$

if I exsits in the metric D, then it is said definite integral of $x(s)$, i.e. $I=\int_a^b x(s)\,ds$.

If $x:[a,b]\to\mathbb{E}^1$ be continuous in the metric D, then its definite integral over $[a,b]$ exsits . Furthermore,

$$\underline{\left(\int_a^b x(s,\alpha)ds\right)}=\int_a^b\underline{x}(s,\alpha)ds,$$
$$\overline{\left(\int_a^b x(s,\alpha)ds\right)}=\int_a^b\overline{x}(s,\alpha)ds.$$

Similarly

$$\underline{\left(\int_0^{w(t)} x(s,\alpha)\,ds\right)}=\int_0^{w(t)}\underline{x(s,\alpha)}\,ds,$$
$$\overline{\left(\int_0^{w(t)} x(s,\alpha)\,ds\right)}=\int_0^{w(t)}\overline{x(s,\alpha)}\,ds.$$

Lemma 1. *[1]. If $x,y:[a,b]\subseteq\mathbb{R}\to\mathbb{E}^1$ are fuzzy continuous functions, then the function $X:[a,b]\to\mathbb{R}_+$ by $X(s)=D\left(x(s),y(s)\right)$ is continuous on $[a,b]$ and*

$$D\left(\int_a^b x(s)ds,\int_a^b y(s)ds\right)\leqslant\int_a^b D\left(x\left(s\right),y\left(s\right)\right)ds.$$

3. Numerical method based on Legendre collocation polynomials

In this section, at first we express the properties of Legendre polynomials, then approximate the solution of FVIE using Legendre polynomials.

3.1. The properties of Legendre polynomials

The Legendre polynomials are defined on the interval $[-1, 1]$ using of the following recurence formula:

$$L_0(t) = 1,$$

$$L_1(t) = t,$$

$$L_{j+1}(t) = \frac{2j+1}{j+1} t L_j(t) - \frac{j}{j+1}(t), \qquad j = 1, 2, \cdots, \qquad -1 \leqslant t \leqslant 1.$$

for using these polynomials on the interval $[0, l]$ we define the so-called shifted Legendre polynomials of degree p as follow:

$$\psi_p = L_p(\frac{2}{l}t - 1), \qquad p = 0, 1, 2, \cdots.$$

The space $L^2[0, l]$ is presented with the following inner product and norm

$$\langle f, g\rangle = \int_0^l f(t)g(t)dt, \qquad \| f\|_2 = \sqrt{\langle f, f\rangle}.$$

The set of shifted Legendre polynomials forms a complete orthogonal space with L^2-norm and orthogonality is guaranteed by

$$\int_0^l \psi_p(t)\,\psi_q(t)dt = \begin{cases} \dfrac{l}{2p+1}, & p = q, \\ 0, p \neq j. \end{cases}$$

Any function $x(t) \in L^2[0, l]$ can be expressed in terms of shifted Legendre polynomials as

$$x(t) = \sum_{p=0}^{\infty} a_p \psi_p(t), \tag{1}$$

where the coefficients a_p are given by

$$a_p = \frac{2p+1}{l} \int_0^l x(t)\psi_p(t)dt, \qquad p = 0, 1, 2, \cdots.$$

If we consider only the first $(n+1)$-terms of (1), then we obtain

$$x(t) \simeq x_n(t) = \sum_{p=0}^{n} a_p \psi_p(t).$$

3.2. Legendre collocation polynomial method

Our objective in this section is to describe the Legendre collocation polynomial method for the numerical solution of FVIE. We explain our method for the following FVIE

$$B(t)\, x(t,\alpha) = f(t,\alpha) + \mu \int_0^{w(t)} k(t,s)\, x(s,\alpha)\, ds, \tag{2}$$

the functions $B(t)$, $w(t)$, $f(t,\alpha)$ and $k(t,s)$ are known for $0 \leqslant t,\ s \leqslant l$ and $0 \leqslant \alpha \leqslant 1$. $k(t,s)$ is kernel of integral, $f(t,\alpha)$ is a fuzzy function and $x(t,\alpha)$ is the unknown function.

