Caspian Journal of Applied Mathematics, Ecology and Economics
V. 3, No 2, 2015, December
ISSN 1560-4055

Parabolic Fractional Maximal Operator with Rough Kernels in Parabolic Local Generalized Morrey Spaces

A.S. Balakishiyev*, E.A. Gadjieva, N.Z. Orucov

Abstract. Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part, $A_t = t^P$, $t > 0$, $\gamma = \mathrm{tr}P$ is the homogeneous dimension on $\mathbb{R}^n$ and Ω is an A_t-homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by the corresponding parabolic metric. We study the parabolic fractional maximal operator $M^P_{\Omega,\alpha}$, $0 \le \alpha < \gamma$ with rough kernels in the parabolic local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$. We find conditions on the pair (φ_1, φ_2) for the boundedness of the operator $I^P_{\Omega,\alpha}$ from the space $LM^{\{x_0\}}_{p,\varphi_1,P}(\mathbb{R}^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $LM^{\{x_0\}}_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$.

Key Words and Phrases: Parabolic fractional maximal function, parabolic local generalized Morrey space.

2010 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

The boundedness of classical operators of the real analysis, such as the maximal operator and fractional maximal operator, from one weighted Lebesgue space to another one is well studied by now, and there are well known various applications of such results in partial differential equations. Besides Lebesgue spaces, Morrey spaces, both the classical ones (the idea od their definition having appeared in [13]) and generalized ones, also play an important role in the theory of partial differential equations.

In this paper, we find conditions for the boundedness of the parabolic fractional maximal operators with rough kernel from a parabolic local generalized Morrey space to another one, including also the case of weak boundedness.

Note that we deal not exactly with the parabolic metric, but with a general anisotropic metric ρ of generalized homogeneity, the parabolic metric being its particular case, but we keep the term "parabolic in the title and text of the paper, following the existing tradition, see for instance [4].

*Corresponding author.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote the open ball centered at x of radius r by $B(x,r)$, its complement by ${}^{\complement}B(x,r)$ and $|B(x,r)|$ will stand for the Lebesgue measure of $B(x,r)$.

Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t^P \quad (t > 0)$, and set

$$\gamma = \mathrm{tr} P.$$

Then, there exists a quasi-distance ρ associated with P such that (see [5])

$$\begin{aligned}
&(a) \quad \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every} \quad x \in \mathbb{R}^n;\\
&(b) \quad \rho(0) = 0, \quad \rho(x) = \rho(-x) \geq 0\\
&\qquad \text{and} \quad \rho(x-y) \leq k(\rho(x-z) + \rho(y-z));\\
&(c) \quad dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad \text{where} \quad \rho = \rho(x), w = A_{\rho^{-1}} x\\
&\qquad \text{and } d\sigma(w) \text{ is a measure on the unit ellipsoid } \; S_\rho = \{w : \rho(w) = 1\}.
\end{aligned}$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [5]) and a homogeneous group in the sense of Folland-Stein (see [7]).

In the standard parabolic case $P_0 = \mathrm{diag}(1, \ldots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

The balls $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$ with respect to the quasidistance ρ are ellipsoids. For its Lebesgue measure one has

$$|\mathcal{E}(x,r)| = v_\rho r^\gamma,$$

where v_ρ is the volume of the unit ellipsoid. By ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ we denote the complement of $\mathcal{E}(x,r)$.

Everywhere in the sequel $A \lesssim B$ means that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

1.1. Parabolic local generalized Morrey spaces

In the doctoral thesis [8], 1994 by Guliyev (see, also [9], [1]-[3]) introduced the local Morrey-type space $LM_{p\theta,w}$ given by

$$\|f\|_{LM_{p\theta,w}} = \big\| w(r) \, \|f\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. If $\theta = \infty$, it denotes $LM_{p,w} \equiv LM_{p\infty,w}$. In [8] Guliyev intensively studied the classical operators in the local Morrey-type space $LM_{p\theta,w}$ (see also the book [9] (1999)), where these results were presented for the case when the underlying space is the Heisenberg group or a homogeneous group, respectively. Note that, the generalized local (central) Morrey spaces $LM_{p,\varphi}(\mathbb{R}^n) = LM_{p,\varphi}^{\{0\}}(\mathbb{R}^n)$ introduced by Guliyev in [8] (see also, [9], [1]-[3]).

