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Some Questions of Atomic Decompositions and Frames

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Abstract. Frames in Hilbert and Banach spaces are considered and their properties in the context of Noetherian mapping are studied in this paper. Atomic decompositions in Banach spaces are also considered. The concept of $\mathscr{K}$-closeness is introduced. The stability of frame properties and atomic decompositions with respect to $\mathscr{K}$-closeness is proved. The concept of t-frame associated with the tensor product of Hilbert spaces is introduced. All the properties of ordinary frames are extended to this case. Noetherian perturbation of t-frames is considered. The stability of t-frameness with respect to quadratic closeness is proved.

Key Words and Phrases: frame, atomic decomposition, Noetherian mapping, close frames, t-frames, tensor product.

2010 Mathematics Subject Classifications: 42C40, 42C15

1. Introduction

Frame theory has a diverse array of applications in many scientific fields, that's why the global interest in it is growing at a rapid pace. Many monographs and review articles have been dedicated to it (see, e.g., [6-8,10-12,15,24], etc.). This theory dates back to the seminal paper by R.J.Duffin and A.C.Schaeffer [13]. Later, there appeared various generalizations of the concept of frame such as Banach frames, p-frames ([1, 9, 19]), etc., and the methods to build a frame have been developed. One of these methods is a perturbation method. Many results have been obtained using this method in the context of classical Paley-Wiener theorem on the perturbation of orthonormal basis (more details on these results can be found in O.Christensen's [7, 8, 9]).

It should be noted that, unlike the Hilbert case, the definition of a Banach frame in general does not guarantee the atomic decomposition for arbitrary element of the space (or for any element of the closure of the linear span of considered system). In special cases, such decompositions hold. L^p-case has been considered by A. Aldroubi, Q. Sun, W.Sh. Tang in [1], where the concept of p-frame was introduced and the atomic decomposition with respect to shift invariant subspaces of L^p were obtained. This idea has been extended to the general Banach case by O.Christensen and D.T. Stoeva [9]. The above-cited works introduced the concept of q-Riesz basis with respect to the Banach space, which is the

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generalization of the Riesz basis introduced by N.K. Bari in [2]. Similar results have been obtained in [3-5,20-23]. There are various generalizations of the concept of frames, and the number of research works dedicated to this topic increases (we refer the reader to [1,9,19,25-31,42-44]).

It should be noted that the interest in the theory of frames is growing not only because of its wide applications in various fields of science, but also because of its theoretical importance. As a striking example of this fact, we can mention a relationship between the theory of frames and the well-known Kadison-Singer problem of 1959. Slightly modified but equivalent statements of this famous problem have been studied extensively in many areas of mathematics such as theory of frames, operator theory, time-frequency analysis, etc. For the results concerning this problem we refer the reader to [32-39] and references therein.

This work consist of two parts. In Part I we consider the frames in Noetherian mapping in Hilbert and Banach spaces. Atomic decompositions in Hilbert and Banach spaces are also considered. More precisely, we consider the perturbations of atomic decompositions and frames in Hilbert and Banach spaces. The concept of $\mathscr{K}$-closeness is introduced and the stability of atomic decomposition and frame properties with respect to this closeness is proved.

In Part II we consider the tensor product of Hilbert spaces and the bilinear mapping generated by this product. We introduce the concept of *t-frame* using the *Hilbert-valued* scalar product. Theoretically, some facts about *t-frames* can be established using earlier results for *G-frames* obtained in [29, 30]. But, the concept of *t-frame* allows many facts relating to ordinary frames to be extended to the case of *t-frame*. The properties of *t-frame* in Noetherian mapping are also studied. The stability of *t-frameness* with respect to quadratic closeness is proved. The results of this work was published in [3,4,31,42-44].

PART I

Frames In Noetherian Mapping. $\mathscr{K}$-Close Frames.

2. Needful Information

We will use the standard notation. $\mathbb{N}$ will be a set of all positive integers; Banach space will be referred to as B-space; Hilbert space will be referred to as H-space; $\|\cdot\|_X$ will denote a norm in the space X; $(\cdot;\cdot)_X$ will denote a scalar product in X; $L[M]$ will denote the linear span of the set M and $\overline{M}$ will stand for the closure of M; δ_{nk} will be the Kronecker symbol; $\circ$ will be a symbol of composition; X^* will stand for a space conjugated to X; D_T ($\mathscr{R}_T$) will denote a domain (range of definition) of the operator T; I_X will be an identity operator in X; $KerT$ will stand for the kernel of the operator T; $L(X;Y)$ will denote a B-space of bounded operators from X to Y; $\dim X$ will stand as usual for a (linear) dimension of X; and by $X/_{X_0}$ we will denote a factor space with respect to the subspace $X_0 \subset X$. Throughout this paper $\vec{x}$ will be denoted $\vec{x} \equiv \{x_n\}_{n\in\mathbb{N}}$.

Let us recall the definition of Noetherian operator. Let $X,\ Y$ be B-spaces and $T : X \to Y$ be a linear operator. If $\bar{\mathscr{R}}_T = \mathscr{R}_T$ and $\alpha = \dim KerT < +\infty$, $\beta = \dim Y/_{\mathscr{R}_T} < +\infty$, then the operator T is called Noetherian and the number æ $= \alpha - \beta$ is called the index of the operator T. For $\alpha = \beta$, T is called a Fredholm operator.

We will also give the concepts of left and right regularizers. Operator $R_1 \in L(Y;\, X)$ ($R_2 \in L(Y;\, X)$) is called the left (right) regularizer of operator $A \in L(X;\, Y)$, if

$$R_1 A = I_X + T_X \left(AR_2 = I_Y + T_Y\right),$$

where $T_X\,(T_Y)$ is a completely continuous operator in $X\,(Y)$. It is known that (see, e.g., [18]) any left regularizer of Noetherian operator is also its right regularizer, and the converse is also true. By regularizer we will mean the left or the right regularizer.

The following statement is true.

Statement 2.1. *Let $A \in L(X;\, Y)$ be a Noetherian operator and $R \in L(Y;\, X)$ be its regularizer. Let the operator $B \in L(X;\, Y)$ satisfy the condition $\|B\| < \|R\|^{-1}$. Then the operator $A + B$ is also Noetherian and its index is equal to the index of the operator A : æ$(A+B) =$ æ(A).*

More details about these and other facts relating to Noetherian operators can be found in [18].

Let's recall some concepts and facts from the theory of frames. First, let us give a definition of atomic decomposition.

Definition 2.2. *Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars. Let $\{f_k\}_{k\in\mathbb{N}} \subset X$, $\{g_k\}_{k\in\mathbb{N}} \subset X^*$. Then $\left(\{g_k\}_{k\in\mathbb{N}}\ ;\ \{f_k\}_{k\in\mathbb{N}}\right)$ is an atomic decomposition of X with respect to $\mathscr{K}$ if :*
(i) $\{g_k(f)\}_{k\in\mathbb{N}} \in \mathscr{K}$, $\forall f \in X$;
(ii) $\exists A, B > 0$: $A\,\|f\|_X \le \left\|\{g_k(f)\}_{k\in\mathbb{N}}\right\|_{\mathscr{K}} \le B\,\|f\|_X$, $\forall f \in X$;
(iii) $f = \sum_{k=1}^{\infty} g_k(f)\, f_k$, $\forall f \in X$.

The concept of frame is a generalization of the concept of atomic decomposition.

Definition 2.3. *Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars. Let $\{g_k\}_{k\in\mathbb{N}} \subset X^*$, and $S : \mathscr{K} \to X$ be some bounded operator. Then $\left(\{g_k\}_{k\in\mathbb{N}}\ ;\ S\right)$ forms a Banach frame for X with respect to $\mathscr{K}$ if :*
(i) $\{g_k(f)\}_{k\in\mathbb{N}} \in \mathscr{K}$, $\forall f \in X$;
(ii) $\exists A, B > 0$: $A\,\|f\|_X \le \left\|\{g_k(f)\}_{k\in\mathbb{N}}\right\|_{\mathscr{K}} \le B\,\|f\|_X$, $\forall f \in X$;
(iii) $S\ \left[\{g_k(f)\}_{k\in\mathbb{N}}\right] = f$, $\forall f \in X$.

A and B will be called frame bounds.

The following statement is true.

Statement 2.4. *[7] Let X be a B-space and $\mathscr{K}$ be a B-space of the sequences of scalars with a canonical basis $\{\delta_n\}_{n\in\mathbb{N}}$, where $\delta_n \equiv \{\delta_{kn}\}_{k\in\mathbb{N}}$. Let $\{g_k\}_{k\in\mathbb{N}} \subset X^*$ and $S \in L(\mathscr{K}; X)$. Then the following statements are equivalent to each other:*
(i) $\left(\{g_k\}_{k\in\mathbb{N}}\ ;\ S\right)$ *forms a Banach frame for X with respect to $\mathscr{K}$;*
(ii) $\left(\{g_k\}_{k\in\mathbb{N}}\ ;\ \{S(\delta_k)\}_{k\in\mathbb{N}}\right)$ *is an atomic decomposition of X with respect to $\mathscr{K}$.*

Future $\mathscr{K}$ will be called as K-space.

Separately we will consider the Hilbert case of spaces.

A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space H is a Riesz basic sequence if there are constants $A, B > 0$ such that for all scalars $\{a_i\}_{i\in I}$ we have

$$A\sum_{i\in I}|a_i|^2 \leq \left\|\sum_{i\in I} a_i f_i\right\|^2 \leq B\sum_{i\in I}|a_i|^2 .$$

We call $\sqrt{A}$, $\sqrt{B}$ the lower and upper Riesz basis bounds for $\{f_i\}_{i\in I}$, respectively. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans H we call it a Riesz basis for H . So "$\{f_i\}_{i\in I}$ is a Riesz basis for H" means there is an orthonormal basis $\{e_i\}_{i\in I}$ such that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is bounded. That is, $0 < \inf_{i\in I}\|f_i\| \leq \sup_{i\in I}\|f_i\| < \infty$.

Hilbert space frames were introduced by Duffin and Schaeffer [13] to address some very deep problems in nonharmonic Fourier series (see [15]). A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space H is called a frame for H if there are constants $0 < A \leq B < \infty$ (called the lower and upper frame bounds, respectively) such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in I}|\langle f, f_i\rangle|^2 \leq B\|f\|^2 . \tag{1}$$

If we only have the right hand inequality in (1) we call $\{f_i\}_{i\in I}$ a Bessel sequence with Bessel bound B. If $A = B$, we call this an A-tight frame and if $A = B = 1$, it is called a Parseval frame. If all the frame elements have the same norm, this is an equal norm frame and if the frame elements are of unit norm, it is a unit norm frame. Obviously, $\|f_i\|^2 \leq B$. If also $\inf\|f_i\| > 0$, then $\{f_i\}_{i\in I}$ is a bounded frame. The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are the frame coefficients of the vector $f \in H$. If $\{f_i\}_{i\in I}$ is a Bessel sequence, then the *synthesis* operator for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : l_2(I) \to H$ given by $T(e_i) = f_i$ for all $i \in I$. The *analysis* operator for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f) = \sum_{i\in I}\langle f, f_i\rangle e_i$. In particular, $\|T^*f\|^2 = \sum_{i\in I}|\langle f, f_i\rangle|^2$, for all $f \in H$, and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2$. In view of (1) we have

Theorem 2.5. *Let H be a Hilbert space and $T : l_2 \to H$, $T(e_i) = f_i$ be a bounded linear operator. The following are equivalent:*

(1) $\{f_i\}_{i\in I}$ is a frame for H.

