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Boundedness of the Fractional Maximal Operator in Local and Global Morrey-type Spaces on the Heisenberg Group

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Abstract. We study the boundedness of the fractional maximal operator M_α on the Heisenberg group $\mathbb{H}^n$ in local and global Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$ and $GM_{p\theta,w}(\mathbb{H}^n)$, respectively. We give a characterization of strong and weak type boundedness for the operator M_α in local Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$.

Key Words and Phrases: fractional maximal operator, local Morrey-type space, Heisenberg group.

2010 Mathematics Subject Classifications: 42B25, 42B35, 43A15

1. Introduction

In this paper, we establish the norm inequalities for the fractional maximal operator in local Morrey-type spaces on Heisenberg group. The Heisenberg group [6, 7, 15, 17] appears in quantum physics and many fields of mathematics, including harmonic analysis, the theory of several complex variables and geometry. We begin with some basic notation. The Heisenberg group $\mathbb{H}_n$ a non-commutative nilpotent Lie group with the product spaces $\mathbb{R}^{2n+1}$ that have the multiplication

$$xy = \Big(x' + y', x_{2n+1} + y_{2n+1} + 2\sum_{k=1}^{n} x_k y_{n+k} - x_{n+k} y_k\Big),$$

where $x = (x', x_{2n+1})$, and $y = (y', y_{2n+1})$. By the definition, the identity element on $\mathbb{H}_n$ is $0 \in \mathbb{R}^{2n+1}$, while the inverse element of $x = (x', t)$ is $x^{-1} = (-x', -t)$.

The corresponding Lie algebra is generated by the left-invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, X_{n+j} \quad = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, X_{2n+1} = \frac{\partial}{\partial x_{2n+1}},\ j = 1, \dots, n.$$

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The only non-trivial commutator relations are

$$\big[X_j, X_{n+j}\big] = -4X_{2n+1}, \quad j = 1, \ldots, n.$$

The non-isotropic dilation on $\mathbb{H}_n$ is defined as $\delta_t(x', x_{2n+1}) = (tx', t^2 x_{2n+1})$ for $t > 0$. The Haar measure dx on this group coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is easy to check that $d\big(\delta_t x\big) = r^Q dx$. In the above, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. The norm of $x = (x', x_{2n+1}) \in \mathbb{H}_n$ is given by $|x|_{\mathbb{H}} = (|x'|^4 + x_{2n+1}^2)^{1/4}$, where $|x'|^2 = \sum_{k=1}^{2n} x_k^2$. The norm satisfies the triangle inequality and leads to the left-invariant distance $d(x, y) = |xy^{-1}|_{\mathbb{H}}$. With this norm we define the Heisenberg ball, $B(x, r) = \{y \in \mathbb{H}_n : |xy^{-1}|_{\mathbb{H}} < r\}$, where x is the center and r is the radius. The volume of $B(x, r)$ is $d_n r^{2n+2}$, where dC_n is the volume of the unit ball $B_1 \equiv B(e, 1)$. Let $S_H = \{x \in \mathbb{H}_n : |x|_{\mathbb{H}} = 1\}$ be the unit sphere in $\mathbb{H}_n$ equipped with the normalized Haar surface measure $d\sigma$.

The fractional maximal function $M_\alpha f$, $0 < \alpha < Q$ on the Heisenberg groups of a function $f \in L_1^{\rm loc}(\mathbb{H}_n)$ is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x, t)|^{-1+\frac{\alpha}{Q}} \int_{B(x,t)} |f(y)| dy.$$

If $\alpha = 0$, then $M \equiv M_0$ is the maximal operator on the Heisenberg groups. It is well known that the fractional maximal operator on the Heisenberg groups play an important role in harmonic analysis (see [7, 16]).

The main purpose of [10] is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups $\mathbb{G}$ in local Morrey-type space $LM_{p\theta,w_1}(\mathbb{G})$. In a series of papers by Burenkov V., Guliyev H. and Guliyev V. etc. (see, for example [2, 3, 4]) be given some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,w_1}(\mathbb{R}^n)$.

In this paper, we study the boundedness of the fractional maximal operator M_α on the Heisenberg group $\mathbb{H}^n$ in local Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$. Also we give a characterization of strong and weak type boundedness for the operator M_α in local Morrey-type spaces $LM_{p\theta,w}(\mathbb{H}^n)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. For a number p, $p' = p/(p-1)$ denotes the conjugate exponent of p.