We define the parabolic local Morrey space $LM_{p,\lambda,P}(\mathbb{R}^n)$ via the norm

$$\|f\|_{LM_{p,\lambda,P}} = \sup_{t>0} \left(t^{-\lambda} \int_{\mathcal{E}(0,t)} |f(y)|^p dy \right)^{1/p} < \infty,$$

where $1 \le p \le \infty$ and $0 \le \lambda \le \gamma$.

If $\lambda = 0$, then $LM_{p,0,P}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = \gamma$, then $LM_{p,\gamma,P}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > \gamma$, then $LM_{p,\lambda,P} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WLM_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space of functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\lambda,P}} = \sup_{t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(0,r))} < \infty,$$

where $WL_p(\mathcal{E}(0,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(\mathcal{E}(0,r))} = \sup_{t>0} t \, |\{y \in \mathcal{E}(0,r) : |f(y)| > t\}|^{1/p} .$$

Note that $WL_p(\mathbb{R}^n) = WLM_{p,0,P}(\mathbb{R}^n)$,

$$LM_{p,\lambda,P}(\mathbb{R}^n) \subset WLM_{p,\lambda,P}(\mathbb{R}^n) \text{ and } \|f\|_{WLM_{p,\lambda,P}} \le \|f\|_{LM_{p,\lambda,P}} .$$

If $P = I$, then $LM_{p,\lambda}(\mathbb{R}^n) \equiv LM_{p,\lambda,I}(\mathbb{R}^n)$ is the local Morrey space.

We introduce the parabolic local generalized Morrey spaces following the known ideas of defining local generalized Morrey spaces ([8, 10, 11] etc).

Definition 1.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The space $LM_{p,\varphi,P} \equiv LM_{p,\varphi,P}(\mathbb{R}^n)$, called the parabolic local generalized Morrey space, is defined by the norm*

$$\|f\|_{LM_{p,\varphi,P}} = \sup_{t>0} \varphi(0,t)^{-1} \, |\mathcal{E}(0,t)|^{-\frac{1}{p}} \, \|f\|_{L_p(\mathcal{E}(0,t))}.$$

Definition 1.2. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The space $WLM_{p,\varphi,P} \equiv WLM_{p,\varphi,P}(\mathbb{R}^n)$, called the weak parabolic local generalized Morrey space, is defined by the norm*

$$\|f\|_{WLM_{p,\varphi,P}} = \sup_{t>0} \varphi(0,t)^{-1} \, |\mathcal{E}(0,t)|^{-\frac{1}{p}} \, \|f\|_{WL_p(\mathcal{E}(0,t))}.$$

If $P = I$, then $LM_{p,\varphi}(\mathbb{R}^n) \equiv LM_{p,\varphi,I}(\mathbb{R}^n)$ and $WLM_{p,\varphi}(\mathbb{R}^n) \equiv WLM_{p,\varphi,I}(\mathbb{R}^n)$ are the generalized local Morrey space and the weak generalized local Morrey space, respectively.

Definition 1.3. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{\{x_0\}}_{p,\varphi,P} \equiv LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L^{\rm loc}_p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi,P}}.$$

Also by $WLM^{\{x_0\}}_{p,\varphi,P} \equiv WLM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^{\rm loc}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi,P}} < \infty.$$

According to this definition, we recover the space $LM^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$ under the choice $\varphi(0,r) = r^{\frac{\lambda-\gamma}{p}}$:

$$LM^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n) = LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \ge 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t-homogeneous of degree zero, i.e. $\Omega(A_t x) \equiv \Omega(x)$, $x \in \mathbb{R}^n$, $t > 0$. The parabolic fractional maximal function $M^P_{\Omega,\alpha} f$ by with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L^{\rm loc}_1(\mathbb{R}^n)$ is defined by

$$M^P_{\Omega,\alpha} f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{\gamma}} \int_{\mathcal{E}(x,t)} |\Omega(x-y)|\, |f(y)| dy.$$