Let $\big(\underline{f}(t), \overline{f}(t)\big)$ and $(\underline{x}(t), \overline{x}(t))$, $0 \leqslant \alpha \leqslant 1$ are parametric form of $f(t)$ and $x(t)$ respectively, then we obtain parametric form of (2) as the following system:

$$\begin{cases} B(t)\, \underline{x}(t,\alpha) = \underline{f}(t,\alpha) + \mu \int_0^{w(t)} \underline{k(t,s)\, x(s,\alpha)}\, ds, \\ \\ B(t)\, \overline{x}(t,\alpha) = \overline{f}(t,\alpha) + \mu \int_0^{w(t)} \overline{k(t,s)\, x(s,\alpha)}\, ds. \end{cases} \tag{3}$$

Suppose $k(t,s)$ be continuous and for fix s, $k(t,s)$ changes its sign in finite points as t_m where $t_m \in [0, w(t)]$. Without loss of generality, let $k(t,s)$ be nonnegative over $[0, t_1]$ and negative over $[t_1, w(t)]$, then the system of (3) yeilds

$$\begin{cases} B(t)\, \underline{x}(t,\alpha) = \underline{f}(t,\alpha) + \mu \int_0^{t_1} k(t,s)\underline{x(s,\alpha)}\, ds + \mu \int_{t_1}^{w(t)} k(t,s)\overline{x(s,\alpha)}\, ds, \\ \\ B(t)\, \overline{x}(t,\alpha) = \overline{f}(t,\alpha) + \mu \int_0^{t_1} k(t,s)\overline{x(s,\alpha)}\, ds + \mu \int_{t_1}^{w(t)} k(t,s)\underline{x(s,\alpha)}\, ds. \end{cases}$$

Refferring to section (2), we have the following equations:

$$\begin{cases} B(t)\, x^c(t,\alpha) = f^c(t,\alpha) + \mu \int_0^{t_1} k(t,s)\, x^c(s,\alpha)\, ds + \mu \int_{t_1}^{w(t)} |\, k(t,s)|\, x^d(s,\alpha)\, ds, \\ \\ B(t)\, x^d(t,\alpha) = f^d(t,\alpha) + \mu \int_0^{t_1} |\, k(t,s)|\, x^d(s,\alpha)\, ds + \mu \int_{t_1}^{w(t)} k(t,r)\, x^c(s,\alpha)\, ds. \end{cases} \tag{4}$$

Clearly, we must solve two VIEs in crisp form. According to section (3.1), the functions $x^c(t,\alpha)$ and $x^d(t,\alpha)$ can be approximated as follows:

$$x^c(t,\alpha) \simeq x_n^c(t,\alpha) = \sum_{p=0}^{n} c_p \psi_p(t), \tag{5}$$

$$x^d(t,\alpha) \simeq x_n^d(t,\alpha) = \sum_{p=0}^{n} d_p \psi_p(t). \tag{6}$$

The coefficients c_p and $d_p, p = 0, 1, \cdots, n$ are unknown. $x(t,\alpha)$ is approximated by $x_n(t,\alpha) = (\underline{x}_n(t,\alpha), \overline{x}_n(t,\alpha))$ where $\underline{x}_n(t,\alpha) = x_n^c(t,\alpha) - x_n^d(t,\alpha)$ and $\overline{x}_n(t,\alpha) = x_n^c(t,\alpha) + x_n^d(t,\alpha)$. Substituting equations (5) and (6) into system (4), we have:

$$\begin{cases} B(t)\sum\limits_{p=0}^{n} c_p\psi_p(t) = & f^c(t,\alpha) + \mu\int_0^{t_1} k(t,s)\left(\sum\limits_{p=0}^{n} c_p\psi_p(s)\right) ds \\ & +\quad \mu\int_{t_1}^{w(t)} |\, k(t,s)\,| \left(\sum\limits_{p=0}^{n} d_p\psi_p(s)\right) ds, \\ B(t)\sum\limits_{p=0}^{n} d_p\psi_p(t) = & f^d(t,\alpha) + \mu\int_0^{t_1} |\, k(t,s)\,| \left(\sum\limits_{p=0}^{n} d_p\psi_p(s)\right) ds \\ & +\quad \mu\int_{t_1}^{w(t)} k(t,s)\left(\sum\limits_{p=0}^{n} c_p\psi_p(s)\right) ds. \end{cases} \tag{7}$$

Let

$$u_p(t) = B(t)\psi_p(t) - \mu\int_0^{t_1} k(t,s)\,\psi_p(s)ds,$$

$$v_p(t) = -\mu\int_{t_1}^{w(t)} |\, k(t,s)\,|\psi_p(s)ds,$$

then system (7) can be rewritten as:

$$\begin{cases} \sum\limits_{p=0}^{n} c_p u_p(t) + \sum\limits_{p=0}^{n} d_p v_p(t) = f^c(t,\alpha), \\ \sum\limits_{p=0}^{n} d_p u_p(t) + \sum\limits_{p=0}^{n} c_p v_p(t) = f^d(t,\alpha). \end{cases} \tag{8}$$

The $n+1$ roots of the shifted Legendre polynomial $\psi_{n+1}(t)$ can be considerd as collocation nodes $t_q\,(q = 0, 1, \cdots, n)$ in equations of system (8).