2. Local and global Morrey-type spaces on the Heisenberg group

Let $0 < p, \theta \le \infty$. Denote by Ω_θ a set of all non-negative measurable functions $w(r)$ on $(0, \infty)$ such that $w(t) \ne 0$ on the set of positive measure and $\|w(r)\|_{L_\theta(t_1,\infty)} < \infty$ for some $t_1 > 0$. The set $\Omega_{p,\theta}$ consists of the functions $w(r) \in \Omega_\theta$ such that $\|w(r) r^{Q/p}\|_{L_\theta(0,t_2)} < \infty$

for some $t_2 > 0$ (see [2]). Let $w_1 \in \Omega_\theta$, $w_2 \in \Omega_{\theta,p}$. Recall that in 1994 the doctoral thesisis [10] (see also [11]) by Guliyev introduced the local Morrey-type space $LM_{p\theta,w_1}$ and in [1] (see also [2, 3, 4]) by Burenkov, Guliyev introduced the global Morrey-type space $GM_{p\theta,w_1}$.

Definition 1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w}(\mathbb{H}^n)$, $GM_{p\theta,w}(\mathbb{H}^n)$, the local Morrey-type spaces, the global Morrey-type spaces on the Heisenberg group respectively, the spaces of all functions $f \in L_p^{\rm loc}(\mathbb{H}^n)$ with finite quasinorms*

$$\|f\|_{LM_{p\theta,w}(\mathbb{H}^n)} = \big\|w(r)\|f\|_{L_p(B(0,t))}\big\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w}(\mathbb{H}^n)} = \sup_{x\in\mathbb{H}^n} \big\|w(r)\|f\|_{L_p(B(x,t))}\big\|_{L_\theta(0,\infty)}$$

respectively.

Note that

$$\|f\|_{LM_{p\infty,1}(\mathbb{H}^n)} = \|f\|_{GM_{p\infty,1}(\mathbb{H}^n)} = \|f\|_{L_p(\mathbb{H}^n)}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p}(\mathbb{H}^n)} \equiv M_{p,\lambda}(\mathbb{H}^n)$, $0 \le \lambda \le Q$.

For a measurable set $\mathbb{H}^n$ and a function v non-negative and measurable on $\mathbb{H}^n$, let $L_{p,v}(\mathbb{H}^n)$ be the weighted L_p-space of all functions f measurable on $\mathbb{H}^n$ for which $\|f\|_{L_{p,v}(\mathbb{H}^n)} = \|vf\|_{L_p(\mathbb{H}^n)} < \infty$.

If $0 < p \le \theta \le \infty$, then $\|f\|_{LM_{p\theta,w}(\mathbb{H}^n)} \le \|f\|_{L_{p,W}(\mathbb{H}^n)}$, and if $0 < \theta \le p \le \infty$, then $\|f\|_{L_{p,W}(\mathbb{H}^n)} \le \|f\|_{LM_{p\theta,w}(\mathbb{H}^n)}$, where for all $x \in \mathbb{H}^n$ $W(x) = \|w\|_{L_\theta(|x|_{\mathbb{H}},\infty)}$.

In particular, for $0 < p \le \infty$ $\|f\|_{LM_{pp,w}(\mathbb{H}^n)} = \|f\|_{L_{p,V}(\mathbb{H}^n)}$, where for all $x \in \mathbb{H}^n$ $V(x) = \|w\|_{L_p(|x|_{\mathbb{H}},\infty)(\mathbb{H}^n)}$.

We shall use the following theorem stating necessary and sufficient conditions for the validity of the following inequality

$$\|M_\alpha f\|_{L_{p_2,v_2}(\mathbb{H}^n)} \le c\|f\|_{L_{p_1,v_1}(\mathbb{H}^n)} \tag{1}$$

where v_1 and v_2 are functions non-negative and measurable on $\mathbb{H}^n$ and $c > 0$ is independent of f (see [5, 14]).

Given a set $\Omega \subset \mathbb{H}^n$, χ_Ω will denote the characteristic function of Ω.

Theorem 1. *Let $0 \le \alpha < Q$, $1 < p_1 \le p_2 < \infty$. Moreover, let v_1, v_2 be non-negative and measurable on $\mathbb{H}^n$. Then inequality (1) holds if, and only if, the following equivalent conditions are satisfied*

$$\mathcal{J} = \sup_{B\subset\mathbb{H}^n} |B|^{\frac{\alpha}{n}-1} \left\|v_1^{-1}\right\|_{L_{p_1'}(B)} \|v_2\|_{L_{p_2}(B)} < \infty \tag{2}$$

and

$$\sup_{B\subset\mathbb{H}^n} \left\|M_\alpha\left(\chi_B v_1^{p_1/(1-p_1)}\right)\right\|_{L_{p_2,v_2}(B)} \left\|v_1^{1/(1-p_1)}\right\|_{L_{p_1}(B)}^{-1} < \infty. \tag{3}$$

Moreover, the sharp (minimal possible) constant c^* in (1), satisfies the inequality $c\mathcal{J} \le c^* \le c\mathcal{J}$, where $c, c^* > 0$ are independent of v_1 and v_2.