If $\Omega \equiv 1$, then $M^P_\alpha \equiv M^P_{1,\alpha}$ is the parabolic fractional maximal operator. If $\alpha = 0$, then $M^P_\Omega \equiv M^P_{\Omega,0}$ is the parabolic maximal operator with rough kernel. If $P = I$, then $M_{\Omega,\alpha} \equiv M^I_{\Omega,\alpha}$ is the fractional maximal operator with rough kernel, and $M \equiv M^I_{\Omega,0}$ is the Hardy-Littlewood maximal operator with rough kernel. It is well known that the parabolic fractional maximal operators play an important role in harmonic analysis (see [7, 14]).

We prove the boundedness of the parabolic fractional maximal operators $M^P_{\Omega,\alpha}$ with rough kernel from one parabolic local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi_1,P}(\mathbb{R}^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/\gamma$, and from the space $LM^{\{x_0\}}_{1,\varphi_1,P}(\mathbb{R}^n)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$, $1 \le q < \infty$, $1 - 1/q = \alpha/\gamma$.

2. Preliminaries

Let v be a weight on $(0,\infty)$. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)|g(t)|$$

and write $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions. By $\mathfrak{M}^+(0,\infty;\uparrow)$ we denote the cone of all functions in $\mathfrak{M}^+(0,\infty)$ non-decreasing on $(0,\infty)$ and introduce also the set

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a non-negative continuous function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [2].

Theorem 2.1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{2.1}$$

We are going to use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \; 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

The following theorem in the case $w = 1$ was proved in [3].

Theorem 2.2. *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H^*_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2.2}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{2.3}$$

Moreover, if C^* is the minimal value of C in (2.2), then $C^* = B$.

Remark 2.1. *In (2.2) and (2.3) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.*

3. Parabolic fractional maximal operator with rough kernels in the spaces $LM^{\{x_0\}}_{p,\varphi,P}(\mathbb{R}^n)$

In [12] was proved the (p,p)-boundedness of the operator M^P_Ω and the (p,q)-boundedness of the operator $M^P_{\Omega,\alpha}$.

Theorem 3.1. *[12] Let $\Omega \in L_s(S_\rho)$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Then the operator M^P_Ω is bounded in the space $L_p(\mathbb{R}^n)$, $p > s'$.*

Corollary 3.1. *[12] Suppose that $0 \le \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$, is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$ and $1/p - 1/q = \alpha/\gamma$. Then the fractional maximal operator M^P_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$ and from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

The following lemma is valid.

Lemma 3.1. *Suppose that $x_0 \in \mathbb{R}^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then for any ball $\mathcal{E} = \mathcal{E}(x_0, r)$ in $\mathbb{R}^n$ and $f \in L^{\rm loc}_p(\mathbb{R}^n)$ there hold the inequalities*

$$\|M^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E}(x_0,r))} \lesssim \|f\|_{L_p(\mathcal{E}(x_0,2kr))} + r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\gamma+\alpha} \|\Omega(x_0-\cdot) f(\cdot)\|_{L_1(\mathcal{E}(x_0,t))}, \quad p > 1,$$

$$\|M^P_{\Omega,\alpha} f\|_{WL_q(\mathcal{E}(x_0,r))} \lesssim \|f\|_{L_1(\mathcal{E}(x_0,2kr))} + r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\gamma+\alpha} \|\Omega(x_0-\cdot) f(\cdot)\|_{L_1(\mathcal{E}(x_0,t))}, \quad p = 1. \tag{3.1}$$