(2) The operator $T$is bounded, linear, and onto.

(3) The operator T^* is an (possibly into) isomorphism.

Moreover, if $\{f_i\}_{i\in I}$ is a Riesz basis, then the Riesz basis bounds are $\sqrt{A}$, $\sqrt{B}$, where A, B are the frame bounds for $\{f_i\}_{i\in I}$.

It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto. The frame operator for the frame is the positive, self-adjoint invertible operator $S = TT^*$:

$H \to H$. That is

$$Sf = TT^* f = T\left(\sum_{i\in I} \langle f, f_i\rangle e_i\right) = \sum_{i\in I} \langle f, f_i\rangle Te_i = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

In particular

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2 .$$

Regarding frame sequences, we have the following theorem.

Theorem 2.6. *The removal of a vector f_j from a frame $\{f_k\}_{k\in N}$ for H leaves either a frame or an incomplete set. More precisely*
if $< f_j, S^{-1}f_j >\neq 1$, then $\{f_k\}_{k\neq j}$ is a frame for H; if $< f_j, S^{-1}f_j = 1$, then $\{f_k\}_{k\neq j}$ is incomplete.

Regarding perturbation of frames, we have the following theorem.

Theorem 2.7. *Let $\{f_k\}_{k\in N}$ be a frame for H, and let $\{g_k\}_{k\in N}$ be a sequence in H . If $K : l_2 \to H$, $K\left(\{c_k\}_{k\in N}\right) = \sum_{k=1}^{\infty} c_k (f_k - g_k)$, is a well-defined compact operator, then $\{g_k\}_{k\in N}$ is a frame sequence.*

This theorem has the following immediate corollary.

Corollary 2.8. *Let $\{f_k\}_{k\in N}$ be a frame for H, and $\{g_k\}_{k\in N}$ a sequence in H. If $g_k = f_k$ except for a finite set of $k \in N$, then $\{g_k\}_{k\in N}$ is a frame sequence.*

In this paper, all of these results are generalized to the case of t-frame. Moreover, we considered the most general case of perturbation, namely, Noetherian perturbation. Of course, these results are special cases of Noetherian perturbation. More details about these results can be found in the monographs by O. Christensen [7, 8] and Heil Ch. [15].

3. Main Results

3.1. Hilbert case. First let's consider the case of H-space. Let X (Y) be an H-space with a norm $\|\cdot\|_X$ $(\|\cdot\|_Y)$, the system $\{x_n\}_{n\in\mathbb{N}} \equiv \vec{x}$ form a frame for it and $A; B > 0$ be the corresponding frame bounds. Let $T \in L(X; Y)$ be a Noetherian operator. Then it is clear that $\mathscr{R}_T$ is closed. It is known that (see, e.g., [24, 6, 7, 8]) $\exists \{x_n^*\}_{n\in\mathbb{N}} \subset X^* : x = \sum_{n=1}^{\infty} (x; x_n^*)_X x_n$, $\forall x \in X$. Assume $\overline{L\left[\{y_n\}_{n\in\mathbb{N}}\right]} \equiv Y_1 \subset Y$, where $y_n = Tx_n$, $\forall n \in \mathbb{N}$. It is absolutely clear that $Y_1 \equiv \mathscr{R}_T$. Represent X as a direct sum $X = KerT\dot{+}X_1$. Consider the restriction of T on X_1, and denote it by T_1, i.e. $T_1 = T/_{X_1}$. It is clear that $T_1 \in L(X_1; Y_1)$, and it is bounded invertible as $\mathscr{R}_{T_1} = \mathscr{R}_T$ (invertibility follows from the Banach theorem). Following [7], we call T_1^{-1} a pseudoinverse of T. Take $\forall y \in Y_1$. Let $x = T_1^{-1}y \in X_1$. Consequently

$$x = \sum_{n=1}^{\infty} \left(T_1^{-1}y; x_n^*\right)_X x_n = \sum_{n=1}^{\infty} \left(y; \left(T_1^{-1}\right)^* x_n^*\right)_Y x_n.$$

It follows directly that

$$y = Tx = \sum_{n=1}^{\infty} (y;\, y_n^*)_Y \, y_n,$$

where $y_n^* = \left(T_1^{-1}\right)^* x_n^* \in Y$, $\forall n \in \mathbb{N}$. Let us show that the system $\vec{y}$ forms a frame for Y_1. Projectors generated by the decomposition $X = KerT \dot{+} X_1$ are denoted by P_0 and P_1, respectively. It is obvious that the projectors P_0 and P_1 are continuous.

We have $x = P_0 x + P_1 x$. Consequently

$$Tx = TP_1 x = T_1 P_1 x \Rightarrow P_1 x = T_1^{-1} T x = T_1^{-1} y,$$

where $y = Tx$. Thus

$$\sum |(y;\, y_n)_Y|^2 = \sum |(y;\, Tx_n)_Y|^2 = \sum |(T^* y;\, x_n)_X|^2 \leq$$

$$\leq B \, \|T^* y\|_X^2 \leq B \, \|T^*\|^2 \, \|y\|_Y^2 \, , \forall y \in \mathscr{R}_T.$$

On the other hand, let $y \in \mathscr{R}_T \Rightarrow \exists x \in X : Tx = y$. Hence $Tx = Tx_1$, where $x_1 = P_1 x$. We have

$$\|y\|_Y^2 = |(y;\, y)_Y| = |(Tx_1;\, y)_Y| = \left| \left(T \left(\sum_{n=1}^{\infty} (x_1;\, x_n^*)_X \, x_n;\, y \right)_Y \right) \right| \leq$$

$$\leq \sum_{n=1}^{\infty} |(x_1;\, x_n^*)_X| \, |(y_n;\, y)_Y| \leq \left\| \{(x;\, x_n^*)_X\}_{n\in\mathbb{N}} \right\|_{l_2} \left\| \{(y;\, y_n)_Y\}_{n\in\mathbb{N}} \right\|_{l_2} . \qquad (2)$$

Taking into account that $x_1 = T_1^{-1} y$, from the condition $(ii\ )$ of Definition 2.3 we obtain

$$\left\| \{(x_1;\, x_n^*)_X\}_{n\in\mathbb{N}} \right\|_{l_2} \leq A^{-1} \left\| T_1^{-1} y \right\| \leq A^{-1} \left\| T_1^{-1} \right\| \, \|y\|_Y \, .$$

As a result, it follows from (2) that

$$\|y\|_Y^2 \leq A^{-1} \left\| T_1^{-1} \right\| \left\| \{(y;\, y_n)_Y\}_{n\in\mathbb{N}} \right\|_{l_2} .$$

Thus, the following theorem is valid.

Theorem 3.1. *Let $X; Y$ be H-spaces and $T \in L(X; Y)$ be some Noetherian operator. If $\vec{x} \equiv \{x_n\}_{n\in\mathbb{N}} \subset X$ forms a frame (is an atomic decomposition) for X, then $\vec{y} \equiv \{Tx_n\}_{n\in\mathbb{N}}$ is a frame sequence (sequence of atomic decomposition) in Y.*

This theorem has the following corollaries.

Corollary 3.2. *Let $T \in L(X; Y)$ be a Fredholm operator. If $\vec{x}$ forms a frame (is an atomic decomposition) for X, then $T\vec{x} = \vec{y}$ (i.e. $y_n = Tx_n$, $\forall n \in \mathbb{N}$) also forms a frame (is an atomic decomposition) for Y, if $\vec{y}$ is complete in it.*

In fact, if all the conditions of this corollary are fulfilled, then it is not difficult to see that the operator T is bounded invertible as $\mathscr{R}_T = X$. The rest follows directly from the definitions of atomic decomposition and frame.

Corollary 3.3. *Let $T = I_X + K$, where K is a compact operator in X, and the system $\vec{x}$ forms a frame (is an atomic decomposition) for X. Then the system $\vec{y} = T\vec{x}$ also forms a frame (is an atomic decomposition) for X.*

The following corollary is also holds.

Corollary 3.4. *Let $T \in L(X; Y)$ be a Noetherian operator and the system $\vec{x}$ be a frame sequence in X. Then the system $T\vec{x}$ is also frame sequence in Y.*

If we take $\overline{L[\vec{x}]}$ as X, then the latter corollary will follow from Theorem 3.1.

3.2. Banach case. Let $X; Y$ be B-spaces and $\mathscr{K}$ be some B-space of the sequences of scalars with a norm $\|\cdot\|_{\mathscr{K}}$. Assume that the couple $\{\vec{x}^*; \vec{x}\}$ is an atomic decomposition of X with respect to $\mathscr{K}$, where $K : \mathscr{K} \to X$ is a decomposition operator defined as follows

$$K\vec{\lambda} = \sum_{n=1}^{\infty} \lambda_n x_n, \vec{\lambda} \in \mathscr{K}.$$

Let $T \in L(X; Y)$ be a Noetherian operator. Assume $\vec{y} = T\vec{x}$. Let us show that $\exists \vec{y}^* \subset Y^*$: $\{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $\mathscr{R}_T \equiv Y_1$ with respect to $\mathscr{K}$. Take $\forall y \in \mathscr{R}_T$. Then $\exists x \in X : Tx = y$. Let $X = KerT \dot{+} X_1$. Put $T_1 = T/_{X_1}$. It is obvious that $T_1 \in L(X_1; Y_1)$ is bounded invertible operator: $T_1^{-1} \in L(Y_1; X_1)$. Thus, $\mathscr{R}_{T_1} = \mathscr{R}_T$: $Tx = T_1 x_1$, where $x = x_0 + x_1$, $x_0 \in KerT$, $x_1 \in X_1$. Let $x_1 = T_1^{-1} y$. We have

$$x_1 = \sum_{n=1}^{\infty} x_n^*(x_1) x_n \Rightarrow y = T\left(\sum_{n=1}^{\infty} x_n^*(x_1) x_n\right) =$$

$$= \sum_{n=1}^{\infty} x_n^*(x_1) y_n = \sum_{n=1}^{\infty} y_n^*(y) y_n,$$

where $y_n^* = \left(T_1^{-1}\right)^* x_n^*$. Consequently, $\forall y \in \mathscr{R}_T$ we have

$$y = \sum_{n=1}^{\infty} y_n^*(y) y_n.$$

Since $y_n^*(y) = \left[\left(T_1^{-1}\right)^* x_n^*\right](y) = x_n^*\left(T_1^{-1} y\right)$ and $T_1^{-1} y \in X$, it is clear that $\{y_n^*(y)\}_{n\in\mathbb{N}} \in \mathscr{K}$, $\forall y \in \mathscr{R}_T$. We have

$$\left\|\{y_n^*(y)\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} = \left\|\left\{x_n^*\left(T_1^{-1} y\right)\right\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} \le B\left\|T_1^{-1} y\right\|_X \le B\left\|T_1^{-1}\right\|\|y\|_Y.$$