Thus we obtain the following system of equations:

$$\begin{cases} \sum\limits_{p=0}^{n} c_p u_p(t_q) + \sum\limits_{p=0}^{n} d_p v_p(t_q) = f^c(t_q,\alpha), & q = 0, 1, \cdots, n, \\ \sum\limits_{p=0}^{n} d_p u_p(t_q) + \sum\limits_{p=0}^{n} c_p v_p(t_q) = f^d(t_q,\alpha), & q = 0, 1, \cdots, n. \end{cases} \tag{9}$$

The approximate solution of $x^c(t,\alpha)$ and $x^d(t,\alpha)$ are obtained after computing the values c_p and d_p from (9).

3.3. Convergence property of Legendre collocation method

We will show in this section that, under mild conditions on the kernel function k, the approximate solution obtained by Legendre collocation method do convergence toward exact solution of FVIE as the degree of Legendre polynomial approaches ∞.

Theorem 1. *For arbitrary $t_p \in [a,b]$, $p = 0,1,\cdots,n$, let $x(t_p,\alpha)$ and $x_n(t_p,\alpha)$ are the exact and approximate solutions of equation* (2) *for $t = t_p$, respectively.*
In integral equation (2)*, if $k(t,s)$, $0 \leqslant t,\ s \leqslant l$, be continuous and is bounded by M, then $x_n(t_p,\alpha) \to x(t_p,\alpha)$ for all $p = 0,1,\cdots,n$, as $n \to \infty$.*

Proof. The convergence in $\mathbb{E}^1$ space with Hausdorff metric is defined by

$$\lim_{n\to\infty} x_n(t_p,\alpha) = x(t_p,\alpha) \Longleftrightarrow \lim_{n\to\infty} D(x(t_p,\alpha),\, x_n(t_p,\alpha)) = 0.$$

$$D\left(x\left(t_p,\alpha\right), x_n\left(t_p,\alpha\right)\right) =$$

$$= D\left(\int_0^{w(t_p)} k(t,s)\,x(s,\alpha)\,ds,\ \int_0^{w(t_p)} k(t,s)\left(\sum_{p=0}^{n} a_p\psi_p(s)\right) ds\right)$$

$$\leq M\,D\left(\int_0^{w(t_p)} x(s,\alpha)\,ds,\ \int_0^{w(t_p)}\left(\sum_{p=0}^{n} a_p\psi_p(s)\right) ds\right),$$

where M is upperbound of $k(t,s)$. Using lemma (1) we have

$$D\left(x\left(t_p,\alpha\right), x_n\left(t_p,\alpha\right)\right) \leqslant M\int_0^{w(t_p)} D\left(x(s,\alpha), \sum_{p=0}^{n} a_p\psi_p(s)\right) ds,$$

So

$$\lim_{n\to\infty} D\left(x\left(t_p,\alpha\right), x_n\left(t_p,\alpha\right)\right) \leqslant M \lim_{n\to\infty}\left(\int_0^{w(t_p)} D\left(x(s,\alpha), \sum_{p=0}^{n} a_p\psi_p(s)\right) ds\right),$$

or

$$\lim_{n\to\infty} D\left(x\left(t_p,\alpha\right), x_n\left(t_p,\alpha\right)\right) \leqslant M\int_0^{w(t_p)} D\left(x(s,\alpha), \lim_{n\to\infty}\sum_{p=0}^{n} a_p\psi_p(s)\right) ds.$$

We know

$$x(s,\alpha) = \lim_{n\to\infty}\sum_{p=0}^{n} a_p\psi_p(s)$$

then

$$\lim_{n\to\infty} D\left(x(t,\alpha), \sum_{p=0}^{n} a_p\psi_p(t)\right) \longrightarrow 0,$$

Finally, we conclude that

$$\lim_{n\to\infty} D\left(x\left(t_p,\alpha\right), x_n\left(t_p,\alpha\right)\right) \longrightarrow 0,$$

so the proof is completed.