3. Boundedness of the fractional maximal operator in local Morrey-type spaces on Heisenberg group

Let $0 < p, \theta \le \infty$. Denote by Ω_θ a set of all non-negative measurable functions $w(r)$ on $(0,\infty)$ such that $w(t) \ne 0$ on the set of positive measure and $\|w(r)\|_{L_\theta(t_1,\infty)} < \infty$ for some $t_1 > 0$. Let $w_1 \in \Omega_\theta$, $w_2 \in \Omega_{\theta,p}$. Recall that in 1994 the doctoral thesisis [10] (see also [11]) by Guliyev V.S. introduced the local Morrey-type space $LM_{p\theta,w_1}(\mathbb{H}^n)$ is given by

$$\|f\|_{LM_{p\theta,w_1}(\mathbb{H}^n)} = \|w_1(r)\|f\|_{B(0,r)}\|_{L_\theta(0,\infty)}.$$

To obtain necessary and sufficient conditions on w_1 and w_2 under which M_α is bounded for other parameter values and to obtain simpler conditions for the case $p = \theta_1 = \theta_2$ we reduce the problem of the boundedness of M_α in the local Morrey-type spaces to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions.

Lemma 1. *Let $0 \le \alpha < Q$, $1 < p_1 \le p_2 < \infty$ and $-\infty < \gamma < \infty$. Then the inequality*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c(r)\|f\|_{L_{p_1,(|x|_{\mathbb{H}}+r)^\gamma}(\mathbb{H}^n)}, \tag{4}$$

where $c(r) > 0$ is independent of f holds for all $f \in L^{\mathrm{loc}}_{p_1}(\mathbb{H}^n)$ if and only if

$$\gamma \ge -\frac{Q}{p_2} \quad \text{and} \quad Q\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \le \alpha \le \frac{Q}{p_1} + \gamma. \tag{5}$$

If (5) holds, then the minimal constant $c(r)$ in (4) satisfies

$$c(r) \asymp r^{\alpha - Q(1/p_1 - 1/p_2) - \gamma}.$$

Proof. We apply Theorem 1 to the pair of functions $v_2(x) = \chi_{B(0,r)}(x), v_1(x) = (|x|_{\mathbb{H}} + r)^\gamma$. Then

$$\begin{aligned}
\mathcal{I}(v_1,v_2) &= \sup_{R>0} R^{\alpha-Q}\left(\int_0^R t^{Q-1}\chi_{(0,r)}(t)dt\right)^{1/p_2}\left(\int_0^R t^{Q-1}(t+r)^{-\gamma p_1'}dt\right)^{1/p_1'}\\
&= r^{Q/p_2+Q/p_1'-\gamma}\sup_{R>0} R^{\alpha-Q}\left(\int_0^{\frac{R}{r}} \tau^{Q-1}\chi_{(0,1)}(\tau)d\tau\right)^{1/p_2}\left(\int_0^{\frac{R}{r}} \tau^{Q-1}(\tau+1)^{-\gamma p_1'}d\tau\right)^{1/p_1'}\\
&= r^{\alpha+Q/p_2-Q/p_1-\gamma}\sup_{\rho>0} \rho^{\alpha-Q}\left(\int_0^{\rho} \tau^{Q-1}\chi_{(0,1)}(\tau)d\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{Q-1}(\tau+1)^{-\gamma p_1'}d\tau\right)^{1/p_1'}\\
&\equiv r^{\alpha+Q/p_2-Q/p_1-\gamma}K,
\end{aligned}$$

where $K = \max\{K_1, K_2\}$,

$$K_1 = \sup_{0<\rho\le 1} \rho^{\alpha-Q}\left(\int_0^{\rho} \tau^{Q-1}\chi_{(0,1)}(\tau)d\tau\right)^{1/p_2}\left(\int_0^{\rho} \tau^{Q-1}(\tau+1)^{-\gamma p_1'}d\tau\right)^{1/p_1'}$$

and

$$K_2 = \sup_{1<\rho\le\infty} \rho^{\alpha-Q} \left(\int_0^\rho \tau^{Q-1} \chi_{(0,1)}(\tau) d\tau \right)^{1/p_2} \left(\int_0^\rho \tau^{Q-1} (\tau+1)^{-\gamma p_1'} d\tau \right)^{1/p_1'}.$$

Next,

$$K_1 < \infty \Leftrightarrow \sup_{0<\rho\le 1} \rho^{\alpha+Q/p_2-Q/p_1} < \infty \Leftrightarrow \alpha + Q/p_2 - Q/p_1 \ge 0.$$

Moreover,

$$K_2 < \infty \Longleftrightarrow \sup_{1<\rho<\infty} \rho^{\alpha-Q} \left(\int_1^\rho \tau^{Q-1-\gamma p_1'} d\tau \right)^{1/p_1'} < \infty.$$

If $\gamma > Q/p_1'$, then $\int_1^\infty \tau^{Q-1-\gamma p_1'} d\tau < \infty$ and $K_2 < \infty$ since $\alpha < Q$.