Similarly we obtain

$$\left\|\{y_n^*(y)\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} = \left\|\left\{x_n^*\left(T_1^{-1} y\right)\right\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} \ge A\left\|T_1^{-1} y\right\|_X \ge A\left\|T_1\right\|^{-1}\|y\|_Y.$$

Consequently, the following relation is valid

$$A_1 \|y\|_Y \leq \left\| \{y_n^* (y)\}_{n\in\mathbb{N}} \right\|_{\mathscr{K}} \leq B_1 \|y\|_Y , \forall y \in \mathscr{R}_T, \tag{3}$$

where $A_1 = A \|T_1\|^{-1}$, $B_1 = B \left\|T_1^{-1}\right\|$. Thus, we get the validity of

Theorem 3.5. *Let $T \in L(X; Y)$ be a Noetherian operator and $\{\vec{x}^*; \vec{x}\}$ be an atomic decomposition of X with respect to $\mathscr{K}$. Then $\exists \vec{y}^* \subset Y^* : \{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $\mathscr{R}_T$ (X, if $\overline{L[\vec{y}]} = X$) with respect to $\mathscr{K}$, where $\vec{y} = T\vec{x}$.*

The similar result is true with respect to the frame. Let $\{\vec{x}^*; S\}$ form a frame for X with respect to $\mathscr{K}$ and $T \in L(X; Y)$ be a Noetherian operator. Let $T_p = T_1^{-1}$ be a pseudoinverse operator of T. Assume $\vec{y}^* = T_p^* \vec{x}^*$. It is absolutely clear that $\vec{y}^* \subset Y^*$. Similar to the previous case, we can show that $\vec{y}^*(y) \in \mathscr{K}$, $\forall y \in \mathscr{R}_T$ and the relation (3) holds. Let $S_1 = TS$. We have

$$S_1 [\vec{y}^* (y)] = S_1 \left[\{y_n^* (y)\}_{n\in\mathbb{N}}\right] = S_1 \left[\left\{\left(T_p^* x_n^*\right)(y)\right\}_{n\in\mathbb{N}}\right] =$$

$$= S_1 \left[\{x_n^* (T_p y)\}_{n\in\mathbb{N}}\right] = T \left(S \left[\{x_n^* (T_p y)\}_{n\in\mathbb{N}}\right]\right) =$$

$$= TT_p y = y, \forall y \in \mathscr{R}_T.$$

It is clear that $S_1 \in L(\mathscr{K}; Y)$. Thus, the following theorem is true.

Theorem 3.6. *Let $T \in L(X; Y)$ be a Noetherian operator, $S \in L(\mathscr{K}; X)$, $\vec{x}^* \subset X^*$ and $\{\vec{x}^*; S\}$ form a frame for X with respect to $\mathscr{K}$. Then the pair $\{\vec{y}^*; TS\}$ forms a frame for $\mathscr{R}_T$, where $\vec{y}^* \equiv \{y_n^*\}_{n\in\mathbb{N}} = \left\{T_p^* x_n^*\right\}_{n\in\mathbb{N}} = T_p^* \vec{x}^*$, and T_p is a pseudoinverse operator of T.*

4. $\mathscr{K}$-Close atomic decompositions and frames

4.1. Hilbert case. Quadratically close frames. Let X be an H-space and the system $\vec{x}$ form a frame for it.

Systems $\vec{x}; \vec{y} \subset X$ are called quadratically close in X, if $\sum_{n=1}^{\infty} \|x_n - y_n\|_X^2 < +\infty$.

The following easily provable lemma is true.

Theorem 4.1. *Let the system $\vec{x}$ form a frame for X and the system $\vec{y}$ be quadratically close to $\vec{x}$. Then the system $\vec{y}$ forms a frame for $\overline{L[\vec{y}]}$.*

Proof. Let $n_0 \in \mathbb{N} : \sum_{n=n_0+1}^{\infty} \|x_n - y_n\|_X^2 < A$, where A is a constant in condition (ii), Definition 2.3. Assume

$$z_n \equiv \begin{cases} x_n, \ n = \overline{1, n_0}, \\ y_n, \ n > n_0 \ . \end{cases}$$

It is absolutely clear that $\sum_{n=1}^{\infty} \|x_n - z_n\|_X^2 < A$. We have

$$\left\| \sum c_k (x_n - z_n) \right\|_X \leq \mu \left(\sum |c_k|^2 \right)^{\frac{1}{2}} ,$$

where $\mu = \left(\sum_{n=n_0+1}^{\infty} \|x_n - z_n\|_X^2\right)^{\frac{1}{2}}$. By Theorem 15.1.1 of [7] we obtain that the system $\vec{z} \equiv \{z_n\}_{n\in\mathbb{N}}$ also forms a frame for X. As a result, it follows from Theorem 15.2.1 of [7] that the system $\vec{y}$ is a frame sequence. The theorem is proved.

4.2. Banach case. $\mathscr{K}$-close frames. Consider the case of B-space. Let X be a B-space and $\vec{x}^* \subset X^*$.

$\vec{x}^*$ is called q-Besselian if

$$\|\vec{x}^*(x)\|_{l_q} \le M \|x\|_X, \forall x \in X,$$

where $M > 0$ is an absolute constant.

Systems $\vec{x}; \vec{y} \subset X$ are called p-close if $\sum_{n=1}^{\infty} \|x_n - y_n\|_X^p < +\infty$.

Assume that $\mathscr{K}$ is some K-space with a canonical basis $\{\delta_n\}_{n\in\mathbb{N}}$. Then it is absolutely clear that the conjugate space $\mathscr{K}^*$ can be identified with the K-space of elements $\vec{\vartheta} \equiv \{\vartheta_n\}_{n\in\mathbb{N}}$, generated by the functionals $\vartheta^* \in \mathscr{K}^*$, where $\vartheta_n = \vartheta^*(\delta_n)$, $n \in \mathbb{N}$. Thus, every element $\vec{\vartheta} \in \mathscr{K}^*$ generates a (continuous) functional by the following expression

$$\vec{\vartheta}(\vec{x}) = \sum_{n=1}^{\infty} x_n \vartheta_n, \forall \vec{x} \in \mathscr{K}.$$

Now we introduce the following concepts.

Definition 4.2. *System $\vec{x}^* \subset X^*$ is called $\mathscr{K}^*$-Besselian if*

$$\|\vec{x}^*(x)\|_{\mathscr{K}^*} \le B \|x\|_X, \forall x \in X, \tag{4}$$

where $B > 0$ is a constant.

Definition 4.3. *The systems $\vec{x}; \vec{y} \subset X$ are called $\mathscr{K}$-close if*

$$\left\|\{\|x_n - y_n\|_X\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} < +\infty. \tag{5}$$

Let $\vec{x}^* \subset X^*, \vec{y} \subset X$ be some systems. Assume

$$X_{\vec{y}} \equiv \left\{ y \in X : \exists x \in X \Rightarrow y = \sum_{n=1}^{\infty} x_n^*(x)\, y_n \right\}.$$

It is absolutely clear that $X_{\vec{y}}$ is a linear subspace of X.

In the sequel, we will need the following

Lemma 4.4. *Let $\{\vec{x}^*; \vec{x}\}$ be an atomic decomposition of X with respect to $\mathscr{K}$ and the system $\vec{y} \subset X$ differ from the system $\vec{x}$ by a finite number of elements, i.e. $y_n = x_n$, $\forall n \ge n_0 + 1$, where $n_0 \in \mathbb{N}$ is some number. Then $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $X_{\vec{y}}$ with respect to $\mathscr{K}$.*

Proof. Consider the operator

$$T_0x = \sum_{n=1}^{\infty} x_n^*(x)(x_n - y_n) = \sum_{n=1}^{n_0} x_n^*(x)(x_n - y_n).$$

It is clear that T_0 is a finite dimensional operator, and, as a result, $T = I_X - T_0$ is a Fredholm operator. It is easy to see that $Tx = \sum_{n=1}^{\infty} x_n^*(x)\, y_n, \forall x \in X$. Let $y \in \mathscr{R}_T \Rightarrow \exists x \in X : Tx = y \Rightarrow y = \sum_{n=1}^{\infty} x_n^*(x)\, y_n \Rightarrow y \in X_{\vec{y}}$. Vice versa, let $y \in X_{\vec{y}} \Rightarrow \exists x \in X : y = \sum_{n=1}^{\infty} x_n^*(x)\, y_n = Tx \Rightarrow y \in \mathscr{R}_T$. Thus, $\mathscr{R}_T = X_{\vec{y}} \Rightarrow X_{\vec{y}}$ is a closure subspace. Let $X = KerT \dot{+} X_1$ and $T_1 = T/_{X_1}$. It is absolutely clear that the operator $T_1 \in L(X_1; \mathscr{R}_T)$ is bounded invertible. Let $y_n^* = \left(T_1^{-1}\right)^* x_n^*, \forall n \in \mathbb{N}$. Proceeding in the same way as in the proof of Theorem 3.5, we get the proof of Lemma 4.4.

Remark 4.5. *It should be noted that, generally speaking, $X_{\vec{y}} \neq \overline{L[\vec{y}]}$. In fact, let X be B-space with the basis $\vec{f}$ and $\vec{f}^*$ is an appropriate conjugate system.*

Let $\mathscr{K}_{\vec{f}}$ be a space of coefficients of basis $\vec{f}$. Assume $x_1^* = f_1^*; x_2^* = 0, x_n^* = f_{n-1}^*, \forall n \geq 3$; and $x_1 = f_1, x_2 = f_1, x_n = f_{n-1}, \forall n \geq 3$. It is easy to see that $\left\{\vec{x}^*; \vec{x}; K; \mathscr{K}_{\vec{f}}\right\}$ is an atomic decomposition of X. Accept $y_1 = 0, y_2 = f_1, y_n = f_{n-1}, \forall n \geq 3$. We have $card\{n : x_n \neq y_n\} = 2$. It is clear that $\overline{L[\vec{y}]} = X$, but $X_{\vec{y}} = L\left[\{f_n\}_{n\geq 2}\right] \neq X$.

