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Some Differential Properties of Generalized Nikolskii-Morrey Type Spaces

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Abstract. In the paper a generalized Nikolski-Morrey type spaces were introduced and studied. With help a integral representation are obtained Sobolev type inequalities for functions from this spaces.

Key Words and Phrases: Nikolskii-Morrey type spaces, integral representation, embedding theorems, generalized Holder condition.
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1. Introduction

In the paper, we introduce a generalized Nikolski-Morrey type spaces

$$\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi). \tag{1}$$

and help of method inetgral representation we study differential-difference properties of functions from this spaces. Let $G \subset R^n$;$1 \leq p^i < \infty$; $l^i \in (0,\infty)^n$,$i = 0,1,\dots,n$; $l^0_j \geq 0, l^i_j \geq 0 (i \neq j = 1,2,\dots,n), l^i_i \geq 0 (i = 1,2,\dots,n)$; $\beta \in [0,1]^n$; $[t]_1 = \min\{1,t\}$, and let vector-functions $\varphi(t) = (\varphi_1(t),\dots,\varphi_n(t))$, with Lebesgue measurable functions $\varphi_j(t) > 0, (t > 0)$, $\lim\limits_{t\to+0} \varphi_j(t) = 0$, $\lim\limits_{t\to+\infty} \varphi_j(t) = L \leq \infty$, $j = 1,2,\dots,n$. Denote by $\mathbb{A}$ the set of vector functions φ. Let $m^0 = (m^0_1,\dots,m^0_n), m^0_j \in N_0(j = 1,\dots,n)$, $m^i = (m^i_1,\dots,m^i_n), m^i_j \in N_0(i \neq j = 1,\dots,n), m^i_i \in N(i = 1,\dots,n)$ $k^0 = (k^0_1,\dots,k^0_n), k^i_j \in N_0(j = 1,\dots,n, i = 1,\dots,n)$.

Definition 1. *The space type* $\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$ *we denote the spaces of all functions* $f \in L^{loc}(G)$ $(m^i_j > l^i_j - k^i_j \geq 0, i \neq j = 1,\dots,n; m^i_i > l^i_i - k^i_i \geq 0, i = 1,2,...,n)$ *with the finite norm*

$$\|f\|_{\bigcap\limits_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)} = \sum_{i=0}^{n} \sup_{0<h<h_0} \frac{\left\|\Delta^{m_i}\left(\varphi(h), G_{\varphi(h)}\right) D^{k_i} f\right\|_{p^i,\varphi,\beta}}{\prod\limits_{j=1}^{n} \varphi_j(h)^{l^i_j - k^i_j}}, \tag{2}$$

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where

$$\|f\|_{p^i,\varphi,\beta;G} = |f|_{L_{p^i,\varphi,\beta}(G)} = \sup_{x\in G, t>0} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p^i, G_{\varphi(t)}(x)} \right), \tag{3}$$

$\left|\varphi([t]_1)\right|^{-\beta} = \prod_{j=1}^{n} (\varphi_j([t]_1))^{-\beta_j}$, $\Delta^{m_i}\left(\varphi(h), G_{\varphi(h)}\right) f$ =?, h_0 *it is positive fixed number, and let for any* $x \in R^n$

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n \right\},$$

For any $t > 0$,suppose $|\varphi([t]_1)| \leq C$, then the embeddings $\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi) \to \bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)$ and hold, i.e.

$$\|f\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)} \leq c\|f\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)}, \tag{4}$$

Note that the spaces $\bigcap_{i=0}^{n} L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)$ and is Banach space. The space (1) when $l^0 = (0, \ldots, 0)$, $l^i = (0, \ldots, 0, l_i, 0, \ldots, 0)$, $p^i = p (i = 0, 1, \ldots, n)$ coincides with the space $H^l_{p,\varphi,\beta}(G_\varphi)$ introduced and studied in [1], in the case $\beta_j = 0\, (j = 1, ..., n)$ it coincides with generalized Nikolski space $\bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)$.The spaces of such type with different norms introduced and studied [3]-[13].

