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Boundary Value Problems for Cauchy-Riemann Inhomogeneous Equation with Nonlocal Boundary Conditions in a Rectangle

N.A. Aliyev, L.F. Fatullayeva*, N.B. Mamedova

Abstract. A boundary value problem for a first order elliptic type equation with nonlocal boundary conditions in a rectangular domain is considered. The problem statement is such that four points of the boundary simultaneously move along the boundaries(every point is situated in one of sides of the rectangle). These points move so that the Carleman conditions are fulfilled, i.e. the neighboring points either move away from one boundary point or they approach to one of the boundary points. Carleman has called such problems the well-psed problems.

Key Words and Phrases: Cauchy-Riemann equation, nonlocal boundary condition, necessary condition, singularity, regularization, Fredholm property.

2010 Mathematics Subject Classifications: 35F15, 35C60

1. Introduction

As is known from the course of mathematical functions equations and partial equations, boundary value problems with local conditions are mainly considered for elliptic type equations [7], [8], [10],[11] .

Further, a boundary value problem with local boundary conditions Dirichlet condition was considered for a first order elliptic type equation (Cauchy-Riemann equation) though such problems are ill-posed [2],[4].

Note that for an ordinary linear differential equation, the number of both initial and boundary conditions coincide with the order of the equation under consideration [12],[5], while for a partial equation the number of initial conditions coincides with the highest order of time derivative contained in the considered equation. As for a local boundary condition (if the number of space variables is greater than a unit with arbitrary boundaries) their number coincides with the half of higher derivatives in space variables contained in the considered equations [9],[6].

Note that linear local boundary conditions with global addends (integrals) are also encountered in the the paper [6], while nonlocal ones in our case [14](with sewing of

*Corresponding author.

boundary values) for many-dimensional boundary value problems are encountered in [3]. Note that [14] contains over 250 works devoted mainly to boundary value problems.

In [14] one can find boundary value problems both for elliptic (both of even and odd orders), parabolic type equations and also for mixed and composite type equations.

There are also boundary value problems for fractional derivative ordinary and partial equations.

Finally, note that we considered both the Cauchy problem and a boundary value problem for a linear differential equation with continuously alternating order of derivative [1].

2. Problem statement

Let us consider the following boundary value problem:

$$\frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} = f(x), \quad x_2 \in (a_k, b_k), \quad k = 1, 2; \tag{1}$$

$$\alpha_{j1}(t)u(a_1 + t(b_1 - a_1), a_2) + \alpha_{j2}(t)(b_1, b_2 + t(a_2 - b_2)) + \alpha_{j3}(t)u(b_1 + t(a_1 - b_1), b_2)$$
$$+\alpha_{j4}(t)u(a_1, a_2 + t(b_2 - a_1)) = \alpha_j(t), \quad j = 1, 2; \ t \in [0, 1], \tag{2}$$

where $x = (x_1, x_2)$, $b_k > a_k > 0$, $k = 1, 2; i = \sqrt{-1}$ $f(x)$ for $x_k \in (a_k, b_k)$, $k = 1, 2$; $\alpha_{jk}(t)$ $\alpha_j(t)$ for $j = 1, 2$; $k = \overline{1, 4}$ are continuous functions and boundary conditions (2) that are linear independent.

Remark 1. *As is seen from the statements of of problems(1)-(2) the Carleman conditions [3] are fulfilled i.e. on the boundary four points move simultaneously and the neighboring points move away from one boundary point or they approach to one boundary point.*

Remark 2. *We show that if simultaneously more than point move along the boundary, i.e. the Carleman conditions are not fulfilled, then the problem is ill-posed, i.e. may have no solution or have a non-unique solution.*

Main relations: As is known, the fundamental solution of the Caushy-Riemann equation (1) has the form ([13]:)

$$U(x - \xi) = \frac{1}{\pi}\frac{1}{x_2 - \xi_2 + i(x_1 - \xi_1)}. \tag{3}$$

For determining the main relation, we multiply equation (1) by fundamental solution (3), integrate with respect to the domain $D = \{x = (x_1, x_2) : x_k \in (a_k, b_k), k = 1, 2$ apply the Ostrogradsky-Gauss formula and have:

$$\int_D \frac{\partial u(x)}{\partial x_2} U(x - \xi)dx + i\int_D \frac{\partial u(x)}{\partial x_1} U(x - \xi)dx = \int_D f(x)U(x - \xi)dx$$

$$\int_\Gamma u(x)U(x - \xi)\left[\cos(\nu, x_2) + i\cos(\nu, x_1)\right]dx$$

$$-\int_D f(x)U(x-\xi)dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma, \end{cases} \tag{4}$$

where $\Gamma = \partial D$ is the boundary of the domain D, ν is the external normal to the boundary Γ of domain D.