If $\gamma = Q/p_1'$, then $K_2 < \infty \Leftrightarrow \sup\limits_{1\le\rho<\infty} \rho^{\alpha-Q} \ln \rho < \infty$. Therefore again $K_2 < \infty$ since $\alpha < Q$.

If $\gamma < Q/p_1'$, then

$$K_2 < \infty \Longleftrightarrow \sup_{1\le\rho<\infty} \rho^{\alpha-Q+Q/p_1'-\gamma} < \infty \Longleftrightarrow$$
$$\alpha - Q + \frac{Q}{p_1'} - \gamma \le 0 \Longleftrightarrow \gamma \ge \alpha - \frac{Q}{p_1}.$$

Inequality $\alpha < Q$, implies that $\alpha p_1 - Q < Q(p_1 - 1)$. Hence $K_2 < \infty \Leftrightarrow \gamma \ge \alpha - Q/p_1$.

Corollary 1. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q$. Then there exists $c > 0$ such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{Q/p_2} \left(\int_{\mathbb{H}^n} \frac{|f(x)|^{p_1}}{(|x|_{\mathbb{H}} + r)^{Q-\alpha p_1}} dx \right)^{\frac{1}{p_1}}, \tag{6}$$

for all $r > 0$ and for all $f \in L_{p_1}^{loc}(\mathbb{H}^n)$.

Proof. In the case $1 < p_1 \le p_2 < \infty$ (6) follows by Lemma 1 with $\gamma = \alpha - Q/p_1$.

If $0 < p_2 < p_1 < \infty$, by Hölder's inequality and (6) for $p_2 = p_1$ we have

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le (d_n r^Q)^{1/p_2 - 1/p_1} \|M_\alpha f\|_{L_{p_1}(B(0,r))} \le c r^{Q/p_2} \|M_\alpha f\|_{L_{p_1}(B(0,r))},$$

where d_n is the volume of the unit ball in $\mathbb{H}^n$ and $c > 0$ depends only on Q, p_1 and p_2.

The following lemma was proved in [2].

Lemma 2. *Let $\beta > 0$ and φ be a function non-negative and measurable on $\mathbb{H}^n$. Then for all $r > 0$*

$$\beta\, 2^{-\beta} \int_r^\infty \left(\int_{B(0,t)} \varphi(x) dx \right) \frac{dt}{t^{1+\beta}} \le \int_{\mathbb{H}^n} \frac{\varphi(x) dx}{(|x|_{\mathbb{H}} + r)^\beta} \le \beta \int_r^\infty \left(\int_{B(0,t)} \varphi(x) dx \right) \frac{dt}{t^{1+\beta}}.$$

Corollary 2. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$. Then there exists $c > 0$ such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{Q/p_2} \left(\int_r^\infty \left(\int_{B(0,r)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{Q-\alpha p_1+1}} \right)^{1/p_1} \tag{7}$$

for all $r > 0$ and for all $f \in L_{p_1}^{loc}(\mathbb{H}^n)$.

Proof. Inequality (7) follows from inequality (6) and Lemma 2.

Corollary 3. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$ and $Q\,(1/p_1 - 1/p_2)_+ \le \alpha \le Q/p_1$, then there exists $c > 0$ such that*

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{\alpha - Q(1/p_1 - 1/p_2)} \|f\|_{L_{p_1}(\mathbb{H}^n)} \tag{8}$$

for all $r > 0$ and for all $f \in L_{p_1}(\mathbb{H}^n)$.

Proof. If $0 < p_2 < \infty$, inequality (8) follows by inequality (6). For $0 < p_2 \le \infty$ and $\alpha = Q/p_1$ it also follows directly from the definition of $M_\alpha f$. Indeed, Hölder's inequality implies that

$$\|M_{Q/p_1} f\|_{L_\infty} \le \|f\|_{L_{p_1}(\mathbb{H}^n)}.$$

Hence

$$\|M_{Q/p_1} f\|_{L_{p_2}(B(0,r)} \le d_n^{1/p_2} r^{Q/p_2} \|f\|_{L_{p_1}(\mathbb{H}^n)}.$$