Let $\mathscr{K}$ have a canonical basis and $\{\vec{x}^*; \vec{x}\}$ be an atomic decomposition of X with respect to $\mathscr{K}$. Suppose that $\vec{x}^* \subset X$ is $\mathscr{K}^*$-Besselian and the system $\vec{y} \subset X$ is $\mathscr{K}$-close to $\vec{x}$, i.e. the relations (4) and (5) are true. Assume

$$\vec{\lambda}_{n_0} = \left\{\underbrace{0; \ldots; 0}_{n_0}; \|x_{n_0+1} - y_{n_0+1}\|_X; \ldots\right\}.$$

The basicity of the system $\{\delta_n\}_{n\in\mathbb{N}}$ in $\mathscr{K}$ directly implies $\left\|\vec{\lambda}_{n_0}\right\|_{\mathscr{K}} \to 0$, $n_0 \to \infty$. Take some $n_0 \in \mathbb{N} : \left\|\vec{\lambda}_{n_0}\right\|_{\mathscr{K}} < B^{-1}$. Define the system $\vec{z} \equiv \{z_n\}_{n\in\mathbb{N}}$ as follows

$$z_n = \begin{cases} x_n, & n = \overline{1, n_0}, \\ y_n, & n > n_0. \end{cases}$$

Thus

$$\left\|\{\|x_n - z_n\|_X\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} < B^{-1}. \tag{6}$$

Consider the operator

$$T_0x = \sum_{n=1}^{\infty} x_n^*(x)(x_n - z_n), \forall x \in X,$$

and put $T = I_X - T_0$. We have

$$\|T_0x\|_X \leq \sum_{n=1}^{\infty} |x_n^*(x)| \, \|x_n - z_n\|_X \leq$$

$$\leq \left\| \{|x_n^*(x)|\}_{n\in\mathbb{N}} \right\|_{\mathscr{K}^*} \left\| \{\|x_n - z_n\|_X\}_{n\in\mathbb{N}} \right\|_{\mathscr{K}} . \qquad (7)$$

In the sequel, we will assume that the space $\mathscr{K}^*$ has the following property

$$\alpha)\ \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K}^* \Leftrightarrow \{|\lambda_n|\}_{n\in\mathbb{N}} \in \mathscr{K}^* \wedge \left\|\{\lambda_n\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}^*} = \left\|\{|\lambda_n|\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}^*} .$$

Then, taking into account (5) and (6), from (7) we obtain

$$\|T_0\| \leq B \left\|\{\|x_n - z_n\|_X\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} < 1.$$

Hence the operator $T \in L(X)$ is an automorphism in X. We have

$$Tx = x - T_0 x = x - \sum_{n=1}^{\infty} x_n^*(x)\, x_n + \sum_{n=1}^{\infty} x_n^*(x)\, z_n = \sum_{n=1}^{\infty} x_n^*(x)\, z_n.$$

Take $\forall z \in X$. Consequently, $\exists!\, x \in X : Tx = z$. As a result, we obtain

$$z = Tx = \sum_{n=1}^{\infty} x_n^*(x)\, z_n = \sum_{n=1}^{\infty} x_n^*\left(T^{-1}z\right) z_n =$$

$= \sum_{n=1}^{\infty} z_n^*(z)\, z_n$, where $z_n^* = \left(T^{-1}\right)^* x_n^*$, $\forall n \in \mathbb{N}$. From the expression $z_n^*(z) = x_n^*\left(T^{-1}z\right)$, $\forall n \in \mathbb{N}$, it follows directly that $\{z_n^*(z)\}_{n\in\mathbb{N}} \in \mathscr{K}$, $\forall z \in X$. We have

$$\left\|\{z_n^*(z)\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} = \left\|\left\{x_n^*\left(T^{-1}z\right)\right\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} \leq B\left\|T^{-1}z\right\|_X \leq B\left\|T^{-1}\right\|_{X\to X}\|z\|_X ,$$

$$\left\|\{z_n^*(z)\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} = \left\|\left\{x_n^*\left(T^{-1}z\right)\right\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} \geq$$
$$\geq A\left\|T^{-1}z\right\|_X \geq A\,\|T\|_{X\to X}^{-1}\|z\|_X ,\forall z \in X.$$

Thus, $\{\vec{z}^*; \vec{z}\}$ is an atomic decomposition of X with respect to $\mathscr{K}$. By virtue of Lemma 4.4 we obtain that $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $X_{\vec{y}}$ *with respect to* $\mathscr{K}$. Thus, the following theorem is true.

Theorem 4.6. *Let K-space $\mathscr{K}$ have the canonical basis and possess the property α). If $\{\vec{x}^*; \vec{x}\}$ is an atomic decomposition of X with respect to $\mathscr{K}$ and the system $\vec{y} \subset X$ is $\mathscr{K}$-close to $\vec{x}$, then $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $X_{\vec{y}}$ with respect to $\mathscr{K}$.*

This theorem has the following

Corollary 4.7. *Let K-space $\mathscr{K}$ have the canonical basis and possess the property α). If $\{\vec{x}^*; \vec{x}\}$ is a sequence of atomic decomposition in X with respect to $\mathscr{K}$ and the system $\vec{y} \subset X$ is $\mathscr{K}$-close to $\vec{x}$, then $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; \vec{y}\}$ is also an atomic decomposition of $X_{\vec{y}}$ with respect to $\mathscr{K}$.*

The scheme of the proof of Theorem 4.6 is applicable to the more general case. Namely, the following theorem is true.

Theorem 4.8. *Let X be a B-space, $\mathscr{K}_k$, $k = 1, 2$; be K-spaces, $\mathscr{K}_2$ have a canonical basis and have the property α). Let $\{\vec{x}^*; \vec{x}\}$ be an atomic decomposition of X with respect to $\mathscr{K}_1$, where the system $\vec{x}^* \subset X^*$ is $\mathscr{K}_2^*$-Besselian and the system $\vec{y} \subset X$ is $\mathscr{K}_2$-close to $\vec{x}$. Then $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $X_{\vec{y}}$ with respect to $\mathscr{K}_1$.*

It follows

Corollary 4.9. *Let the spaces X, $\mathscr{K}_k$, $k = 1, 2$; satisfy the conditions of Theorem 4.8. Let $\{\vec{x}^*; \vec{x}\}$ be an atomic decomposition of $\overline{L[\vec{x}]}$ with respect to $\mathscr{K}_1$, where $\vec{x}^* \subset X^*$ is $\mathscr{K}_2^*$-Besselian and the system $\vec{y} \subset X$ is $\mathscr{K}_2$-close to $\vec{x}$. Then $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; \vec{y}\}$ is an atomic decomposition of $X_{\vec{y}}$ with respect to $\mathscr{K}_1$.*

Now let's consider frame perturbation in B-spaces. Let X be a B-space, $\mathscr{K}$ be some K-space and the pair $\{\vec{x}^*; S\}$ form a frame for X with respect to $\mathscr{K}$, where $\vec{x}^* \equiv \{x_n^*\}_{n\in\mathbb{N}} \subset X^*$, $S \in L(\mathscr{K}; X)$. Let the system $\vec{x}^*$ be $\mathscr{K}$-Besselian, i.e. let the inequality (12) hold and $S_1 \in L(\mathscr{K}; X)$ be some operator. We have

$$\|S[\vec{x}^*(x)] - S_1[\vec{x}^*(x)]\|_X \le$$

$$\le \|S - S_1\|_{\mathscr{K}\to X} \|\vec{x}^*(x)\|_{\mathscr{K}} \le B\|S - S_1\|_{\mathscr{K}\to X} \|x\|_X . \tag{8}$$

Put $S_0 = S\circ\vec{x}^* - S_1\circ x^*$. If $\|S - S_1\| < B^{-1}$, then it follows from (8) that $\|S_0\| < 1$, and, as a result, the operator $T = I_X - S_0$ is invertible in X. Take $\forall y \in X \Rightarrow \exists!\, x \in X : Tx = y$. We have

$$y = Tx = x - S_0x = x - S[\vec{x}^*(x)] - S_1[\vec{x}^*(x)] =$$

$$= x - x - S_1[\vec{x}^*(x)] = S_1[\vec{x}^*(x)] =$$

$$= S_1\left[\vec{x}^*\left(T^{-1}y\right)\right] = S_1[\vec{y}^*(y)],$$

where $\vec{y}^* \equiv \{y_n^*\}_{n\in\mathbb{N}} \equiv \left\{\left(T^{-1}\right)^* x_n^*\right\}_{n\in\mathbb{N}}$. From the relation

$$\vec{y}^*(y) = \{y_n^*(y)\}_{n\in\mathbb{N}} \equiv \left\{\left[\left(T^{-1}\right)^* x_n^*\right](y)\right\}_{n\in\mathbb{N}} \equiv \left\{x_n^*\left(T^{-1}y\right)\right\}_{n\in\mathbb{N}},$$

it follows that $\vec{y}^*(y) \in \mathscr{K}$ as $T^{-1}y \in X$. On the other hand

$$\|\vec{y}^*(y)\|_{\mathscr{K}} \equiv \left\|\left\{x_n^*\left(T^{-1}y\right)\right\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} \le B\left\|T^{-1}y\right\|_X \le B\left\|T^{-1}\right\|_{X\to X}\|y\|_X ,$$

$$\|\vec{y}^*(y)\|_{\mathscr{K}} \equiv \left\|\left\{x_n^*\left(T^{-1}y\right)\right\}_{n\in\mathbb{N}}\right\|_{\mathscr{K}} \ge A\left\|T^{-1}y\right\|_X \ge A\left\|T^{-1}\right\|_{X\to X}\|y\| .$$

Thus, we have proved the following theorem.

Theorem 4.10. *Let X be a B-space, $\mathscr{K}$ be a K-space and the pair $\{\vec{x}^*; S\}$ form a frame for X with respect to $\mathscr{K}$. If the operator $S_1 \in L(\mathscr{K}; X)$ satisfies the condition $\|S - S_1\|_{\mathscr{K}\to X} < B^{-1}$, then $\exists \vec{y}^* \subset X^* : \{\vec{y}^*; S_1\}$ also forms a frame for X with respect to $\mathscr{K}$.*

5. Some Applications

5.1. Perturbation in the sense of Littlewood-Paley. By $l_{p;\,p-2}$ we denote a K-space of sequences with a norm

$$\left\|\{\lambda_n\}_{n\in\mathbb{N}}\right\|_{p;\,p-2} \equiv \left(\sum_{n=1}^{\infty} n^{p-2}\,|\lambda_n|^p\right)^{1/p}, \quad 1<p<+\infty.$$

The classical Paley theorem can be stated as follows.