The basic relation (4), consists of two parts. The first part corresponding to $\xi \in D$ gives the general solution of equation (1) determined in domain D, the second part corresponding to $\xi \in \Gamma$ is a necessary condition.

For giving necessary conditions at first we write the main relation (4) in the expanded form, i.e.

$$-\frac{1}{2\pi}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{a_2-\xi_2+i(a_1-\tau(b_1-a_1)-\xi_1)}d\tau + \frac{i}{2\pi}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{b_2+\tau(a_2-b_2)-\xi_2+i(b_1-\xi_1)}d\tau$$

$$+\frac{1}{2\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{b_2-\xi_2+i(b_1+\tau(a_1-b_1)-\xi_1)}d\tau - \frac{i}{2\pi}\int_0^1 \frac{u(a_1,a_2+\tau(b_2-a_1))}{a_2+\tau(b_2-a_2)-\xi_2+i(a_1-\xi_1)}d\tau$$

$$-\frac{1}{2\pi}\frac{f(x)}{x_2-\xi_2+i(x_1-\xi_1)}dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma. \end{cases} \tag{5}$$

The necessary conditions:

$$u(a_1+\tau(b_1-a_1),a_2) = \frac{i}{\pi(b_1-a_1)}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{\tau-t}d\tau$$

$$+\frac{i}{\pi}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{(a_2-b_2)(\tau-t)+i(b_1-a_1)(1-t)}d\tau + \frac{1}{\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{b_2-a_2+i(b_1-a_1)(1-\tau-t)}d\tau$$

$$-\frac{i}{\pi}\int_0^1 \frac{u(a_1,a_2+\tau(b_2-a_1))}{(b_2-a_2)\,\tau-i(b_1-a_1)t}d\tau - \frac{1}{\pi}\int_D \frac{f(x)}{x_2-a_2+i(x_1-a_1-t(b_1-a_1))}dx. \tag{6}$$

$$u(b_1,b_2+t(a_2-b_2)) = -\frac{1}{\pi}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{(a_2-b_2)(1-t)+i(a_1-b_1)(1-\tau)}d\tau$$

$$+\frac{i}{\pi(a_2-b_2)}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{\tau-t}d\tau + \frac{1}{\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{(b_2-a_2)\,t+i(a_1-b_1)\tau}d\tau$$

$$-\frac{i}{\pi}\int_0^1 \frac{u(a_1,a_2+\tau(a_2-b_2))}{(a_2-b_2)(1-\tau-t)+i(a_1-b_1)}d\tau - \frac{1}{\pi}\int_D \frac{f(x)}{x_2-b_2-t(a_2-b_2)+i(x_1-b_1)}dx, \tag{7}$$

$$u(b_1+t(a_1-b_1),b_2)=-\frac{1}{\pi}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{a_2-b_2+i(a_1-b_1)(1-\tau-t)}d\tau$$

$$+\frac{i}{\pi}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{(a_2-b_2)\tau+i(b_1-a_1)t}d\tau-\frac{1}{\pi(a_1-b_1)}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{\tau-t}d\tau$$

$$-\frac{i}{\pi}\int_0^1 \frac{u(a_1,a_2+\tau(b_2-a_2))}{(a_2-b_2)(1-\tau)+i(a_1-b_1)(1-t)}d\tau-\frac{1}{\pi}\int_D \frac{f(x)}{x_2-b_2+i(x_1-b_1-t(a_1-b_1))}dx, \tag{8}$$

$$u(a_1,a_2+t(b_2-a_2))=-\frac{1}{\pi}\int_0^1 \frac{u(a_1+\tau(b_1-a_1),a_2)}{(a_2-b_2)t+i(b_1-a_1)\tau}d\tau$$

$$+\frac{1}{\pi}\int_0^1 \frac{u(b_1,b_2+\tau(a_2-b_2))}{(b_2-a_2)(1-\tau-t)+i(b_1-a_1)}d\tau+\frac{1}{\pi}\int_0^1 \frac{u(b_1+\tau(a_1-b_1),b_2)}{(b_2-a_2)(1-t)-i(a_1-b_1)(1-\tau)}d\tau$$

$$-\frac{i}{(b_2-a_2)\pi}\int_0^1 \frac{u(a_1,a_2+\tau(b_2-a_2))}{\tau-t}d\tau-\frac{1}{\pi}\int_D \frac{f(x)}{x_2-a_2-t(b_2-a_2)+i(x_1-a_1)}dx. \tag{9}$$

This establishes the following statement:

Theorem 1. *If $f(x)$ is a continuous function, then every solution of equation (1), determined in domain D satisfies necessary singular conditions (6)-(9)*

Remark 3. *As was mentioned above, every solution of equation (1) determined in domain D is found from the main relation (5) for $\xi \in D$, i.e. the first expression of the main relation (5)*