Let H be the Hardy operator

$$Hg = \int_0^r g(t)dt, \quad 0 < r < \infty.$$

Lemma 3. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$, $0 < \theta \le \infty$ and $w \in \Omega_\theta$. Then there exists $c > 0$ such that*

$$\|M_\alpha f\|_{LM_{p_2\theta,w}} \le c \|Hg\|_{L_{\theta/p_1,v}(0,\infty)}^{1/p_1}$$

for all $f \in L_{p_1}^{\text{loc}}(\mathbb{H}^n)$, where

$$g(t) = \int_{B\left(0,t^{1/(\alpha p_1 - Q)}\right)} |f(y)|^{p_1} dy \tag{9}$$

and

$$v(r) = \left[w\left(r^{1/(\alpha p_1 - Q)} \right) r^{(Q/p_2 + 1/\theta)/(\alpha p_1 - Q) - 1/\theta} \right]^{p_1}. \tag{10}$$

Proof. By Corollary 2

$$\|M_\alpha f\|_{LM_{p_2\theta,w}} = \left\| w(r)\|M_\alpha f\|_{L_{p_2}(B(0,r))}\right\|_{L_\theta(0,\infty)}$$
$$\leq c\left\| w(r) r^{Q/p_2}\left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1}dx\right)\frac{dt}{t^{Q-\alpha p_1+1}}\right)^{1/p_1}\right\|_{L_\theta(0,\infty)}$$
$$= c(Q-\alpha p_1)^{-1/p_1}\left\| w(r) r^{Q/p_2}\left(\int_0^{r^{\alpha p_1 - Q}} \left(\int_{B(0,\tau^{1/(\alpha p_1-Q)})} |f(x)|^{p_1}dx\right)d\tau\right)^{1/p_1}\right\|_{L_\theta(0,\infty)}$$

$$= c(Q-\alpha p_1)^{-1/p_1}\left(\int_0^\infty \left(w(r)r^{Q/p_2}\right)^\theta \left(\int_0^{r^{\alpha p_1-Q}} g(\tau)d\tau\right)^{\theta/p_1} dr\right)^{\frac{1}{\theta}}$$
$$= c\left(\int_0^\infty \left(w\left(\rho^{1/(\alpha p_1-Q)}\right)\rho^{Q/(p_2(\alpha p_1-Q))}\right)^\theta \rho^{1/(\alpha p_1-Q)-1}\left(\int_0^\rho g(\tau)d\tau\right)^{\theta/p_1} d\rho\right)^{\frac{1}{\theta}}$$
$$= c\|Hg\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)},$$

where $c>0$ depends only on Q, p_1, p_2 and α.

Corollary 4. *Let $1<p_1<\infty$, $0<p_2<\infty$, $Q\left(1/p_1-1/p_2\right)_+ \leq \alpha < Q/p_1$, $0<\theta\leq\infty$ and $w\in\Omega_{p_1,\theta}$. Then there exists $c>0$ such that*

$$\|M_\alpha f\|_{GM_{p_2\theta,w}} \leq c\sup_{x\in\mathbb{H}^n}\|H\left(g(x,\cdot)\right)\|^{1/p_1}_{L_{\theta/p_1,v}(0,\infty)}$$

for all $f\in L^{\rm loc}_{p_1}(\mathbb{H}^n)$, where v is defined by (10) *and*

$$g(x,t) = \int_{B\left(x,t^{1/(\alpha p_1-Q)}\right)} |f(y)|^{p_1}dy = \int_{B(0,t^{1/(\alpha p_1-Q)})} |f(y^{-1}\cdot x)|^{p_1}dy. \tag{11}$$

Theorem 2. *Let $1<p_1<\infty$, $0<p_2<\infty$, $Q\left(1/p_1-1/p_2\right)_+ \leq \alpha < Q/p_1$, $0<\theta_1,\theta_2\leq\infty$, $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$. Assume that H is bounded from $L_{\theta_1/p_1,v_1}(0,\infty)$ to $L_{\theta_2/p_1,v_2}(0,\infty)$ on the cone of all non-negative functions φ non-increasing on $(0,\infty)$ and satisfying $\lim\limits_{t\to\infty}\varphi(t)=0$, where*

$$v_1(r) = \left[w_1\left(r^{1/(\alpha p_1-Q)}\right)r^{1/((\alpha p_1-Q)\theta_1)-1/\theta_1}\right]^{p_1}, \tag{12}$$

$$v_2(r) = \left[w_2\left(r^{1/(\alpha p_1-Q)}\right)r^{(Q/p_2+1/\theta_2)/(\alpha p_1-Q)-1/\theta_2}\right]^{p_1}. \tag{13}$$

Then M_α is bounded from $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ and from $GM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $GM_{p_2\theta_2,w_2}(\mathbb{H}^n)$. (In the latter case we assume that $w_1\in\Omega_{p_1,\theta_1}$, $w_2\in\Omega_{p_2,\theta_2}$.)