Proof. Given a ball $\mathcal{E} = \mathcal{E}(x_0, r)$, we split the function f as $f = f_1 + f_2$, where $f_1 = f\chi_{\mathcal{E}(x_0,2kr)}$ and $f_2 = f\chi_{\complement(\mathcal{E}(x_0,2kr))}$, and then

$$\|M^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E})} \le \|M^P_{\Omega,\alpha} f_1\|_{L_q(\mathcal{E})} + \|M^P_{\Omega,\alpha} f_2\|_{L_q(\mathcal{E})}.$$

Let $p > 1$. By Corollary 3.1

$$\|M^P_{\Omega,\alpha} f_1\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(\mathcal{E}(x_0,2kr))}.$$

To estimate $M^P_{\Omega,\alpha} f_2(y)$, observe that if $\mathcal{E}(y,t) \cap \complement(\mathcal{E}(x_0, 2kr)) \ne \emptyset$, where $y \in \mathcal{E}$, then $t > r$. Indeed, if $z \in \mathcal{E}(y,t) \cap \complement(\mathcal{E}(x_0, 2kr))$, then $t > \rho(y-z) \ge \frac{1}{k}\rho(x_0-z) - \rho(x_0-y) > 2r - r = r$.

On the other hand, $\mathcal{E}(y,t) \cap \complement(\mathcal{E}(x_0, 2kr)) \subset \mathcal{E}(x_0, 2kt)$. Indeed, for $z \in \mathcal{E}(y,t) \cap \complement(\mathcal{E}(x_0, 2kr))$ we get $\rho(x_0-z) \le k\rho(y-z) + k\rho(x_0-y) < k(t+r) < 2kt$.

Hence

$$\begin{aligned} M^P_{\Omega,\alpha} f_2(y) &= \sup_{t>0} \frac{1}{|\mathcal{E}(y,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(y,t)\cap\complement(\mathcal{E}(x_0,2kr))} |f(z)||\Omega(x_0-z)|dz \\ &\le (2k)^{\gamma-\alpha} \sup_{t>r} \frac{1}{|\mathcal{E}(x_0,2kt)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x_0,2kt)} |f(z)||\Omega(x_0-z)|dz \\ &= (2k)^{\gamma-\alpha} \sup_{t>2kr} \frac{1}{|\mathcal{E}(x_0,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x_0,t)} |f(z)||\Omega(x_0-z)|dz. \end{aligned}$$

Therefore, for all $y \in \mathcal{E}$ we have

$$M^P_{\Omega,\alpha} f_2(y) \le (2k)^{\gamma-\alpha} \sup_{t>2kr} \frac{1}{|\mathcal{E}(x_0,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x_0,t)} |f(z)||\Omega(x_0-z)|dz. \tag{3.2}$$

Thus

$$\|M^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E})} \lesssim \|f\|_{L_p(\mathcal{E}(x_0,2kr))} + |\mathcal{E}|^{\frac{1}{q}} \sup_{t>2kr} \frac{1}{|\mathcal{E}(x_0,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x_0,t)} |f(z)||\Omega(x_0-z)|dz.$$

Let $p = 1$. We have

$$\|M^P_{\Omega,\alpha} f\|_{WL_q(\mathcal{E})} \le \|M^P_{\Omega,\alpha} f_1\|_{WL_q(\mathcal{E})} + \|M^P_{\Omega,\alpha} f_2\|_{WL_q(\mathcal{E})}.$$

By Corollary 3.1 we get

$$\|M^P_{\Omega,\alpha} f_1\|_{WL_q(\mathcal{E})} \lesssim \|f\|_{L_1(\mathcal{E}(x_0,2kr))}.$$

Then by (3.2) we arrive at (3.1) and complete the proof.

Similarly to Lemma 3.1 and Theorem 3.1 the following lemma may be proved.