Regularization: Proceeding from (6)-(9), we create the following linear combination:

$$\alpha_{j1}(t)(b_1-a_1)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)(a_2-b_2)u(b_1,b_2+t(a_2-b_2))$$

$$+\alpha_{j3}(t)(b_1-a_1)u(b_1+t(a_1-b_1),b_2)+\alpha_{j4}(t)(a_2-b_2)u(a_1,a_2+t(b_2-a_2))$$

$$=\frac{i}{\pi}\int_0^1 [\alpha_{j1}(\tau)u(a_1+\tau(b_1-a_1),a_2)+\alpha_{j2}(\tau)u(b_1,b_2+t(a_2-b_2))$$

$$+\alpha_{j3}(\tau)u(b_1+\tau(a_1-b_1),b_2)+\alpha_{j4}(\tau)u(a_1,a_2+\tau(b_2-a_2))]\,\frac{d\tau}{\tau-t}+...,\tag{10}$$

where, when obtaining (10) it was supposed that

$$\alpha_{jk}(t)\in H^{\mu}(0,1),\quad j=1,2;\quad k=\overline{1,4};\mu\in(0,1),\tag{11}$$

$H^{\mu}(0,1)$ is a Holder class with the exponent $\mu \in (0,1)$, the dots $(\cdots)$ denotes the sum of nonsingular addends.

Taking boundary condition (2) into account in (10), we get

$$\alpha_{j1}(t)(b_1-a_1)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)(a_2-b_2)u(b_1,b_2+t(a_2-b_2))$$
$$+\alpha_{j3}(t)(b_1-a_1)u(b_1+t(a_1-b_1),b_2)+\alpha_{j4}(t)(a_2-b_2)u(a_1,a_2+t(b_2-a_2))$$
$$=\frac{i}{\pi}\int_0^1\frac{\alpha_j(\tau)}{q-t}dt+...,j=1,2;\ t\in[0,1]. \quad (12)$$

As is seen from (12), as the first part does not contain an unknown function, then it exists in the Cauchy sense.

If we suppose

$$\alpha_j(t)\in C^{(1)}(0,1),\quad j=1,2;\quad \alpha_j(0)=\alpha_j(1)\quad j=1,2; \quad (13)$$

then the integral in the right hand side of (12) exists in the ordinary sence .

This establishes

Theorem 2. *Under conditions of theorem 1, if conditions (11) and (12) hold, then relations (13) are regular.*

Fredholm property: Now combining the given boundary condition (2) with regular expressions (12), we have:

$$\alpha_{j1}(\tau)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)u(b_1,b_2+t(a_2-b_2))$$
$$+\alpha_{j3}(\tau)u(b_1+t(a_1-b_1),b_2)+\alpha_{j4}(t)u(a_1,a_2+t(b_2-a_2))=\alpha_j(t),\quad j=1,2;\ t\in[0,1],$$
$$\alpha_{j1}(t)(b_1-a_1)u(a_1+t(b_1-a_1),a_2)+\alpha_{j2}(t)(a_2-b_2)u(b_1,b_2+t(a_2-b_2))$$
$$+\alpha_{j3}(t)(b_1-a_1)u(b_1+t(a_1-b_1),b_2)$$
$$+\alpha_{j4}(t)(a_2-b_2)u(a_1,a_2+t(b_2-a_2))=...j=1,2;\ t\in[0,1]. \quad (14)$$

Let

$$\Delta(t)=\begin{vmatrix}\alpha_{11}(t) & \alpha_{12}(t) & \alpha_{13}(t) & \alpha_{14}(t)\\ \alpha_{21}(t) & \alpha_{22}(t) & \alpha_{23}(t) & \alpha_{24}(t)\\ \alpha_{11}(t)(b_1-a_1) & \alpha_{12}(t)(a_2-b_2) & \alpha_{13}(t)(b_1-a_1) & \alpha_{14}(t)(a_2-b_2)\\ \alpha_{21}(t)(b_1-a_1) & \alpha_{22}(t)(a_2-b_2) & \alpha_{23}(t)(b_1-a_1) & \alpha_{24}(t)(a_2-b_2)\end{vmatrix}\neq 0, \quad (15)$$

Then from (14) we get a system of normal form of Fredholm integral equations of second kind with nonsingular kernels.

We get the following statement:

Theorem 3. *Let the condition of theorem 3 hold, then if condition (15) is valid, the stated boundary value problem (1)-(2) is Fredholm.*

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N.A. Aliyev
Baku State University, Baku AZ 1148, Azerbaijan

L.F. Fatullayeva
Baku State University, Baku AZ 1148, Azerbaijan
E-mail:laura_fat@rambler.ru

N.B. Mamedova
Baku State University, Baku AZ 1148, Azerbaijan
E-mail: nt.mammadova@mail.ru

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