Proof. By Lemma 3 applied to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$

$$\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} \le c\|Hg\|^{1/p_1}_{L_{\theta_2/p_1,v_2}(0,\infty)},$$

where $c>0$ is independent of f.

Since g is non-negative, non-increasing on $(0,\infty)$ and $\lim_{t\to+\infty} g(t)=0$ and H is bounded from $L_{\theta_1/p_1,v_1}(0,\infty)$ to $L_{\theta_2/p_1,v_2}(0,\infty)$ on the cone of functions containing g, we have

$$\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} \le c\|g\|^{1/p_1}_{L_{\theta_1/p_1,v_1}(0,\infty)},$$

where $c>0$ is independent of f.

Hence

$$\begin{aligned}\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} &\le c\left(\int_0^\infty v_1(t)^{\theta_1/p_1}\|f\|^{\theta_1}_{L_{p_1}\left(B\left(0,t^{1/(\alpha p_1-Q)}\right)\right)}dt\right)^{1/\theta_1}\\ &= c\,Q^{\frac{1}{\theta_1}}\left(\int_0^\infty v_1(r^{\alpha p_1-Q})^{\theta_1/p_1} r^{\alpha p_1-Q-1}\|f\|^{\theta_1}_{L_{p_1}(B(0,r))}dr\right)^{1/\theta_1}\\ &= c\,Q^{\frac{1}{\theta_1}}\left(\int_0^\infty \Big(w_1(r)\|f\|_{L_{p_1}(B(0,r))}\Big)^{\theta_1}dr\right)^{1/\theta_1}\\ &= c\,Q^{\frac{1}{\theta_1}}\|f\|_{LM_{p_1\theta_1,w_1}(\mathbb{H}^n)},\end{aligned}$$

where $c>0$ is independent of f.

In order to obtain explicit sufficient conditions on weight functions ensuring the boundedness of M_α, first we shall apply the following statement.

Lemma 4. *[2] Let $0<\theta_1\le\infty$, $0<\theta_2\le\infty$, v_1 and v_2 be functions positive and measurable on $(0,\infty)$. Then the condition*

$$\left\| v_2(r)\left\| t^{-(1-\theta_1)_+/\theta_1} v_1^{-1}(t)\right\|_{L_{\theta_1/(\theta_1-1)_+}(0,r)}\right\|_{L_{\theta_2}(0,\infty)}<\infty \tag{14}$$

is a sufficient conditions for the boundedness of H from $L_{\theta_1,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$ in the case $1\le\theta_1\le\infty$ and the boundedness H from $L_{\theta_1,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$ on the cone of all non-negative functions φ non-increasing on $(0,\infty)$ in the case $0<\theta_1<\infty$.

If $\theta_1=\infty$, then condition (14) is also necessary for the boundedness of H from $L_{\infty,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$.

Theorem 2 and Lemma 4 imply a sufficient condition for the boundedness of M_α from $LM_{p_1\infty,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

Theorem 3. *Let $1<p_1<\infty$, $0<p_2<\infty$, $Q\,(1/p_1-1/p_2)_+\le\alpha<Q$, $0<\theta_2\le\infty$, $w_2\in\Omega_{\theta_2}$.*

1. For $\alpha<Q/p_1$, let $w_1\in\Omega_{\theta_1}$ and

$$\left\| w_2(r) r^{Q/p_2}\left\| w_1^{-1}(t)t^{\alpha-Q/p_1-1/\min\{p_1,\theta_1\}}\right\|_{L_s(r,\infty)}\right\|_{L_{\theta_2}(0,\infty)}<\infty. \tag{15}$$

where $s = p_1\theta_1/(\theta_1 - p_1)_+$. *(If* $\theta_1 \le p_1$, *then* $s = \infty$.*) Then* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

2. For $\alpha = Q/p_1$, *let*

$$w_2(r)r^{\alpha-Q(1/p_1-1/p_2)} \in L_{\theta_2}(0,\infty). \tag{16}$$

Then M_α *is bounded from* $L_{p_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

Corollary 5. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$, $0 < \theta_2 \le \infty$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_{\theta_2}$ *and let*

$$\left\| w_2(r)r^{Q/p_2}\left(\int_r^\infty \frac{dt}{w_1^{p_1}(t)t^{Q+1-\alpha p_1}}\right)^{1/p_1}\right\|_{L_{\theta_2}(0,\infty)} < \infty. \tag{17}$$