Lemma 1. *Let* $G \subset R^n$, $1 \leq p^i \leq \infty$, *and* $f \in \bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)$. *Then we can construct the sequence* $h_s = h_s(x)$ $(s = 1, 2, \ldots)$ *of infinitely differentiable finite in* R^n *functions for which*

$$\lim_{s\to\infty} \|f - h_s\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)} = 0. \tag{5}$$

Proof. Let $G = \bigcup_{\lambda=1}^{M} G^\lambda$ and for obtaining equality (5) we estimate the norm

$$\|f - h_s\|_{\bigcap_{i=0}^{n} L^{<l^i>}_{p^i}(G_\varphi)} = \sum_{i=0}^{n} \omega_i^{l_i}(f - h_s). \tag{6}$$

$$\omega_i^{l_i}(f - h_s) = \sup_{0<h<h_0} \frac{\left\|\Delta^{m_i}\left(\varphi(h), G_{\varphi(h)}\right) D^{k_i} f\right\|_{p^i,\varphi,\beta}}{\prod_{j=1}^{n} \varphi_j(h)^{l_i - k_i}} \tag{7}$$

The sequence $h_s(x)$ $(s = 1, 2, \ldots)$ is determined by the equality

$$h_s(x) = F(x, \varphi(t))|_{t=\frac{1}{s}} = \sum_{\lambda} \eta_\lambda(x) f_{\varphi^\lambda(t)}(x),$$

here the averaging functions are determined as follows:

$$f_{\varphi^\lambda(t)}(x) = \int\limits_{R^n} f(x+\varphi^\lambda(t)y)K_\lambda(y)dy,$$

where $K_\lambda(y) \in C_0^\infty(R^n)$ $(\lambda = 1, 2, \ldots, M)$ $\sup pK_\lambda(\cdot) \subset [-1; 1]$

$$\int\limits_{R^n} K_\lambda(y)dy = 1,$$

the functions $\eta_\lambda = \eta_\lambda(x)$ $(\lambda = 1, 2, \ldots, M)$ determine the expansion of a unit in the domain G, i.e.

1)$1 \le \eta_\lambda(x) \le 1$ in R^n;

2) $\eta_\lambda(x) = 0$ in $G \setminus G_\lambda$ for all $\lambda = 1, 2, \ldots, M$;

3) $|D^\alpha \eta_\lambda(x)| \le C_\lambda$, $C_\lambda = const$ for all $\lambda = 1, 2, \ldots, M$ and $\alpha \ge 0$.

Obviously,

$$\|f(\cdot) - h_s(\cdot)\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)} \le \sum_{\lambda}^{M} \|\eta_\lambda(\cdot)(f(\cdot) - f_{\varphi^\lambda(t)}(\cdot))\| \le$$
$$\le C\sum_{\lambda}^{M} \|(f(\cdot) - f_{\varphi^\lambda(t)}(\cdot))\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)}, \qquad (8)$$

As much as small for rather small, t, as a consequence of continuity of L_p- average functions, belonging to the space $L_p(G_\varphi^\lambda)$,from (6),(7) and (8) it follows

$$\|f(\cdot) - h_s(\cdot)\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)} < \varepsilon,$$

in other words,

$$\lim_{s\to\infty} \|f - h_s\|_{\bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)} = 0.$$

Assuming that $\varphi_j(t)$ $(j = 1, 2, ..., n)$ are also differentiable on $[0, T]$, we can show that for $f \in \bigcap\limits_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x \in U \subset G)$

$$D^\nu f(x) = (-1)^{|\nu|+|l^0|} \prod_{j=1}^{n} (\varphi_j(T))^{-\nu_j - 1} \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} K_0^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(T, x))}{\varphi(t)}\right)$$
$$\times \zeta_i \left(\frac{u}{\varphi_i(T)}, \frac{\rho_i(\varphi_i(T, x))}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(T), x)\right) \Delta_i^{m^0}(\varphi_i(\delta) u)$$