Paley theorem. *An arbitrary uniformly bounded orthonormal system $\vec{\varphi} \equiv \{\varphi_n\}_{n\in\mathbb{N}}$ in $L_p\,(a,\,b)$, $1<p\le 2$, is $l_{p;\,p-2}$-Besselian.*

It is absolutely clear that the space $l_{p;\,p-2}$, $1<p\le 2$, has a canonical basis and possesses the property α). It is not difficult to see that the conjugate of $l_{p;\,p-2}$ is the space $l_{q;\,q-2}$ with a norm

$$\left\|\{\lambda_n\}_{n\in\mathbb{N}}\right\|_{q,\,q-2} \equiv \left(\sum_{n=1}^{\infty} n^{q-2}\,|\lambda_n|^q\right)^{1/q}.$$

Taking into account Corollary 4.9, we obtain

Corollary 5.1. *If the system $\vec{\psi} \equiv \{\psi_n\}_{n\in\mathbb{N}} \subset L_p\,(a,\,b)$ is $l_{q;\,q-2}$-close to $\vec{\varphi}$, i.e.*

$$\sum_{n=1}^{\infty} n^{q-2}\,\|\varphi_n-\psi_n\|_p^q < +\infty,$$

then $\exists \vec{\psi}^* \subset L_q\,(a,\,b) : \left\{\vec{\psi}^*;\,\vec{\psi}\right\}$ is an atomic decomposition of $X_{\vec{\psi}}$ with respect to $l_{p;\,p-2}$, where

$$\|f\|_p = \left(\int_a^b |f\,(t)|^p\,dt\right)^{1/p}.$$

5.2. Frames of eigenfunctions of a Sturm-Liouville operator. Consider the following Cauchy problem

$$\left.\begin{aligned} &-y''\,(x)+q\,(x)\,y\,(x)=\lambda^2 y\,(x)\,, \quad x\in(0,\,\pi)\,,\\ &y\,(0)=1,\; y'\,(0)=\sigma\,, \end{aligned}\right\} \tag{9}$$

where $q\,(x)\in L_1\,(0,\,\pi)$ is a real function, $\sigma\in R$. This spectral problem can be understood in the sense of V.A.Ilyin [16]. We are interested to find out: for which sequences $\{\lambda_n\}_{n\in\mathbb{N}} \subset R$ the system $\{y_{\lambda_n}\,(x)\}_{n\in\mathbb{N}}$, as a solution of the problem (9), forms a frame for $L_p \equiv L_p\,(0,\,\pi)$? Note that a similar question in the context of Riesz basicity has been earlier studied in [14].

Let $\vec{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \subset R$ be some sequence and consider the system of cosines $c_{\vec{\lambda}} \equiv \{\cos\lambda_n x\}_{n\in\mathbb{N}}$. As is known (see, e.g., [17]), the following relation holds

$$y_\lambda\,(x)=\cos\lambda x+\int_0^x K\,(x;\,t)\cos\lambda t\,dt,$$

where $K(x;\,t)$ is a continuous function on $[0,\,\pi]$. By K we denote the operator defined as follows

$$[Kf](x) = \int_0^x K(x;\,t)\,f(t)\,dt.$$

It is absolutely clear that K is the Volterra operator, and hence the operator $I_{L_p}+K$ is bounded invertible in L_p. Then, the relation $y_\lambda(x) = \left(I_{L_p}+K\right)\cos\lambda x$ and the results of previous section imply that the system $\{y_{\lambda_n}(x)\}_{n\in\mathbb{N}}$ is an atomic decomposition of L_p (forms a frame for L_p) if and only if the system of cosines $\{\cos\lambda_n x\}_{n\in\mathbb{N}}$ has the same property. Thus, the following theorem is true.

Theorem 5.2. *Let K-space $\mathscr{K}$ have the canonical basis and possess the property α). Let $q\in L_1$ and $\{\lambda_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be some sequence. Then the system $\{y_{\lambda_n}(x)\}_{n\in\mathbb{N}}$, as a solution of Cauchy problem (9), is an atomic decomposition of L_p (forms a frame for L_p) with respect to $\mathscr{K}$ if and only if the system of cosines $\{\cos\lambda_n x\}_{n\in\mathbb{N}}$ has the same property.*

Part II

Frames In Noetherian Mapping. $\mathscr{K}$-Close Frames.

6. Needful Information

Let's recall some concepts and facts concerning the tensor product of Hilbert spaces. Let $X;Y$ be some H-spaces and $Z = X\otimes Y$ be their tensor product. For simplicity, the tensor product $x\otimes y$ of elements $x\in X$ and $y\in Y$ will be denoted by $xy = x\otimes y$. Let $M\subset Y$ be some set. Assume

$$L_t[M] \equiv \left\{ z\in Z : \exists\,\{x_k; y_k\}_1^m \subset X\times M \Rightarrow z = \sum_{k=1}^m x_k y_k \right\}.$$

$L_t[M]$ is called a *t-span* of set M. Let $\vec{y}\subset Y$ be some system. Define

$$\Lambda^{(t)} \equiv \left\{ \vec{x}\subset X : \sum_{k=1}^\infty x_k y_k < +\infty \right\},$$

where $\sum(\,\cdot\,) < +\infty$ means the convergence of series in Z.

System $\vec{y}\subset Y$ is said to be t- complete in Z, if for $\forall z\in Z$, $\exists\left\{x_k^{(n)}\right\}_{k=1}^{m_n}\subset X$, $\forall n\in N$:

$$\lim_{n\to\infty}\sum_{k=1}^{m_n} x_k^{(n)} y_k = z.$$

System $\{y_n\}_{n\in N}\subset Y$ is said to be a t-basis with respect to the triple $X;Y;Z$, if for $\forall z\in Z$ there exists a unique $\{x_n\}_{n\in N}\subset X$:$z = \sum_{n=1}^\infty x_n y_n$.

System $\{y_n\}\subset Y$ is said to be t-independent if for every finite set of $\{x_n\}$ the equality $\sum_n x_n y_n = 0$ holds only when $x_n = 0\,,\ \forall n$.

Thus

$$(x_1 \otimes y_1;\, x_2 \otimes y_2)_Z = (x_1;\, x_2)_X\, (y_1;\, y_2)_Y\,, \forall x_k \in X,\ \forall y_k \in Y,\, k = 1,\, 2,$$

where $(\,\cdot\,;\,\cdot\,)_Z$ is a scalar product in Z and $\|\cdot\|_Z^2 = (\,\cdot\,;\,\cdot\,)_Z$.

Let us introduce the concept of *t-scalar* product for the pair $(y;\, z) \in Y \times Z$. Take $\forall x \in X$ and consider the linear functional $\vartheta_{(y;\,z)}(x) = (x \otimes y;\, z)_Z$. We have

$$\left|\vartheta_{(y;\,z)}(x)\right| \le \|x \otimes y\|_Z\, \|z\|_Z = \|y\|_Y\, \|z\|_Z\, \|x\|_X\,.$$

Consequently, $\vartheta_{(y;\,z)} \in X^* \equiv X$. As a result, $\exists!\ \tilde{x} \in X$: $\vartheta_{(y;\,z)}(x) = (x;\, \tilde{x})_X$, $\forall x \in X$. $\tilde{x}$ will be called a *t-scalar* product of the elements y and z, and we will denote it by $\tilde{x} =< y;\, z >_X$.

7. t-Besselian systems

Let us introduce the following definition.

Definition 7.1. *System $\vec{y} \subset Y$ is called t-Besselian, if $\exists M > 0$:*

$$\sum_{n=1}^{\infty} \|\langle y_n;\, z\rangle_X\|_X^2 \le M\, \|z\|_Z^2\,, \forall z \in Z. \tag{10}$$

Let us prove the following theorem.

Theorem 7.2. *Let $\vec{y} \subset Y$ be some system and let the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ be convergent in Z for $\forall \vec{x} \subset X$. Then the expression*

$$T\vec{x} = \sum_{n=1}^{\infty} x_n \otimes y_n$$

defines a bounded operator, i.e. $T \in L(l_2(X);\, Z)$. The conjugate operator $T^* \in L(Z;\, l_2(X))$ has the form $T^* z = \{< y_n;\, z >_X\}_{n \in N}$. Moreover

$$\sum_{n=1}^{\infty} \|< y_n;\, z >_X\|_X^2 \le \|T\|^2\, \|z\|_Z^2\,, \forall z \in Z.$$

Proof. Consider the bounded operators T_n:

$$T_n \vec{x} = \sum_{k=1}^{n} x_k \otimes y_k, \forall n \in N. \tag{11}$$

It is clear that $T_n \vec{x} \overset{n}{\to} T\vec{x}$, $\forall \vec{x} \in l_2(X)$. Then, by the Banach-Steinhaus theorem we obtain $\sup_n \|T_n\| < +\infty \Rightarrow T \in L(l_2(X);\, Z)$. Let's find the expression for $T^* z$. Consider the operators T_n, defined by (11). It is clear that $T_n^* z \mapsto T^* z$, $\forall z \in Z$. We have

$$(T_n^* z)\, \vec{x} = (\vec{x}\,;\, T_n^* z)_X = z\,(T_n \vec{x}) = (T_n \vec{x};\, z)_Z = \left(\sum_{k=1}^{n} x_k \otimes y_k;\, z\right)_Z =$$

$$= \sum_{k=1}^{n} (x_k \otimes y_k; z)_Z = \sum_{k=1}^{n} (x_k, < y_k; z >_X)_X = (\vec{x}; T_n^* z)_X ,$$

where $(\vartheta)\,\vec{x}$ means the value of the functional ϑ on $\vec{x}$ and
$T_n^* z = \{< y_1; z >_X, \ldots, < y_n; z >_X, 0, 0, \ldots\}$. It follows directly that

$$\{< y_k; z >_X\}_{k\in N} \in l_2(X) \wedge T^* z \equiv \{< y_k; z >_X\}_{k\in N} .$$

Thus

$$\sum_{k=1}^{\infty} \|< y_k; z >_X\|_X^2 = \|T^* z\|_{l_2(X)}^2 \le \|T^*\|^2 \|z\|_Z = \|T\|^2 \|z\|_Z^2 , \forall z \in Z.$$

The theorem is proved.

The following theorem is true.

Theorem 7.3. *The sequence $\vec{y} \equiv \{y_n\}_{n\in N} \subset Y$ is t-Besselian if and only if $T \in L(l_2(X); Z)$ and $\|T\| \le \sqrt{M}$, where $T\vec{x} = \sum_{k=1}^{\infty} x_k \otimes y_k$, $\forall \vec{x} \in l_2(X)$, and M is a constant from* (10).

Proof. Let $\vec{y}$ be *t-Besselian.* Take $\forall n; m \in N$: $n < m$. We have

$$\left\| \sum_{k=n}^{m} x_k \otimes y_k \right\|_Z = \sup_{\|z\|_Z=1} \left| \left(\sum_{k=n}^{m} x_k \otimes y_k; z \right)_Z \right| =$$

$$= \sup_{\|z\|_Z=1} \left| \sum_{k=n}^{m} (x_k; < y_k; z >_X)_X \right| \le \sup_{\|z\|_Z=1} \sum_{k=n}^{m} \|x_k\|_X \|< y_k; z >_X\|_X \le$$

$$\le \left(\sum_{k=n}^{m} \|x_k\|_X^2 \right)^{1/2} \sup_{\|z\|_Z=1} \left(\sum \|< y_k; z >_X\|^2 \right)^{1/2} \le \sqrt{M} \left(\sum_{k=n}^{m} \|x_k\|_X^2 \right)^{1/2} . \qquad (12)$$

As $\vec{x} \in l_2(X)$, it follows that the sequence $\{\sum_{k=1}^{n} x_k \otimes y_k\}_{n\in N}$ is fundamental in Z, and, as a result, the series $\sum_{k=1}^{\infty} x_k \otimes y_k$ is convergent in Z. Taking $n = 1$ in (12) and passing to the limit as $m \to \infty$, we obtain $\|T\| \le \sqrt{M}$. The converse follows from Theorem 7.2. Theorem is proved.