Then M_α *is bounded from* $LM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ *and from* $GM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $GM_{p\theta_2,w_2}(\mathbb{H}^n)$. *(In the latter case we assume that* $w_1 \in \Omega_{p_1,\infty}$, $w_2 \in \Omega_{p_2,\theta_2}$.*)*

Corollary 6. *Let* $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$, $w_1 \in \Omega_\infty$, $w_2 \in \Omega_\infty$ *and let for some* $c > 0$ *for all* $r > 0$

$$\int_r^\infty \frac{dt}{w_1^{p_1}(t)t^{Q+1-\alpha p_1}} \le \frac{c}{w_2^{p_1}(r)r^{\frac{Qp_1}{p_2}}}. \tag{18}$$

Then M_α *is bounded from* $LM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\infty,w_2}(\mathbb{H}^n)$ *and from* $GM_{p_1\infty,w_1}(\mathbb{H}^n)$ *to* $GM_{p_2\infty,w_2}(\mathbb{H}^n)$. *(In the latter case we assume that* $w_1 \in \Omega_{p_1,\infty}$, $w_2 \in \Omega_{p_2,\infty}$.*)*

Remark 1. *Note that, the Corollary 6 was proved in [8], see also [9, 12, 13].*

For the majority of cases the necessary and sufficient conditions for the validity of

$$\|H\varphi\|_{L_{\frac{\theta_2}{p_1},v_2}(0,\infty)} \le c\|\varphi\|_{L_{\frac{\theta_1}{p_1},v_1}(0,\infty)}, \tag{19}$$

where $c > 0$ is independent of φ, for all non-negative decreasing functions φ are known, for detailed information see [18], [19]. Application of any of those conditions gives sufficient conditions for the boundedness of the fractional maximal operator from $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ and from $GM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $GM_{p_2\theta_2,w_1}(\mathbb{H}^n)$.

However, there is no guarantee that the application of the necessary and sufficient conditions on v_1 and v_2 ensuring the validity of (19) implies the necessary and sufficient conditions for the boundedness of M_α from $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

Fortunately for certain values of the parameters this is the case, namely for $1 < p_1 < \infty$, $0 < p_2 < \infty$, $Q\,(1/p_1 - 1/p_2)_+ \le \alpha < Q/p_1$, $0 < \theta_1 \le \theta_2 < \infty$, $\theta_1 \le p_1$.

Note that in this case the necessary conditions (coinciding with the sufficient ones) for the validity of inequality (19) for decreasing functions are obtained by taking $\varphi = \chi_{(0,t)}$ with an arbitrary $t > 0$.

Since in the proof of Theorem 2 inequality (19) is applied to the function $\varphi = g$, where g is given by (9), it is natural to choose, as test functions, functions f_t, $t > 0$, for which $\int_{B\left(0,u^{1/(\alpha p_1-Q)}\right)}|h_t(y)|^{p_1}dy$ is equal or close to $B(t)\chi_{(0,t)}(u)$, $u>0$, where $B(t)$ is independent of u. The simplest choice of f satisfying this requirement is

$$f_t(x)=\chi_{B(0,2t)\setminus B(0,t)}(x), \quad x\in\mathbb{H}^n,\ t>0. \tag{20}$$

Note that,

$$\|f_t\|_{L_{p_1}(B(0,r))}=0,\ \ 0<r\le t, \quad \|f_t\|_{L_{p_1}(B(0,r))}\le ct^{n/p_1}, \quad t<r<\infty, \tag{21}$$

where $c>0$ depends only on Q and p_1.

For functions F,G defined on $(0,\infty)\times(0,\infty)$ we shall write $F\asymp G$ if there exist $c,c'>0$ such that $cF(r,t)\le G(r,t)\le c'F(r,t)$ for all $r,t\in(0,\infty)$.

Lemma 5. *If* $0\le\alpha<Q$, $0<p<\infty$, *then*

$$\|M_\alpha f_t\|_{L_p(B(0,r))}\asymp t^\alpha r^{Q/p}\begin{cases}\left(\frac{t}{r+t}\right)^{\min\{Q-\alpha,Q/q\}}, & p\ne\frac{Q}{Q-\alpha},\\ \left(\frac{t}{r+t}\right)^{Q/p}\ln\left(e+\frac{r}{t}\right), & p=\frac{Q}{Q-\alpha}.\end{cases}$$

Theorem 1. (1) *Let* $1<p_1\le\infty$, $0<p_2\le\infty$, $0\le\alpha<Q$, $0<\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$. *If* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$, *then there exists a constant* $C_1>0$ *such that for all* $t>0$,

$$t^{\alpha-\frac{Q}{p_1}+\min(Q-\alpha,Q/p_2)}\left\|\frac{w_2(r)r^{Q/p_2}}{(t+r)^{\min(Q-\alpha,Q/p_2)}}\right\|_{L_{\theta_2}(0,\infty)}\le C_1\|w_1\|_{L_{\theta_1(t,\infty)}}.$$