$$\times f\left(x+y+u_1+\ldots+u_n\right) dydu+\sum_{i=1}^{n}(-1)^{|\nu|+|l^i|}\int_0^T\int_{R^n}\int_{-\infty}^{+\infty}K_i^{(\nu)}\times$$

$$\times\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi\left(t,x\right)\right)}{\varphi(t)}\right)\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i\left(t,x\right)\right)}{\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi(t),x\right)\right)\Delta_i^{m^i}\left(\varphi_i\left(\delta\right)u\right)$$

$$\times f\left(x+y+u_1+\ldots+u_n\right) dydu\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdudy, \tag{9}$$

Let $\Phi_i\left(\cdot,y\right)\in C_0^\infty\left(R^n\right)$ be such that

$$S\left(\psi_i\right)\subset I_{\varphi(t)}=\left\{y:|y_j|<\frac{1}{2}\varphi_j(t),\quad j=1,2,...,n\right\}.$$

for any $0<T\leq 1$ assume that

$$V=\bigcup_{0<t\leq T}\left\{y:\frac{y}{\varphi(t)}\in S\left(\psi_i\right)\right\}.$$

It is clear that $V\subset I_{\varphi(t)}$ and suppose that $U+V\subset G$.

Lemma 2. *Let $1\leq p^i\leq p\leq r\leq\infty$; $0<\eta,t<T\leq 1$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, $\nu_j\geq 0$ are integers, $j=1,2,...,n$; $\Delta_i^{m^i}\left(h\right)\in L_{p^i,\varphi,\beta}(G)$ and let*

$$F\left(x\right)=\prod_{j=1}^{n}(-1)^{|\nu_j|-1}\int_{R^n}\int_{-\infty}^{+\infty}K_0^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)$$

$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i\left(\varphi_i(t),x\right)}{2\varphi_i(t)},\frac{1}{2}\rho_i'\left(\varphi_i(t),x\right)\right)$$

$$\times\Delta^{m^0}\left(\varphi_i\left(\delta\right)u\right)f\left(x+y+u\right)dxdudy \tag{10}$$

$$F_\eta^i\left(x\right)=\int_0^\eta L_i\left(x,t\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-2}\prod_{j\in m^i}\frac{\varphi_j'(t)}{\varphi_j(t)}dt \tag{11}$$

$$F_{\eta T}^i\left(x\right)=\int_\eta^T L_i\left(x,t\right)\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{\nu_j-2}\prod_{j\in m^i}\frac{\varphi_i'(t)}{\varphi_i(t)}dt \tag{12}$$

$$Q_T^i=\int_0^T\prod_{j=1}^{n}\left(\varphi_j(t)\right)^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j\in l^i}\frac{\varphi_i'(t)}{\left(\varphi_i(t)\right)^{1-l_i}}dt<\infty$$

where

$$L_i\left(x,t\right)=\int_{R^n}\int_{-\infty}^{+\infty}M_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}\right)$$

$$\times \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t), x\right)}{2\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi_i(t), x\right) \right) \Delta_i^{m_i}\left(\varphi_i\left(\delta\right) u\right) f\left(x + y + ue_i\right) dudy \quad (13)$$

Then for any $\overline{x} \in U$ *the following inequalities are true*

$$\sup_{\overline{x} \in U} \|F\|_{qU_{\psi(\xi)}(\overline{x})} \leq C_1 \left\| \prod_{j=1}^{n} \left(\varphi_i(t)\right)^{-l_j^0} \Delta^{m^0}\left(\varphi_i(T), G_{\varphi(T)}\right) f \right\|_{p^0,\varphi,\beta;G}$$

$$\times \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-\nu_j - (1-\beta_j p)\left(\frac{1}{p^i} - \frac{1}{p}\right)} \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p}{q}}, \quad (14)$$