4. Numerical examples

To show the efficiency of the proposed numerical method we use it for three FVIEs, we compute the error of the proposed method by

$$\gamma^c(t,\alpha) = \| e^c(t,\alpha)\|_2, \qquad \gamma^d(t,\alpha) = \| e^d(t,\alpha)\|_2, \tag{10}$$

where

$$e^c(t,\alpha) = |\, x^c(t,\alpha) - x_n^c(t,\alpha)|, \qquad e^d(t,\alpha) = |\, x^d(t,\alpha) - x_n^d(t,\alpha)|,$$

where $x^c(t,\alpha)$ and $x^d(t,\alpha)$ are the exact solutions of the system (4), $x_n^c(t,\alpha)$ and $x_n^d(t,\alpha)$ are the approximated solutions.

Example 1. *Consider the fuzzy Volterra integral equation* (2) *and let*

$\underline{f}(t,\alpha) = (\cosh(t) + 1 - \cosh^2(t))(\alpha^2 + \alpha),$

$\overline{f}(t,\alpha) = (\cosh(t) + 1 - \cosh^2(t))(4 - \alpha^3 - \alpha), \qquad 0 \leqslant \alpha \leqslant 1,$

$w(t) = t$ *and kernel* $k(t,s) = \sinh(t)$, $0 \leqslant t \leqslant 1$, $\mu = 1$.

The exact solution is

$$\underline{x}(t,\alpha) = (\cosh(t))(\alpha^2 + \alpha),$$
$$\overline{x}(t,\alpha) = (\cosh(t))(4 - \alpha^3 - \alpha), \qquad 0 \leqslant \alpha \leqslant 1.$$

So the exact solution of $x^c(t,\alpha)$ *and* $x^d(t,\alpha)$ *are*

$$x^c(t,\alpha) = \frac{1}{2}(\cosh(t))(4 - \alpha^3 + \alpha^2),$$
$$x^d(t,\alpha) = \frac{1}{2}(\cosh(t))(4 - \alpha^3 - \alpha^2 - 2\alpha), \qquad 0 \leqslant \alpha \leqslant 1.$$

It is clear that

$$f^c(t,\alpha) = \frac{(\cosh(t) + 1 - \cosh^2(t))(4 + \alpha^2 - \alpha^3)}{2},$$

$$f^d(t,\alpha) = \frac{(\cosh(t) + 1 - \cosh^2(t))(4 - \alpha^3 - \alpha^2 - 2\alpha)}{2}, \qquad 0 \leqslant \alpha \leqslant 1.$$

So we have the following system of crisp VIEs

$$\begin{cases} B(t)\, x^c(t,\alpha) = f^c(t,\alpha) + \mu \int_0^t \sinh(t)\, x^c(s,\alpha)\, ds, \\ \\ B(t)\, x^d(t,\alpha) = f^d(t,\alpha) + \mu \int_0^t |\sinh(t)|\, x^d(s,\alpha)\, ds, \end{cases}$$
$$0 \leqslant t \leqslant 1,\ 0 \leqslant \alpha \leqslant 1.$$

The functions $x^c(t,\alpha)$ and $x^d(t,\alpha)$ can be approximated by proposed method. We obtain the error of the computed solution by introduced error formula. Tables 1, 2 show the convergence behavior for $n = 2, 3, \cdots, 7$ and $\alpha = 0, 0.1, 0.5, 0.9$.

Table 1: The error of $x^c(t,0)$, $x^c(t,0.1)$, $x^c(t,0.5)$, $x^c(t,0.9)$

n	$\gamma^c(t,0)$	$\gamma^c(t,0.1)$	$\gamma^c(t,0.5)$	$\gamma^c(t,0.9)$
2	3.48×10^{-2}	3.49×10^{-2}	3.59×10^{-2}	3.55×10^{-2}
3	4.7×10^{-3}	4.7×10^{-3}	4.9×10^{-3}	4.8×10^{-3}
4	1.1147×10^{-4}	1.1172×10^{-4}	1.1495×10^{-4}	1.1373×10^{-4}
5	1.0145×10^{-5}	1.0167×10^{-5}	1.0462×10^{-5}	1.0350×10^{-5}
6	1.7023×10^{-7}	1.7062×10^{-7}	1.7555×10^{-7}	1.7368×10^{-7}
7	1.165×10^{-8}	1.1677×10^{-8}	1.2015×10^{-8}	1.1887×10^{-8}

