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A Boundary Value Problem for an Irrational Order Partial Equation

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Abstract. It is known that as the investigation of the solutions of problems stated for partial equations is difficult than the investigation of problems stated for ordinary equations, the solution of problems stated for fractional order equations is more difficult than the investigation of the solutions of problems stated for irrational order equations. Here we will be engaged in finding the solution of a boundary value problem for a partial equation whose order is an irrational number.

Key Words and Phrases: irrational order partial discrete equation, a boundary value problem for such an equation, finding of the solution in the form of unknown coefficient series corresponding to Mittag-Loffler functions.

2010 Mathematics Subject Classifications: 26A33, 35R11, 35E12

1. Introduction

The problems for natural number order partial equations have been studied enough [1], [3]. Recently, the problems for rational order equations are the most considered ones and the problems stated for partial equations are considered under various boundary conditions [4], [5]. Finally, the problems for ordinary and partial equations whose order of derivative are real positive members (mainly the irrational numbers) are considered [6], [7], [8].

For the sake of completeness of historical analysis we should note that the most interesting is the investigation of mathematical problems stated for ordinary and partial equations with continuous variable order derivative.

2. Problem statement

Let's consider the following boundary value problem

$$D_x^{\sqrt{3}}u\left(x,y\right) = D_y^{\sqrt{2}}u\left(x,y\right), \; x \in (0,1), \; y \in (0,1), \tag{1}$$

$$\alpha_0 u\left(0,t\right) + \alpha_1 u\left(1,t\right) + \beta_0 u\left(t,0\right) + \beta_1 u\left(t,1\right) = \varphi\left(t\right), \; t \in [0,1], \tag{2}$$

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where (1) is a differential irrational order partial differential equation with discretely variable order derivative, (2) is a boundary condition stated for it.

The coefficients of the boundary condition are real numbers, the right side is a real valued continuous function. The problem is considered in a unique square.

Let's turn back to Euler's theory. It is known that the function constructed by Euler is in the form

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{3}$$

Here the derivative of each term (first order) gives the previous term, the derivative of the first term gives a zero.

At the same time for the function

$$h_{\frac{1}{n}}(x) = \frac{x^{\frac{1-n}{n}}}{\left(\frac{1-n}{n}\right)!} + \frac{x^{\frac{2-n}{n}}}{\left(\frac{2-n}{n}\right)!} + \frac{x^{\frac{3-n}{n}}}{\left(\frac{3-n}{n}\right)!} + ... = \sum_{k=1}^{\infty} \frac{x^{\frac{k-n}{n}}}{\left(\frac{k-n}{n}\right)!}, \tag{4}$$

being the analogue of the Mittag-Loffler function, beginning from the second one, as the $\frac{1}{n}$ order derivative of each term is the previous term, and the $\frac{1}{n}$ order derivative of the first term is

$$(-1)! = \infty,$$

it is equal to zero. Note that the factorials taking part in the denominator of (4) are obtained by means of the Euler gamma function.

So, we sought for the solution of equation (1) in the form

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n} \frac{x^{-1+(m-1)\sqrt{3}}}{\left(-1+(m-1)\sqrt{3}\right)!} \frac{y^{-1+n\sqrt{2}}}{\left(-1+n\sqrt{2}\right)!}, \tag{5}$$

here $u_{m,n}$ are unknown coefficients. For determining these unknown constants, we take into account

$$D_x^{\sqrt{3}} u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n} \frac{x^{-1+(m-1)\sqrt{3}}}{\left[-1+(m-1)\sqrt{3}\right]!} \frac{y^{-1+n\sqrt{2}}}{\left(-1+n\sqrt{2}\right)!} =$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} u_{m+1,n} \frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!} \frac{y^{-1+n\sqrt{2}}}{\left(-1+n\sqrt{2}\right)!} =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m+1,n} \frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!} \frac{y^{-1+n\sqrt{2}}}{\left(-1+n\sqrt{2}\right)!}$$

and

$$D_y^{\sqrt{2}} u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n+1} \frac{x^{-1+m\sqrt{2}}}{\left(-1+m\sqrt{2}\right)!} \frac{y^{-1+n\sqrt{2}}}{\left(-1+n\sqrt{2}\right)!},$$

and as the functions

$$\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+n\sqrt{3}}}{\left(-1+n\sqrt{3}\right)!}\;, \quad m,n=\overline{1,\infty},$$

are linear independent, we get the infinitely many equations

$$u_{m+1,n}=u_{m,n+1}, \quad m,n=\overline{1,\infty}. \tag{6}$$

From the obtained system (6) we get the result

$$u_{m,n}=u_{m+n-1,1}, \; m,n=\overline{1,\infty}, \tag{7}$$

i.e. all the coefficients $u_{m,n}$ are expressed by the coefficients $u_{k,1}$. Then the solution of equation (1) accepts the form

$$u\left(x,y\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{m+n-1,1}\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+n\sqrt{2}}}{\left(-1+n\sqrt{2}\right)!}, \tag{8}$$

We can show this expression in the form

$$u\left(x,y\right)=\sum_{k=1}^{\infty}u_{k,1}\sum_{m=1}^{k}\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+(k+1-m)\sqrt{2}}}{\left[-1+\left(k+1-m\right)\sqrt{2}\right]!}. \tag{9}$$

Now give a boundary condition for equation (1). As the obtained expression (9) satisfies equation $U_{k,1}$ regardless the coefficients $x>0, y>0$ if for

$$\begin{cases} u\left(x,0\right)=0, \;\; x>0, \\ u\left(0,y\right)=0, \;\; y>0, \end{cases} \tag{10}$$

we consider this equation in the first quarter, we see that the solution satisfying the boundary conditions is of infinite number.