For the *t-Besselness* of the system, it suffices that the relation (10) hold with respect to a dense set in Z, i.e. the following lemma is true.

Lemma 7.4. *Let $\vec{y} \subset Y$, and let $Z_0 \subset Z$ be a dense set in Z. If $\exists M > 0$:*

$$\sum_{n=1}^{\infty} \|< y_n; z >_X\|_X^2 \le M \|z\|_Z^2 , \ \forall z \in Z_0,$$

then the system $\vec{y}$ is t-Besselian.

Proof. Let $\vec{y}$ be not *t-Besselian.* Then $\exists w \in Z$:

$$\sum_{n=1}^{\infty} \|< y_n ;\, w >_X\|_X^2 > M \|w\|_Z^2 .$$

Then it is absolutely clear that $\exists n_0 \in N$:

$$\sum_{n=1}^{n_0} \|< y_n ;\, w >_X\|_X^2 > M \|w\|_Z^2 . \tag{13}$$

By the definition of $\tilde{x} =< y;\, w >_X$ we immediately obtain that

$$\|< y;\, w >_X\|_X = \|\tilde{x}\|_X = \left\|\vartheta_{(y;\, w)}\right\| =$$

$$\sup_{\|x\|_X=1} \left|\vartheta_{(y;\, w)}(x)\right| = \sup_{\|x\|_X=1} \left|(x \otimes y;\, w)_Z\right| \le$$

$$\le \|w\|_Z \sup_{\|x\|_X=1} \|x \otimes y\|_Z = \|y\|_Y \|w\|_Z .$$

It follows that $< y;\, w >_X$ depends continuously on y and w. As $\overline{Z}_0 = Z$, from (13) we obtain that $\exists z_0 \in Z_0$:

$$\sum_{n=1}^{n_0} \|< y_n ;\, z_0 >_X\|_X^2 > M \|z_0\|_Z .$$

So we arrive at the contradiction which proves the lemma.

By combining the results of Theorems 7.2 and 7.3, we get the validity of the following theorem.

Theorem 7.5. *With respect to the system $\vec{y} \subset Y$, the following properties are equivalent:*

1) $\vec{y}$ is t-Besselian;

2) the series $\sum_{n=1}^{\infty} x_n \otimes y_n$ is convergent for $\forall \vec{x} \in l_2(X)$;

3) $T \in L(l_2(X);\, Z)$, where $T\vec{x} = \sum_{n=1}^{\infty} x_n \otimes y_n$.

In exactly the same way we prove the following

Lemma 7.6. *Let $\vec{y} \subset Y$, and let $Z_0 \subset Z$ be a dense set in Z. If $\exists m > 0$:*

$$m \|z\|_Z^2 \le \sum_{n=1}^{\infty} \|< y_n ;\, z >_X\|_X^2 , \forall z \in Z_0,$$

then this inequality holds for $\forall z \in Z$.

8. t-frames

Let us introduce the concept of *t-frame*. Let $\vec{y} \equiv \{y_n\}_{n \in N} \subset Y$ be some system.

Definition 8.1. *System $\vec{y} \subset Y$ is called a t-frame in Z if $\exists$ A; $B > 0$:*

$$A \|z\|_Z^2 \leq \sum_{n=1}^{\infty} \|< y_n; z >_X\|_X^2 \leq B \|z\|_Z^2, \ \ \forall z \in Z. \tag{14}$$

Constants A, B are called the bounds of *t-frame*. $\inf B$ and $\sup A$ which satisfy (14) are called the optimal bounds (upper and lower, respectively) of *t-frame*. Similar to the ordinary case, we introduce the following

Definition 8.2. *t-frame is said to be tight if $A = B$. t-frame is said to be exact if it is no longer a t-frame after exclusion of any one of its elements.*

Let $L_t[\vec{y}]$ denote a t-span of the system $\vec{y}$.

Definition 8.3. *System $\vec{y} \subset Y$ is called a t-frame sequence if it forms a t-frame for $\overline{L_t[\vec{y}]}$.*

Example 1. Let Y be an H-space with an orthonormal basis $\vec{e} \equiv \{e_n\}_{n \in N}$. Assume

$$\vec{f} \equiv \{f_n\}_{n \in N} \equiv \left\{ \underbrace{e_1; e_1; \ldots; e_1}_{m_1}; \ldots; \underbrace{e_k; \ldots; e_k}_{m_k}; \ldots \right\},$$

i.e. $f_i = e_n$ for $i \in I_n$, where $I_n = \{m_{n-1}; m_{n-1}+1; \ldots; m_n\}$, $m_0 = 1$. Let us show that $\vec{f}$ forms a *t-frame*. As the system $\vec{e}$ forms a t-basis for Z, it is clear that all possible finite sums of the form $z = \sum_{k \in M} x_k \otimes e_k$ are dense in Z, where $M \subset N$ is a finite set. It is absolutely clear that in this case we have

$$\|z\|_Z^2 = \sum_{k \in M} \|x_k\|^2.$$

Let $i \in I_n$. We have

$$\|< f_i; z >_X\|_X^2 = (< f_i; z >_X; < f_i; z >_X)_X =$$

$$= (x; < f_i; z >_X)_X = (x \otimes f_i; z)_Z, \text{ where } x =< f_i; z >_X.$$

Taking into account the expression for z, we obtain

$$\|< f_i; z >_X\|_X^2 = \sum_{k \in M} (x \otimes f_i; x_k \otimes e_k)_Z =$$

$$= \sum_{k \in M} (x;\, x_k)_X \, (f_i;\, e_k)_Y = (x;\, x_n)_X \, \chi_{M \bigcap I_n}(i)\,,$$

where $\chi_{M_0}(\cdot)$ is a characteristic function of M_0. On the other hand

$$(x;\, x_n)_X = \overline{(x_n;\, < f_i;\, z >_X)_X} = \overline{(x_n \otimes e_n;\, z)_Z} =$$

$$= \sum_{k \in M} (x_k \otimes e_k;\, x_n \otimes e_n)_Z = \|x_n\|_X^2 \, \chi_{M \bigcap I_n}(i)\,.$$

Summing over i we have

$$\sum_i \|< f_i;\, z >_X\|_X^2 = \sum_k m_k \, \|x_k\|_X^2\,.$$

Let $m = \min_k m_k$, $M = \max_k m_k < +\infty$. Consequently

$$m \, \|z\|_Z^2 \le \sum_i \|< f_i;\, z >_X\|_X^2 \le M \, \|z\|_Z^2\,.$$

Then from Lemmas 7.4 and 7.6 we obtain that the system $\vec{f}$ forms a *t-frame* for Z.

As in the ordinary case, the operator $T : l_2(X) \to Z$, defined by the expression

$$T\vec{x} = \sum_{n=1}^{\infty} x_n \otimes y_n, \;\; \vec{x} \in l_2(X)\,,$$

is called a synthesis operator, and the adjoint operator

$$T^* : Z \mapsto l_2(X) : T^* z = \{< y_n;\, z >_X\}_{n \in N}\,,$$

is called an analysis operator. The operator $S : Z \to Z$; $S = TT^*$ is called a *t-frame* operator. Thus

$$Sz = TT^* z = \sum_{n=1}^{\infty} < z;\, y_n >_X \otimes y_n. \tag{15}$$

Similar to the ordinary case, we prove the following

Lemma 8.4. *Let the system $\vec{y} \subset Y$ form a t-frame for Z. Then*

$$S \in L(Z)\,, \; \exists S^{-1} \in L(Z)\,, \; S^* = S \text{ and } S > 0.$$

By virtue of Lemmas 7.4 and 7.6, we get the validity of the following statement.

Statement 8.5. *Let $\vec{y} \subset Y$, and let $Z_0 \subset Z$ be a dense set in Z. If $\exists A;\, B > 0$:*

$$A \, \|z\|_Z^2 \le \sum_{n=1}^{\infty} \|< y_n;\, z >_X\|_X^2 \le B \, \|z\|_Z^2\,, \;\; \forall z \in Z_0,$$

then the system $\vec{y}$ forms a t-frame for Z.

Let the system $\vec{y} \subset Y$ form a *t-frame* for Z. Then, by Lemma 8.4, the operator S is invertible. Take $\forall z \in Z$. We have

$$z = SS^{-1}z = \sum_{n=1}^{\infty} < y_n; S^{-1}z >_X \otimes y_n = \sum_{n=1}^{\infty} S_{y_n} z \otimes y_n, \tag{16}$$

where $S_{y_n} z =< y_n; S^{-1}z >_X$. It is clear that $S_{y_n} \in L(Z; X)$, $\forall n \in N$. Thus

$$\sum_{n=1}^{\infty} \|S_{y_n} z\|_X^2 = \sum_{n=1}^{\infty} \left\|< y_n; S^{-1}z >_X\right\|_X^2 \leq B \left\|S^{-1}z\right\|_Z^2 \leq B \left\|S^{-1}\right\|^2 \|z\|_Z^2,$$

$$\sum_{n=1}^{\infty} \|S_{y_n} z\|_X^2 \geq A \|S\|^{-2} \|z\|_Z^2, \quad \forall z \in Z.$$

The system $\{S_{y_n}\}_{n\in N}$ is called a *t-frame* dual to $\vec{y}$, and $\{S_{y_n} z\}_{n\in N}$ are called *t-frame* coefficients.

Let $\vec{x} \in l_2(X) : z = \sum_{n=1}^{\infty} x_n \otimes y_n$. Assume $\vec{x}_0 = \{x_n - S_{y_n} z\}_{n\in N} \in l_2(X)$. We have

$$T\vec{x}_0 = \sum_{n=1}^{\infty} (x_n - S_{y_n} z) \otimes y_n = \sum_{n=1}^{\infty} x_n \otimes y_n -$$

$$- \sum_{n=1}^{\infty} S_{y_n} z \otimes y_n = 0 \Rightarrow \vec{x}_0 \in KerT.$$

On the other hand

$$T^*\left(S^{-1}z\right) = \left\{< y_n; S^{-1}z >_X\right\}_{n\in N} = \{S_{y_n} z\}_{n\in N} \in l_2(X),$$

because $S^{-1}z \in Z$. Consequently, $\{S_{y_n} z\}_{n\in N} \in \mathscr{R}_{T^*}$. By $\vec{x} = \vec{x}_0 + \{S_{y_n} z\}_{n\in N}$, from $KerT = \mathscr{R}_{T^*}^{\perp}$ we obtain

$$\|\vec{x}\|_{l_2(X)}^2 = \|\vec{x}_0\|_{l_2(X)}^2 + \left\|\{S_{y_n} z\}_{n\in N}\right\|_{l_2(X)}^2. \tag{17}$$

This implies the validity of the following lemma.