Table 2: The error of $x^d(t,0)$, $x^d(t,0.1)$, $x^d(t,0.5)$, $x^d(t,0.9)$

n	$\gamma^d(t,0)$	$\gamma^d(t,0.1)$	$\gamma^d(t,0.5)$	$\gamma^d(t,0.9)$
2	3.48×10^{-2}	3.30×10^{-2}	2.29×10^{-2}	5.8×10^{-3}
3	4.7×10^{-3}	4.5×10^{-3}	3.1×10^{-3}	7.8274×10^{-4}
4	1.1147×10^{-4}	1.0559×10^{-4}	7.3151×10^{-5}	1.8420×10^{-5}
5	1.0145×10^{-5}	9.6094×10^{-6}	6.6574×10^{-6}	1.6764×10^{-6}
6	1.7023×10^{-7}	1.6125×10^{-7}	1.1172×10^{-7}	2.8131×10^{-8}
7	1.1651×10^{-8}	1.1036×10^{-8}	7.6457×10^{-9}	1.9253×10^{-9}

Example 2. *Consider the FVIE (2) with*

$$\underline{f}(t,\alpha) = \alpha(1 - t - \frac{t^2}{2}),$$

$\overline{f}(t,\alpha) = (2-\alpha)(1-t-\frac{t^2}{2}), \quad 0 \leqslant \alpha \leqslant 1,$

$w(t) = t$ *and kernel* $k(t,s) = t-s$, $0 \leqslant t,s \leqslant 1$, $\mu = 1$.

The exact solution of this FVIE is [16]

$$\underline{x}(t,\alpha) = \alpha(1-sinh(t)),$$
$$\overline{x}(t,\alpha) = (2-\alpha)(1-sinh(t)), \quad 0 \leqslant \alpha \leqslant 1.$$

i.e.

$$x^c(t,\alpha) = 1-sinh(t),$$
$$x^d(t,\alpha) = (1-\alpha)(1-sinh(t)), \quad 0 \leqslant \alpha \leqslant 1.$$

Referring to section (2), *we have*

$$f^c(t,\alpha) = 1-t-\frac{t^2}{2},$$
$$f^d(t,\alpha) = (1-\alpha)(1-t-\frac{t^2}{2}), \quad 0 \leqslant \alpha \leqslant 1.$$

By substituting the $f^c(t,\alpha)$ *and* $f^d(t,\alpha)$ *in system* (4) *we obtain a system of crisp VIEs where is solved by the present method.*
Tables 3, 4 show the computed error of $x^c(t,\alpha)$ *and* $x^d(t,\alpha)$ *for different values of* α. *In this example* $x^c(t,\alpha)$ *is independent of* α, *also the results in table 3, confirm independency of* α.

Table 3: The error of $x^c(t,0)$, $x^c(t,0.1)$, $x^c(t,0.5)$, $x^c(t,0.9)$

n	$\gamma^c(t,0)$	$\gamma^c(t,0.1)$	$\gamma^c(t,0.5)$	$\gamma^c(t,0.9)$
2	3.74×10^{-2}	3.74×10^{-2}	3.74×10^{-2}	3.74×10^{-2}
3	1.1×10^{-3}	1.1×10^{-3}	1.1×10^{-3}	1.1×10^{-3}
4	1.2029×10^{-4}	1.2029×10^{-4}	1.2029×10^{-4}	1.2029×10^{-4}
5	2.3639×10^{-6}	2.3639×10^{-6}	2.3639×10^{-6}	2.3639×10^{-6}
6	1.8413×10^{-7}	1.8413×10^{-7}	1.8413×10^{-7}	1.8413×10^{-7}
7	2.7093×10^{-9}	2.7093×10^{-9}	2.7093×10^{-9}	2.7093×10^{-9}