(2) *Let* $1<p_1<\infty$, $0<p_2<\infty$, $0<\theta_1\le\theta_2\le\infty$, $\theta_1\le p_1$, $Q\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+\le\alpha<\frac{Q}{p_1}$, $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$ *and the equality* $\left\|\frac{w_2(r)r^{Q/p_2}}{(t+r)^{Q/p_1-\alpha}}\right\|_{L_{\theta_2(0,\infty)}}\le C_2\|w_1\|_{L_{\theta_1(t,\infty)}}$ $(C_2>0)$ *be true for all* $t>0$; *then* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$. If also $w_1\in\Omega_{p_1,\theta_1}$, $w_2\in\Omega_{p_2,\theta_2}$, then M_α is bounded from $GM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ to $GM_{p_2\theta_2,w_2}(\mathbb{H}^n)$.

(3) *In particular, for* $1<p_1<\infty$, $0<p_2<\infty$, $0<\theta_1\le\theta_2\le\infty$, $\theta_1\le p_1$, $Q\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\le\alpha<\frac{Q}{p_1}$, $w_1\in\Omega_{\theta_1}$, $w_2\in\Omega_{\theta_2}$ *the operator* M_α *is bounded from* $LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$ *to* $LM_{p_2\theta_2,w_2}(\mathbb{H}^n)$ *if and only if for all* $t>0$,

$$\|w_2(r)r^{Q/p_2}(t+r)^{-Q/p_2}\|_{L_{\theta_2(0,\infty)}}\le C_3\|w_1\|_{L_{\theta_1}(t,\infty)}.$$

Here the constant $C_3>0$ *is independent of* t.

Note that, in the Euclidean setting Theorem 1 was proved in [2].

Proof. Sufficiency. It is known [19] that for $\theta_1\le\theta_2\le\infty$ the necessary and sufficient condition for the validity of (19) for all non-negative decreasing on $(0,\infty)$ functions φ has the form: for some $c>0$

$$\|v_2(r)\min\{t,r\}\|_{L_{\theta_2/p_1}(0,\infty)}\le c\|v_1(r)\|_{L_{\theta_1/p_1}(0,t)}$$

for all $t > 0$. Applying this condition to the functions v_1 and v_2 given by (12) and (13) we obtain

$$\left\| w_2(r) \frac{r^{Q/p_2}}{(t+r)^{Q/p_1-\alpha}} \right\|_{L_{\theta_2}(0,\infty)} \le c\|w_1\|_{L_{\theta_1}(t,\infty)}. \tag{22}$$

Indeed, taking into account equalities (12) and (13) and replacing $r^{-\frac{p_2}{Q}}$ by ρ and $t^{-\frac{p_2}{Q}}$ by τ, we get that for some $c > 1$

$$\left\| w_2(\rho)\rho^{Q/p_2} \min\{\tau^{\alpha-Q/p_1}, \rho^{\alpha-Q/p_1}\} \right\|_{L_{\theta_2}(0,\infty)} \le c \left\| w_1 \right\|_{L_{\theta_1}(\tau,\infty)}$$

for all $\tau > 0$.

Hence (22) follows since

$$\rho^{Q/p_2} \min\{\tau^{\alpha-Q/p_1}, \rho^{\alpha-Q/p_1}\} \asymp \frac{\rho^{Q/p_2}}{(\rho+\tau)^{Q/p_1-\alpha}}.$$

Necessity. Assume that, for some $c > 0$ and for all $f \in LM_{p_1\theta_1,w_1}(\mathbb{H}^n)$

$$\|M_\alpha f\|_{LM_{p_2\theta_2,w_2}(\mathbb{H}^n)} \le c\|f\|_{LM_{p_1\theta_1,w_1}(\mathbb{H}^n)}. \tag{23}$$

In (23) take $f = f_t$, where f_t is defined by (20). Then by (21) the right-hand side of (23) does not exceed a constant multiplied by $t^{Q/p_1}\|w_1\|_{L_{\theta_1}(t,\infty)}$. Furthermore by Lemma 5 the left-hand side of inequality (23) is greater than or equal to a constant multiplied by

$$t^{\alpha+\min\{Q-\alpha,Q/p_2\}} \left\| w_2(r) \frac{r^{Q/p_2}}{(t+r)^{\min\{Q-\alpha,Q/p_2\}}} \right\|_{L_{\theta_2}(0,\infty)}.$$

This works foe the case $\alpha = \frac{n}{p_2'}$ too, since $\ln(e + \frac{r}{t}) \ge 1$.

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