$$\sup_{\overline{x} \in U} \left\|F_\eta^i\right\|_{qU_{\psi(\xi)}(\overline{x})} \leq C_2 \left\| \prod_{j=1}^{n} \left(\varphi_i(t)\right)^{-l_j^i} \Delta^{m^i}\left(\varphi_i(T), G_{\varphi(T)}\right) f \right\|_{p^i,\varphi,\beta;G}$$

$$\times |Q_T^i| \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}}, \quad (15)$$

$$\sup_{\overline{x} \in U} \left\|F_{\eta T}^i\right\|_{qU_{\psi(\xi)}(\overline{x})} \leq C_3 \left\| \left(\varphi_i(t)\right)^{-l_j^i} \Delta^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G}$$

$$\times \left|Q_{\eta T}^i\right| \prod_{j=1}^{n} \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j \frac{p^i}{p}}, \quad (16)$$

is hold, where $U_{\psi(\xi)}\left(\overline{x}\right) = \left\{x : |x_j - \overline{x}_j| < \frac{1}{2}\psi_j\left(\xi\right), j = 1, 2, ..., n\right\}$ *and* $\psi \in A$, C_1, C_2 *are the constants independent of* φ, ξ, η *and* T.

Corollary 1.

$$\|F\|_{p,\psi,\beta^1;U} \leq C_1' \left\| \prod_{j=1}^{n} \left(\varphi_i(t)\right)^{-l_j^0} \Delta^{m^0}\left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^0,\varphi,\beta;G}, \quad (17)$$

$$\left\|F_\eta^i\right\|_{p,\psi,\beta^1;U} \leq C_2' \left\| \prod_{j=1}^{n} \left(\varphi_i(t)\right)^{-l_j^i} \Delta^{m^i}\left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G}. \quad (18)$$

$$\left\|F_{\eta,T}^i\right\|_{p,\psi,\beta^1;U} \leq C_3' \left\| \prod_{j=1}^{n} \left(\varphi_i(t)\right)^{-l_j^i} \Delta^{m^i}\left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G}. \quad (19)$$

The proof is similar to the proof of Lemma 2 in [1].

2. Main results

Prove two theorems on the properties of the functions from the space $\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)$.

Theorem 1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \le p^i \le p \le \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ be entire $j = 1, 2, ..., n$, $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ and let $f \in \bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)$. Then the following embeddings hold*

$$D^\nu : \bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi) \to L_{q,\psi,\beta^1}(G)$$

i.e. for $f \in \bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are true

$$\|D^\nu f\|_{p,G} \le$$

$$\le C_1 \sum_{i=1}^{n} \left|Q_T^i\right| \sup_{0<t<t_0} \left\| \prod_{j=1}^{n} (\varphi_i(t))^{l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p,\varphi,\beta;G}, \tag{20}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \le C_2 \|f\|_{\bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi)},\ p^i \le p < \infty. \tag{21}$$

In particular, if

$$Q_{T,0}^i = \int_0^T \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\frac{1}{p}} \prod_{j \in l^i} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty,$$

then $D^\nu f(x)$ is continuous on G, i.e.

$$\sup_{x \in G} |D^\nu f(x)| \le \sum_{i=1}^{n} \left|Q_{T,0}^i\right| \sup_{0<t<t_0} \left\| \prod_{j=1}^{n} (\varphi_i(t))^{l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i,\varphi,\beta;G} \tag{22}$$

$0 < T \le \min\{1, T_0\}$, *T_0 is a fixed number; C_1, C_2 are the constants independent of f, C_1 are independent also on T.*

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$ on G. Indeed, from the condition $Q_T^i < \infty$ for all $(i = 1, 2, ..., n)$ it follows that for $f \in \bigcap_{i=0}^{n} L_{p^i,\varphi,\beta}^{<l^i>}(G_\varphi) \to \bigcap_{i=0}^{n} L_{p^i}^{<l^i>}(G_\varphi)$, there exists $D^\nu f \in L_p(G)$ and for it integral representation (9) with the same kernels is valid.