Example 3. *In equation* (2), *let*

$\underline{f}(t,\alpha) = 3+\alpha,$

Table 4: The error of $x^d(t,0)$, $x^d(t,0.1)$, $x^d(t,0.5)$, $x^d(t,0.9)$

n	$\gamma^d(t,0)$	$\gamma^d(t,0.1)$	$\gamma^d(t,0.5)$	$\gamma^d(t,0.9)$
2	3.47×10^{-2}	3.36×10^{-2}	1.87×10^{-2}	3.7×10^{-3}
3	1.1×10^{-3}	9.9836×10^{-4}	5.5464×10^{-4}	1.1093×10^{-4}
4	1.2029×10^{-4}	1.0826×10^{-4}	6.0146×10^{-5}	1.2029×10^{-5}
5	2.3639×10^{-6}	2.1275×10^{-6}	1.1820×10^{-6}	2.3639×10^{-7}
6	1.8413×10^{-7}	1.6571×10^{-7}	9.2063×10^{-8}	1.8413×10^{-8}
7	2.7093×10^{-9}	2.4384×10^{-9}	1.3546×10^{-9}	2.7093×10^{-10}

$\overline{f}(t,\alpha) = 8-2\alpha, \qquad 0 \leqslant \alpha \leqslant 1,$

$w(t)=t$ and kernel $k(t,s)=t-s$, $0\leqslant t,s\leqslant 1$, $\mu=1$.

The exact solution in this case is given by [16]

$$\underline{x}(t,\alpha) = (3+\alpha)cosh(t),$$
$$\overline{x}(t,\alpha) = (8-2\alpha)cosh(t), \qquad 0\leqslant \alpha\leqslant 1$$

that means,

$$x^c(t,\alpha) = \frac{1}{2}(11-\alpha)(cosh(t)),$$
$$x^d(t,\alpha) = \frac{1}{2}(5-3\alpha)(cosh(t)), \qquad 0\leqslant \alpha\leqslant 1.$$

we have

$$f^c(t,\alpha) = \frac{11-\alpha}{2},$$
$$f^d(t,\alpha) = \frac{5-3\alpha}{2}, \qquad 0\leqslant \alpha\leqslant 1.$$

Tables 5, 6 show the computed error by (10) *for $n=2,3,\cdots,7$.*

5. Conclusion

In this paper, we introduced a staright approach to approximate the solution of FVIE based on the Legendre polynomial. We showed the convergency of method, also our achieving results show that Legendre polynomials approximation method for solving FVIE, is very effective and it produces high accurate answers. The numerical examples support this claim.

Table 5: The error of $x^c(t,0)$, $x^c(t,0.1)$, $x^c(t,0.5)$, $x^c(t,0.9)$

n	$\gamma^c(t,0)$	$\gamma^c(t,0.1)$	$\gamma^c(t,0.5)$	$\gamma^c(t,0.9)$
2	9.67×10^{-2}	9.58×10^{-2}	9.23×10^{-2}	8.88×10^{-2}
3	1.30×10^{-2}	1.29×10^{-2}	1.25×10^{-2}	1.20×10^{-2}
4	3.0836×10^{-4}	3.0556×10^{-4}	2.9435×10^{-4}	2.8313×10^{-4}
5	2.7955×10^{-5}	2.7701×10^{-5}	2.6685×10^{-5}	2.5668×10^{-5}
6	4.7033×10^{-7}	4.6606×10^{-7}	4.4895×10^{-7}	4.3185×10^{-7}
7	3.2115×10^{-8}	3.1823×10^{-8}	3.0656×10^{-8}	2.9488×10^{-8}

Table 6: The error of $x^d(t,0)$, $x^d(t,0.1)$, $x^d(t,0.5)$, $x^d(t,0.9)$

n	$\gamma^d(t,0)$	$\gamma^d(t,0.1)$	$\gamma^d(t,0.5)$	$\gamma^d(t,0.9)$
2	1.76×10^{-2}	4.13×10^{-2}	3.08×10^{-2}	2.02×10^{-2}
3	2.4×10^{-3}	5.6×10^{-3}	4.2×10^{-3}	2.7×10^{-3}
4	5.6066×10^{-5}	1.3175×10^{-4}	9.8115×10^{-5}	6.4476×10^{-5}
5	5.0828×10^{-6}	1.1945×10^{-5}	8.8949×10^{-6}	5.8452×10^{-6}
6	8.5515×10^{-8}	2.0096×10^{-7}	1.4965×10^{-7}	9.8342×10^{-8}
7	5.8392×10^{-9}	1.3722×10^{-8}	1.0219×10^{-8}	6.7150×10^{-9}

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M. Arab Ameri
Department of Mathematics, Faculty of mathematics,
University of Sistan and Baluchestan, Iran
Email: *arabameri@math.usb.ac.ir*

M. Shahrezaee
University of Sistan and Baluchestan, Iran

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