Indeed, the functions

$$u_k\left(x,y\right)=\sum_{m=1}^{k}\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+(k+1-m)\sqrt{2}}}{\left[-1+\left(k+1-m\right)\sqrt{2}\right]!}, \;\; k=\overline{1,\infty} \tag{11}$$

satisfy the problem (1), (10).

Here, as for $m=\overline{1,k}, \; k=\overline{1,\infty}$, the orders of x and y are always positive, the functions (11) satisfy the conditions (10). As

$$D_x^{\sqrt{3}}u_k\left(x,y\right)=\sum_{m=1}^{k}\frac{x^{-1+(m-1)\sqrt{3}}}{\left[-1+\left(m-1\right)\sqrt{3}\right]!}\frac{y^{-1+(k+1-m)\sqrt{2}}}{\left[-1+\left(k+1-m\right)\sqrt{2}\right]!}=$$

$$=\sum_{m=0}^{k-1}\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+(k-m)\sqrt{2}}}{\left[-1+\left(k-m\right)\sqrt{2}\right]!}=\sum_{m=1}^{k-1}\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+(k-m)\sqrt{2}}}{\left[-1+\left(k-m\right)\sqrt{2}\right]!},$$

$$D_y^{\sqrt{2}} u_k(x,y) = \sum_{m=1}^{k} \frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!} \frac{y^{-1+(k-m)\sqrt{2}}}{\left[-1+(k-m)\sqrt{2}\right]!} =$$

$$= \sum_{m=1}^{k-1} \frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!} \frac{y^{-1+(k-m)\sqrt{2}}}{\left[-1+(k-m)\sqrt{2}\right]!},$$

the functions (11) satisfy the equation (1).

Finally, in a square with a side equal to a unit for equation (1) we consider the following boundary condition

$$\alpha_1 u(1,t) + \beta_1 u(t,1) = \varphi(t), \quad t \in [0,1], \tag{12}$$

where α_1 and β_1 are the given constants, $\varphi(t)$ is a continuous function.

If we write the general solution (9) obtained for the given equation (1) in the boundary condition (12), we get

$$\alpha_1 \sum_{k=1}^{\infty} u_{k,1} \sum_{m=1}^{k} \frac{1}{\left(-1+m\sqrt{3}\right)!} \frac{t^{-1+(k+1-m)\sqrt{2}}}{\left[-1+(k+1-m)\sqrt{2}\right]!} +$$

$$+\beta_1 \sum_{k=1}^{\infty} u_{k,1} \sum_{m=1}^{k} \frac{t^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!} \frac{1}{\left[-1+(k+1-m)\sqrt{2}\right]!} = \varphi(t)$$

or

$$\sum_{k=1}^{\infty} u_{k,1} v_k(t) = \varphi(t), \quad t \in [0,1], \tag{13}$$

where

$$v_k(t) = \sum_{m=1}^{k} \left\{ \frac{\alpha_1}{\left(-1+m\sqrt{3}\right)} \frac{t^{-1+(k+1-m)\sqrt{2}}}{\left[-1+(k+1-m)\sqrt{2}\right]!} + \frac{\beta_1}{\left[-1+(k+1-m)\sqrt{2}\right]!} \frac{t^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!} \right\}, \quad k=\overline{1,\infty};\, t \in [0,1]. \tag{14}$$

It is easily seen that the functions (14) are linear independent.

Let's construct such functions $w_s(t)$ biorthogonal with them so that the relations

$$(v_k, w_s) = \delta_{ks} = \begin{cases} 1, & k = s, \\ 0, & k \neq s, \end{cases} \tag{15}$$

be fulfilled. Then from (13) we get:

$$\sum_{k=1}^{\infty} u_{k,1}(v_k, w_s) = (\varphi, w_s), \quad s = \overline{1,\infty}. \tag{16}$$

Here, according to (15),

$$u_{s,1} = (\varphi, w_s), \quad s = \overline{1,\infty}. \tag{17}$$

Having written them in (9), we get the solution of problem (1), (2) in the form

$$u(x,y)=\sum_{k=1}^{\infty}(\varphi,w_k)\sum_{m=1}^{k}\frac{x^{-1+m\sqrt{3}}}{\left(-1+m\sqrt{3}\right)!}\frac{y^{-1+(k+1-m)\sqrt{2}}}{\left[-1+(k+1-m)\sqrt{2}\right]!}. \tag{18}$$

So, we get the following statement:

Theorem. *If* α_1 *and* β_1 *are the given constants,* $\varphi(t)$ *are continuous, the functions* $w_s(t)$ *are biorthogonal to* $v_k(t)$ *in the form* (14) *and satisfy the condition* (15)*, then the solution of boundary value problem is given in the form* (18)*.*

Remark. *In the same way, the solution of problem* (1), (2) *is given in the form* (18) *because the solution of* (18) *for* $x=0$ *and* $y=0$ *becomes zero.*

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