Lemma 3.2. *Let the function $\Omega \in L_s(S_\rho)$, $1 < s \le \infty$, be A_t-homogeneous of degree zero and $x_0 \in \mathbb{R}^n$. Then for $p > s'$ and any ball $\mathcal{E} = \mathcal{E}(x_0, r)$ the inequality*

$$\|M^P_\Omega f\|_{L_p(\mathcal{E}(x_0,r))} \lesssim \|f\|_{L_p(\mathcal{E}(x_0,2kr))} + r^{\frac{\gamma}{p}} \sup_{t>2kr} t^{-\gamma} \|\Omega(x_0-\cdot)f(\cdot)\|_{L_1(\mathcal{E}(x_0,t))}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 3.3. *Suppose that the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero and $x_0 \in \mathbb{R}^n$. Let $0 < \alpha < \gamma$, $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$. Then for $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ there hold the inequalities*

$$\|M^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_p(\mathcal{E}(x_0,t))}, \quad p > 1, \tag{3.3}$$

$$\|M^P_{\Omega,\alpha} f\|_{WL_q(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathcal{E}(x_0,t))}, \quad p = 1. \tag{3.4}$$

Proof. Let $p > 1$ Denote

$$\mathcal{A}_1 := |\mathcal{E}|^{\frac{1}{q}} \left(\sup_{t>2kr} \frac{1}{|\mathcal{E}(x_0,t)|^{1-\alpha/\gamma}} \int_{\mathcal{E}(x_0,t)} |f(z)||\Omega(x_0-z)|dz \right),$$

$$\mathcal{A}_2 := \|f\|_{L_p(\mathcal{E}(x_0,2kr))}.$$

Applying Hölder's inequality, we get

$$\mathcal{A}_1 \lesssim |\mathcal{E}|^{\frac{1}{q}} \sup_{t>2kr} \|f\|_{L_p(\mathcal{E}(x_0,t))} \|\Omega(x_0-\cdot)\|_{L_{\frac{\gamma}{\gamma-\alpha}}(\mathcal{E}(x_0,t))} |\mathcal{E}(x_0,t)|^{\frac{\alpha}{\gamma}-\frac{1}{p}}$$

$$\lesssim |\mathcal{E}|^{\frac{1}{q}} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-\frac{1}{q}} \, \|f\|_{L_p(\mathcal{E}(x_0,t))}.$$

On the other hand,

$$|\mathcal{E}|^{\frac{1}{q}} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-\frac{1}{q}} \, \|f\|_{L_p(\mathcal{E}(x_0,t))}$$
$$\gtrsim |\mathcal{E}|^{\frac{1}{q}} \sup_{t>2kr} |\mathcal{E}(x_0,t)|^{-\frac{1}{q}} \, \|f\|_{L_p(\mathcal{E}(x_0,2kr))} \approx \mathcal{A}_2.$$

Since $\|M^P_{\Omega,\alpha} f\|_{L_q(\mathcal{E})} \le \mathcal{A}_1 + \mathcal{A}_2$, by Lemma 3.1, we arrive at (3.3).

Let $p = 1$. The inequality (3.4) directly follows from (3.1).

Similarly to Lemma 3.3 and Theorem 3.1 the following lemma is also proved.

Lemma 3.4. *Suppose that the function $\Omega \in L_s(S_\rho)$, $1 < s \le \infty$, is A_t-homogeneous of degree zero and $x_0 \in \mathbb{R}^n$. Then for $p > s'$ and any ball $\mathcal{E} = \mathcal{E}(x_0, r)$, the inequality*

$$\|M^P_\Omega f\|_{L_p(\mathcal{E}(x_0,r))} \lesssim r^{\frac{\gamma}{q}} \sup_{t>2kr} t^{-\frac{\gamma}{p}} \|f\|_{L_p(\mathcal{E}(x_0,t))}$$

holds for $f \in L^{\rm loc}_p(\mathbb{R}^n)$.

Theorem 3.2. *Suppose that $x_0 \in \mathbb{R}^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} t^{\alpha-\frac{\gamma}{p}} \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\, \tau^{\frac{\gamma}{p}} \le C\, \varphi_2(x_0,r), \tag{3.5}$$

where C does not depend on x_0 and r. Then the operator $M^P_{\Omega,\alpha}$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}(\mathbb{R}^n)$ to $LM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}(\mathbb{R}^n)$ to $WLM^{\{x_0\}}_{q,\varphi_2,P}(\mathbb{R}^n)$ for $p = 1$.