Lemma 8.6. *Let $\vec{y} \subset Y$ form a t-frame for Z. Then*

$$\left\|\{S_{y_n} z\}_{n\in N}\right\|_{l_2(X)} = \min\left\{\|\vec{x}\|_{l_2(X)} : z = \sum_{n=1}^{\infty} x_n \otimes y_n, \quad \vec{x} \in l_2(X)\right\}.$$

Take $\forall n_0 \in N$ and consider the mapping $S_{n_0} : X \to X$ defined by

$$S_{n_0} x =< S^{-1}(x \otimes y_{n_0}); y_{n_0} >_X, \quad \forall x \in X.$$

Assume $\vec{y}_{n_0} \equiv \vec{y} \backslash \{y_{n_0}\}$. We have

Theorem 8.7. *Let the system $\vec{y} \subset Y$ form a t-frame for Z. Then: 1) if $Ker\,(I_X - S_{n_0}) \neq \{0\}$, then $\vec{y}_{n_0}$ is t-non-complete in Z, but it forms a t-frame for $\overline{L_t\,[\vec{y}_{n_0}]}$; 2) if $Ker\,(I_X - S_{n_0}) = \{0\}$ and $(I_X - S_{n_0})^{-1} \in L\,(X)$, then $\vec{y}_{n_0}$ forms a t-frame for Z.*

Proof. Let $Ker\,(I_X - S_{n_0}) \neq \{0\}$ and $x_0 \neq 0 : \; x_0 = S_{n_0} x_0$. As $\vec{y}$ forms a *t-frame* for Z, we have

$$z_0 = x_0 \otimes y_{n_0} = \sum_{n=1}^{\infty} < y_n; \; S^{-1} z_0 >_X \otimes y_n.$$

On the other hand

$$z_0 = \sum_{n=1}^{\infty} (\delta_{nn_0} x_0) \otimes y_n.$$

By virtue of (17) we obtain

$$\|x_0\|_X^2 = \left\| \{\delta_{nn_0} x_0 - S_{y_n} z_0\}_{n \in N} \right\|_{l_2(X)}^2 + \left\| \{S_{y_n} z_0\}_{n \in N} \right\|_{l_2(X)}^2 \Rightarrow$$

$$\Rightarrow S_{y_n} z_0 = 0, \, \forall n \neq n_0.$$

It is clear that $z_0 \neq 0$ for $y_{n_0} \neq 0$. Consequently, the system $\vec{y}_{n_0}$ is t-non-complete in Z. Let us show that in this case $S_{n_0} = I_X$. Assume the converse. Then $\exists x_1 \in X \backslash \{0\}$: $S_{n_0} x_1 \neq x_1$. Consider

$$x_1 \otimes y_{n_0} = \sum_{n=1}^{\infty} < S^{-1} (x_1 \otimes y_{n_0}) \; ; \, y_n >_X \otimes y_n \Rightarrow$$

$$\Rightarrow (x_1 - S_{n_0} x_1) \otimes y_{n_0} = \sum_{n \neq n_n} S_{y_n} (x_1 \otimes y_{n_0}) \otimes y_n. \tag{18}$$

Let

$$a_n = \|x_1 - S_{n_0} x_1\|_X^{-2} \left(S_{y_n} (x_1 \otimes y_{n_0}) \; ; \; x_1 - S_{n_0} x_1 \right)_X, \;\; \forall n \neq n_0.$$

Scalar multiplication of both sides of (18) by $(x_1 - S_{n_0} x_1)$ yields

$$y_{n_0} = \sum_{n \neq n_0} a_n y_n.$$

Since $\forall z \in Z$ can be expanded as

$$z = \sum_{n=1}^{\infty} S_{y_n} z \otimes y_n = \sum_{n \neq n_0} S_{y_n} z \otimes y_n +$$

$$+ S_{y_{n_0}} z \otimes \left(\sum_{n \neq n_0} a_n y_n \right) = \sum_{n \neq n_0} \left(S_{y_n} z + a_n S_{y_{n_0}} z \right) \otimes y_n,$$

we obtain that the system $\vec{y}_{n_0}$ is t-complete in Z, which contradicts the condition of the theorem. So $S_{n_0} = I_X$.

Take $\forall x \in X \backslash \{0\}$. We have

$$x \otimes y_k = \sum_{n=1}^{\infty} S_{y_n} (x \otimes y_k) \otimes y_n, \ \ \forall k \neq n_0.$$

If $S_{y_{n_0}} (x \otimes y_k) \neq 0$, then, similar to the previous case, we find that the system $\vec{y}_{n_0}$ is t-complete in Z, which contradicts the condition of the theorem. Consequently, $S_{y_{n_0}} (x \otimes y_k) = 0, \ \forall x \in X, \ \forall k \neq n_0$. It follows directly that $S_{y_{n_0}} z = 0, \ \forall z \in L_{n_0}^{(t)}$, where $L_{n_0}^{(t)}$ is a closure of the *t-span* of the system $\vec{y}_{n_0}$, i.e. $L_{n_0}^{(t)} \equiv \overline{L_t [\vec{y}_{n_0}]}$. Thus

$$z = \sum_{n \neq n_0}^{\infty} S_{y_n} z \otimes y_n \, , \ \forall z \in L_{n_0}^{(t)}. \tag{19}$$

We have

$$\|z\|_Z^2 = \sum_{n \neq n_0} (x_n \otimes y_n; \ z)_Z = \sum_{n \neq n_0} (x_n; \ < y_n; \ z >_X)_X \, ,$$

where $x_n = S_{y_n} z, \ \ \forall n \neq n_0$. Hence

$$\|z\|_Z^2 \leq \left(\sum_{n \neq n_0} \|x_n\|_X^2 \right)^{1/2} \left(\sum_{n \neq n_0} \|< y_n; \ z >_X\|_X^2 \right)^{1/2} .$$

From (19) we obtain

$$S^{-1} z = \sum_{n \neq n_0}^{\infty} S^{-1} (x_n \otimes y_n) \Rightarrow$$

$$\Rightarrow \left(S^{-1} z; \ z \right)_Z = \sum_{n \neq n_0} \left(S^{-1} (x_n \otimes y_n) \, ; \ z \right)_Z =$$

$$= \sum_{n \neq n_0} \left(x_n \otimes y_n; \ S^{-1} z \right)_Z = \sum_{n \neq n_0} \left(x_n; \ < y_n; \ S^{-1} z >_X \right)_X .$$

Taking into account the expression

$$x_n = S_{y_n} z =< y_n; \ S^{-1} z >_X,$$

we have

$$\left(S^{-1} z; \ z \right)_Z = \sum_{n \neq n_0} \|x_n\|_X^2 \, .$$

Consequently

$$\|z\|_Z^2 \leq \left| \left(S^{-1} z; \ z \right) \right|^{\frac{1}{2}} \left(\sum_{n \neq n_0} \|< y_n; \ z >_X\|_X^2 \right)^{1/2} \leq$$

$$\leq \left\|S^{-1}\right\|^{\frac{1}{2}} \|z\|_Z \left(\sum_{n\neq n_0} \|< y_n; z >_X\|_X^2\right)^{1/2} \Rightarrow$$

$$\Rightarrow \|z\|_Z^2 \leq \left\|S^{-1}\right\| \sum_{n\neq n_0} \|< y_n; z >_X\|_X^2 , \ \ \forall z \in L_{n_0}^{(t)}.$$

It is absolutely clear that the following relation is valid

$$\sum_{n\neq n_0} \|< y_n; z >\|_X^2 \leq B \|z\|_Z^2 .$$

Thus, the assertion 1) is proved.

Now let's show the validity of the assertion 2). Consider the tensor product of the operators $(I_X - S_{n_0})$ and I_Y : $\tilde{S} = (I_X - S_{n_0}) \otimes I_Y$. It is clear that $\tilde{S}$ is bounded and boundedly invertible operator in $L(Z)$. We have

$$(x_1 - S_{n_0} x_1) \otimes y_{n_0} = \sum_{n\neq n_0} S_{y_n} (x_1 \otimes y_{n_0}) \otimes y_n, \ \ \forall x_1 \in X.$$

Let

$$\tilde{x}_{n_0} = (I_X - S_{n_0})\, x_{\cdot 1} , \ \ \tilde{x}_n = S_{y_n} (x_1 \otimes y_{n_0}) ,$$

i.e.

$$\tilde{x}_{n_0} \otimes y_{n_0} = \sum_{n\neq n_0} \tilde{x}_n \otimes y_n. \tag{20}$$

Take $\forall z \in Z$. Scalar multiplication of both sides of (20) by z yields:

$$(\tilde{x}_{n_0} \otimes y_{n_0}; z)_Z = \sum_{n\neq n_0} (\tilde{x}_n \otimes y_n; z)_Z .$$

Thus

$$(\tilde{x}_{n_0}; < y_{n_0}; z >_X)_X = \sum_{n\neq n_0} (\tilde{x}_n; < y_n; z >_X)_X$$

Take

$$x_1 = (I_X - SS_{n_0})^{-1} < y_{n_0}; z >_X .$$

We have

$$\|< y_{n_0}; z >_X\|_X^2 = \sum_{n\neq n_0} (S_{y_n} (x_1 \otimes y_{n_0}) \, ; < y_n; z >_X)_X \leq$$

$$\leq \left(\sum_{n\neq n_0} \|S_{y_n} (x_1 \otimes y_{n_0})\|_X^2\right)^{1/2} \left(\sum_{n\neq n_0} \|< y_n; z >_X\|_X^2\right)^{1/2} . \tag{21}$$

Taking into account the expression

$$S_{y_n} (x_1 \otimes y_{n_0}) =< S^{-1} (x_1 \otimes y_{n_0}) \, ; y_n >_X,$$

from the *t-frameness* of the system $\vec{y}$ we get

$$\sum_{n\neq n_0} \|S_{y_n}(x_1 \otimes y_{n_0})\|_X^2 = \sum_{n\neq n_0} \left\|< S^{-1}(x_1 \otimes y_{n_0})\ ;\ y_n >_X\right\|_X^2 \leq$$

$$\leq B\left\|S^{-1}(x_1 \otimes y_{n_0})\right\|_Z^2 \leq B\left\|S^{-1}\right\|^2 \|y_{n_0}\|_Y^2 \|x_1\|_X^2 \leq$$

$$\leq \tilde{B}^2 \|< y_{n_0};\, z >_X\|_X^2 ,$$

where

$$\tilde{B} = \sqrt{B}\left\|S^{-1}\right\| \left\|(I_X - S_{n_0})^{-1}\right\| \|y_{n_0}\|_Y .$$

As a result, it follows from (21) that

$$\|< y_{n_0};\, z >_X\|_X^2 \leq \tilde{B}^2 \sum_{n\neq n_0} \|< y_n;\, z >_X\|_X^2 .$$

We have

$$A\|z\|_Z^2 \leq \sum_{n=1}^{\infty} \|< y_n;\, z >_X\|_X^2 \leq \left(1+\tilde{B}^2\right) \sum_{n\neq n_0} \|< y_n;\, z >_X\|_X^2 \leq$$

$$\leq \left(1+\tilde{B}^2\right) B\|z\|_Z^2 .$$

The theorem is proved.