Based around the Minkowsky inequality, from identities (9) we get

$$\|D^\nu f\|_{q,G} \le \|F\|_{q,G} + \sum_{i=1}^{n} \|F_i\|_{p,G}\,. \tag{23}$$

By means of inequality (14) for $U=G$, $M_i=K_i^i$, $t=T$ we get

$$\|F\|_{p,G}\leq C_1|Q_T^0|\left\|\prod_{j=1}^{n}(\varphi_i(t))^{-l_j^0}\,\Delta^{m^0}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p^0,\varphi,\beta;G}, \qquad (24)$$

and by means inequality (15) for $\eta=T$, $M_i=K_i^i$, $U=G$, we get

$$\|F_i\|_{q,G}\leq C_2|Q_T^i|\left\|\prod_{j=1}^{n}(\varphi_i(t))^{-l_j^i}\,\Delta^{m^i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p^i,\varphi,\beta;G}, \qquad (25)$$

Substituting (25) and (24) in (23), we get inequality (20). By means of inequalities (17), (18) and (19) for $\eta=T$ we get inequality (21).

Now let conditions $Q_T^i<\infty$ $(i=1,2,...,n)$ be satisfied, then based around identities (9) from inequality (23) we get

$$\left\|D^\nu f-f_{\varphi(T)}^{(\nu)}\right\|_{\infty,G}\leq C\sum_{i=1}^{n}\left|Q_{T,0}^i\right|\sup_{0<t<t_0}\left\|\frac{\Delta^{m^i}\left(\varphi_i(t),G_{\varphi(t)}\right)f}{\prod\limits_{j=1}^{n}(\varphi_i(t))^{l_j^i}}\right\|_{p^i,\varphi,\beta;G}.$$

As $T\to0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}^{(\nu)}(x)$ is continuous on G and the convergence on $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G.

Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2. Let all the conditions of theorem 1 be fulfilled. Then for $Q_T^i<\infty$ $(i=1,2,...,n)$ the derivative $D^\nu f$ satisfies on G the Holder generalized condition, i.e. the following inequality is valid:

$$\|\Delta(\gamma,G)\,D^\nu f\|_{q,G}\leq C\|f\|_{\bigcap\limits_{i=0}^{n}L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)}\cdot|H(|\gamma|,\varphi;T)|, \qquad (26)$$

where C is a constant independent of f, $|\gamma|$ and T.

In particular, if $Q_{T,0}^i<\infty$, $(i=1,2,\ldots,n)$, then

$$\sup_{x\in G}|\Delta(\gamma,G)\,D^\nu f(x)|\leq C\|f\|_{\bigcap\limits_{i=0}^{n}L^{<l^i>}_{p^i,\varphi,\beta}(G_\varphi)}\cdot|H_0(|\gamma|,\varphi,T)|. \qquad (27)$$

where $H(|\gamma|,\varphi,T)=\max\limits_{i}\left\{|\gamma|,Q_{|\gamma|}^i,Q_{|\gamma|,T}^i\right\}$ $\left(H_0(|\gamma|,\varphi,T)=\max\limits_{i}\left\{|\gamma|,Q_{|\gamma|,0}^i,Q_{|\gamma|,T,0}^i\right\}\right)$

Proof. According to lemma 8.6 from [2] there exists a domain

$$G_\omega\subset G\left(\omega=\zeta r(x),\ \zeta>0\ r(x)=\rho(x,\partial G),\ x\in G\right)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (9) with the same kernels are valid. After same transformations, from (9) and (4) we get