Proof. By Theorem 2.1 and Lemma 3.3 we get

$$\|M^P_{\Omega,\alpha} f\|_{LM^{\{x_0\}}_{q,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>r} t^{-\frac{\gamma}{q}} \, \|f\|_{L_p(\mathcal{E}(x_0,t))}$$
$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{\gamma}{p}} \, \|f\|_{L_p(\mathcal{E}(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{p,\varphi_1,P}},$$

if $p \in (1,\infty)$ and

$$\|M^P_{\Omega,\alpha} f\|_{WLM^{\{x_0\}}_{q,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>r} t^{-\frac{\gamma}{q}} \|f\|_{L_1(\mathcal{E}(x_0,t))}$$
$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\gamma} \, \|f\|_{L_1(\mathcal{E}(x_0,r))} = \|f\|_{LM^{\{x_0\}}_{1,\varphi_1,P}},$$

if $p = 1$.

In the same way, by means of Lemma 3.4 we can obtain the following theorem.

Theorem 3.3. *Suppose that the function $\Omega \in L_s(S_\rho)$, $1 < s \le \infty$ is A_t-homogeneous of degree zero and $x_0 \in \mathbb{R}^n$. Let $p > s'$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} t^{-\frac{\gamma}{p}} \operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\, \tau^{\frac{\gamma}{p}} \le C\, \varphi_2(x_0,r),$$

where C does not depend on x_0 and r. Then the operator M^P_Ω is bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}(\mathbb{R}^n)$ to $LM^{\{x_0\}}_{p,\varphi_2,P}(\mathbb{R}^n)$.

Acknowledgements

The authors would like to express their gratitude to the referees for very valuable comments and suggestions.

The research of E. Gadjieva was partially supported by the grant of Presidium of Azerbaijan National Academy of Science 2015.

References

[1] V.I. Burenkov, H.V. Guliyev, and V.S. Guliyev, *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces*, J. Comput. Appl. Math., **208(1)**, 2007, 280–301.

[2] Burenkov V, Gogatishvili A, Guliyev V.S, Mustafayev R, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ., **55(8-10)**, 2010, 739–758

[3] V. Burenkov, A. Gogatishvili, V.S. Guliyev, R. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal., **35(1)**, 2011, 67–87.

[4] A.P. Caldereon, A. Torchinsky *Parabolic maximal function associated with a distribution, I.* Advances in Math., **16**, 1975, 1–64.

[5] R.R. Coifman, G. Weiss, *Analyse harmonique non-commutative sur certains espaces homognes.* (French) Étude de certaines intégrales singuliàres. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.

[6] E.B. Fabes, N. Rivère, *Singular integrals with mixed homogeneity*, Studia Math., **27**, 1966, 19–38.

[7] G.B. Folland, E.M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes, 28, Princeton Univ. Press, Princeton, 1982.

[8] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian)

[9] V.S. Guliyev, *Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications*, Casioglu, Baku, 1999, 332 pp. (in Russian)

[10] V.S. Guliyev, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci., (N. Y.) **193(2)**, 2013, 211–227.

[11] V.S. Guliyev, *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerbaijan Journal of Mathematics, **3(2)**, 2013, 79–94.

[12] V. Guliyev, A.S. Balakishiyev, *Parabolic fractional maximal and integral operators with rough kernels in parabolic generalized Morrey spaces*. Journal of Mathematical Inequalities, **9(1)**, 2015, 257–276.

[13] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, 1938, 126–166.

[14] E.M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.

Aydin S. Balakishiyev
Baku State University, Baku, AZ 1148, Azerbaijan
Email: abalakishiyev@yahoo.com

E.A. Gadjieva
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ 1141, Baku, Azerbaijan
Email: egadjieva@gmail.com

Nazim Z. Orucov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ 1141, Baku, Azerbaijan
Email: nazimorucov@gmail.com

Received 12 December 2015
Accepted 29 December 2015