$$\begin{aligned}
&|\Delta(\gamma, G) D^\nu f(x)| \le C_1 \prod_{j=1}^{n} (\varphi_j(t))^{-1-\nu_j} \times \\
&\times \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} \left| K_0^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)}\right) - K_0^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(T)}\right)\right| dydz \times \\
&\times |\Delta^{m^0}(\varphi(\delta)u)(x+y+u_1+\ldots+u_n)| \cdot |\zeta^0\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{2\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi_i(t), x)\right)| dudy + \\
&+ C_2 \sum_{i=1}^{n} \left\{ \int\limits_{0}^{|\gamma|} \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} \left| \zeta^i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho'(\varphi(t), x)\right)\right| \times - \right. \\
&\times \left|\Delta^{m^i}(\varphi_i(\delta)u) f(x+y+u_1+\ldots+u_n)\right| \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j-2} \prod_{j\in m^i} \frac{\varphi_i'(t)}{\varphi_i(t)} dydudt \\
&+ \int\limits_{|\gamma|}^{T} \int\limits_{R^n} \int\limits_{-\infty}^{+\infty} \left| K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho(\varphi(t,x))}{\varphi(t)}\right)\right| \left| \zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)}, \frac{1}{2}\rho_i'(\varphi(t), x)\right)\right| \\
&\left. \times \int\limits_{0}^{1} \left|\Delta^{m^i}(\varphi_i(\delta)u) f(x+y+u_1+\ldots+u_n\gamma)\right| \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j-2} \prod_{j\in m^i} \frac{\varphi_i'(t)}{\varphi_i(t)} dvdudydt \right\}. \\
&= C_1 F(x,\gamma) + C_2 \sum_{i=1}^{n} \left(F_1(x,\gamma) + F_2^i(x,\gamma)\right), \qquad (28)
\end{aligned}$$

where $0 < T \le \{1, T_0\}$ we also assume that $|\gamma| < T$. Consequently, $|\gamma| < \min(\omega, T)$. If $x \in G \setminus G_\omega$ then by definition

$$\Delta(\gamma, G) D^\nu f(x) = 0.$$

Based around (28) we have

$$\begin{aligned}
\|\Delta(\gamma, G) D^\nu f\|_{q,G} \le{}& \left\|F_1^i(\cdot,\gamma)\right\|_{q,G_\omega} \\
&+ \sum_{i=1}^{n} \left(\|E(\cdot,\gamma)\|_{q,G_\omega} + \left\|F_2^i(\cdot,\gamma)\right\|_{q,G_\omega} \right), \qquad (29)
\end{aligned}$$

$$F(x,\gamma) \le \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j-2} \int\limits_{0}^{|\gamma|} d\zeta \int\limits_{R^n} \int\limits_{R^n} |f(x+y+u_1+\ldots+u_n)|$$

$$\times \left| D_j K^{(\nu)} \left(\frac{y}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \Omega^{(\nu)} \left(\frac{z}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \right| dydz.$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, based around the generalized Minkowsky inequality, from inequality (19) for $U = G$, we have

$$\|F(\cdot, \gamma)\|_{p, G_\vartheta} \le C_1 |\gamma| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^0} \Delta^{m^0} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i, \varphi, \beta; G} \quad (30)$$

By means of inequality (16), for $U = G$, $\eta = |\gamma|$ we get

$$\left\|F_1^i(\cdot, \gamma)\right\|_{q, G_\omega} \le C_2 \left| Q^i_{|\gamma|} \right| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i, \varphi, \beta; G} \quad (31)$$

and by means of inequality (10) for $U = G$, $\eta = |\gamma|$ we get

$$\left\|F_2^i(\cdot, \gamma)\right\|_{q, G_\omega} \le C_3 \left| Q^i_{|\gamma|, T} \right| \left\| \prod_{j=1}^{n} (\varphi_i(t))^{-l_j^i} \Delta^{m^i} \left(\varphi_i(t), G_{\varphi(t)}\right) f \right\|_{p^i, \varphi, \beta; G} . \quad (32)$$

From inequalities (29) -(32) we get the required inequality.

Now suppose that $|\gamma| \ge \min(\omega, T)$. Then

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \le 2 \|D^\nu f\|_{p,G} \le C(\vartheta T) \|D^\nu f\|_{p,G} |H(|\gamma|, \varphi; T)| .$$

Estimating for $\|D^\nu f\|_{p,G}$ by means of inequality (20), in this case we get estimation (26).

Theorem 2 is proved.

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