This theorem has the following corollary.

Corollary 8.8. *Let the system $\vec{y} \subset Y$ form a t-frame for Z. Then the system $\vec{y}_F \equiv \vec{y} \backslash \{y_k\}_{k\in F}$ forms a t-frame for $\overline{L_t[\vec{y}_F]}$, where $F \subset N : cardF < +\infty$ is an arbitrary set.*

Let's prove that the converse is also true. Let the system $\vec{y} \subset Y$ form a *t-frame* for $\overline{L_t[\vec{y}]}$. Consider the system $\vec{\vartheta} \equiv \{\vartheta_k\}_{k=1}^m \bigcup \vec{y}$, where $\{\vartheta_k\}_{k=1}^m \bigcup Y$ is some system. Let us show that $\vec{\vartheta}$ forms a *t-frame* for $\overline{L_t\left[\vec{\vartheta}\right]}$. Without loss of generality, we assume that $m=1$. Consider two cases: *i)* $L_t[\vartheta_1] \in \overline{L_t[\vec{y}]}$; *ii)* $L_t[\vartheta_1] \notin \overline{L_t[\vec{y}]}$. Let's start with the case *i)*. Take $\forall z \in \overline{L_t[\vec{y}]}$. We have

$$A\|z\|_Z^2 \leq \sum_{n=1}^{\infty} \|< y_n;\, z >_X\|_X^2 \leq \sum_{n=1}^{\infty} \|< \vartheta_n;\, z >_X\|_X^2 \leq$$

$$\leq \|\vartheta_1\|_Y^2 \|z\|_Z^2 + B\|z\|_Z^2 = B_1 \|z\|_Z^2 ,$$

where $B_1 = B + \|\vartheta_1\|_Y^2$, $\vartheta_{n+1} = y_n$, $\forall n \in N$.

Now consider the case *ii)*. We have

$$\overline{L_t\left[\vec{\vartheta}\right]} = L_t[\vartheta_1] + \overline{L_t[\vec{y}]}.$$

Then it is known that $\exists c > 0 : \forall z \in \overline{L_t\left[\vec{\vartheta}\right]}$ can be represented as $z = z_1 + z_2$, $z_1 \in L_t[\vartheta_1]$, $z_2 \in \overline{L_t[\vec{y}]}$ with $\|z_1\|_Z + \|z_2\|_Z \le m\|z\|_Z$. More details about this result can be found, e.g., in W. Rudin [40].

By $Y_1 : Y_1 \equiv \overline{L[\vec{y}]}$ we denote the closure of the linear span of the (ordinary) system $\vec{y}$ in Y. It is not difficult to see that $Z_1 \equiv \overline{L_t[\vec{y}]} = X \otimes Y_1$. By assumption, $\vec{y}$ forms a *t-frame* for Z_1. Denote the corresponding frame operator by S. So, S_1; $S_1^{-1} \in L(Z_1)$. As $z_2 \in Z_1$, it is clear that the following decomposition is valid

$$z_2 = \sum_{n=1}^{\infty} < y_n;\, S_1^{-1} z_2 >_X \otimes y_n = \sum_{n=1}^{\infty} a_n \otimes y_n,$$

where $a_{n+1} =< y_n;\, S_1^{-1} z_2 >_X$, $n = \overline{1, \infty}$. Let $z_1 = a_1 \otimes \vartheta_1$. We have

$$\|z\|_Z^2 = (z;\, z)_Z =$$

$$= \left(\sum_{n=1}^{\infty} a_n \otimes \vartheta_n;\, z\right)_Z = \sum_{n=1}^{\infty} (a_n \otimes \vartheta_n;\, z)_Z = \sum_{n=1}^{\infty} (a_n;\, < \vartheta_n;\, z >_X)_X \le$$

$$\le \sum_{n=1}^{\infty} \|a_n\|_X \|< \vartheta_n;\, z >_X\|_X \le \left(\sum_{n=1}^{\infty} \|a_n\|_X^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|< \vartheta_n;\, z >_X\|_X^2\right)^{1/2}. \tag{22}$$

We denote the frame bounds of system $\vec{y}$ in Z_1 by $A_1 > 0$ and $B_1 > 0$. Consequently

$$\|a_1\|_X^2 + \sum_{n=2}^{\infty} \|a_n\|_X^2 = \|\vartheta_1\|_Y^{-2} \|z_1\|_Z^2 +$$

$$+ \sum_{n=1}^{\infty} \left\|< y_n;\, S_1^{-1} z_2 >_X\right\|_X^2 \le m^2 \|\vartheta_1\|_Y^{-2} \|z\|_Z^2 + B_1 \left\|S_1^{-1} z_2\right\|_Z^2 \le B_2 \|z\|_Z^2,$$

where $B_2 = m^2 \left(\|\vartheta_1\|_Y^{-2} + B_1 \left\|S_1^{-1}\right\|^2\right)$. Hence, by (22), we obtain that

$$\|z\|_Z^2 \le B_2 \sum_{n=1}^{\infty} \|< \vartheta_n;\, z >_X\|_X^2. \tag{23}$$

Denote by $L_t^{\perp}$ the orthogonal complement of $\overline{L_t[\vec{y}]}$ in Z. Take $\forall z \in \overline{L_t\left[\vec{\vartheta}\right]}$. Let $z = z_0 + z_1$, where $z_0 \in L_t^{\perp}$, $z_1 \in \overline{L_t[\vec{y}]}$. It is obvious that $(x \otimes y_n;\, z_0)_Z = 0$, $\forall x \in X$, $\forall n \in N$. We have

$$\|< \vartheta_1;\, z >_X\|_X^2 \le \|\vartheta_1\|_Y^2 \|z\|_Z^2,$$

$$\|< y_n;\, z >_X\|_X = \left\|\vartheta_{(y_n;\, z)}\right\| = \sup_{\|x\|_X = 1} \left|\vartheta_{(y_n;\, z)}(x)\right| =$$

$$= \sup_{\|x\|_X = 1} |(x \otimes y_n;\, z)_Z| = \sup_{\|x\|_X = 1} |(x \otimes y_n;\, z_1)| = \|< y_n;\, z_1 >_X\|_X.$$

Thus
$$\sum_{n=1}^{\infty} \|< \vartheta_n;\, z >_X\|_X^2 \leq \|\vartheta_1\|_Y^2 \, \|z\|_Z^2 + \sum_{n=1}^{\infty} \|< y_n;\, z_1 >_X\|_X^2 \leq$$
$$\leq \|\vartheta_1\|_Y^2 \, \|z\|_Z^2 + B\, \|z_1\|_Z^2 \leq \left(\|\vartheta_1\|_Y^2 + B\right) \, \|z\|_Z^2 \, . \quad (24)$$

From (23), (24) it follows that the system $\vec{\vartheta}$ forms a *t-frame* for $\overline{L_t\left[\vec{\vartheta}\right]}$. So we have

Lemma 8.9. *Let $\vec{y} \subset Y$ be a t-frame sequence, i.e. let $\vec{y} \subset Y$ form a t-frame for $\overline{L_t\,[\vec{y}]}$. Then the system $\vec{y} \bigcup \{\vartheta\}$ is also a t-frame sequence for $\forall \vartheta \in Y$.*

We also have

Corollary 8.10. *Let $\vec{y} \subset Y$ be a t-frame sequence. Then the system $\vec{y} \bigcup \{\vartheta_k\}_{k\in F}$ is also a t-frame sequence for $\forall F \subset N : cardF < +\infty$.*

Using Corollaries 8.8 and 8.10 , we obtain the validity of the following statement.

Statement 8.11. *Let $\vec{z};\, \vec{y} \subset Y$ and $card\,\{k : z_k \neq y_k\} < +\infty$. Then the system $\vec{z}$ is a t-frame sequence if and only if $\vec{y}$ has the same property.*

9. Noetherian perturbation

9.1. Noetherian closeness

Let $X;\, Y_k,\ k = 1,\, 2,$ be H-spaces, and $T \in L\,(Y_1;\, Y_2)$ be a Noetherian operator. Let $Z_k = X \otimes Y_k,\ k = 1,\, 2$. Suppose the system $\vec{y}^{(1)}$ forms a *t-frame* for Z_1. Assume $\vec{y}^{(2)} = T\,\vec{y}^{(1)}$, i.e. $\vec{y}_n^{(2)} = T\,\vec{y}_n^{(1)},\ \forall n \in N$. Denote $\tilde{Y} = \overline{L\left[\vec{y}^{(2)}\right]}$. It is absolutely clear that $\tilde{Y} = \mathscr{R}_T$. Let Y_1 be represented in the form of a direct sum: $Y_1 = KerT \dot{+} \tilde{Y}_1$. Denote by $T_1 = T\Big/_{\tilde{Y}_1}$ the restriction of the operator T on $\tilde{Y}_1$. It is clear that $T_1 \in L\left(\tilde{Y}_1;\, \tilde{Y}\right)$. Besides, T_1 is boundedly invertible as $\mathscr{R}_{T_1} = \mathscr{R}_T$ (the invertibility follows from the Banach theorem). Following [8], we call the operator T_1^{-1} pseudo-inverse of T. Let $\tilde{Z} = X \otimes \tilde{Y}$. Assume $\tilde{T} = I_X \otimes T_1$. It is absolutely clear that $\tilde{T} \in L\left(Z_1;\, \tilde{Z}\right)$, where $Z_1 = X \otimes \tilde{Y}_1$. Besides, this operator is boundedly invertible. Take $\forall \tilde{z} \in \tilde{Z}$. Let $z = \tilde{T}^{-1}\tilde{z} \in Z_1$. Assume that the system $\vec{y}^{(1)}$ forms a *t-frame* for Z. Then it is clear that z has the following decomposition
$$z = \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1}z >_X \otimes y_n^{(1)}.$$

It is not difficult to see that
$$(I_X \otimes T_1)\, z = (I_X \otimes T)\, z,\ \ \forall z \in Z_1.$$

We have

$$\tilde{z} = \sum_{n=1}^{\infty} (I_X \otimes T) \left(< y_n^{(1)};\, S^{-1} z >_X \otimes y_n^{(1)} \right) =$$

$$= \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1} z >_X \otimes T y_n^{(1)} = \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1} z >_X \otimes y_n^{(2)} =$$

$$= \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1} \tilde{T}^{-1} \tilde{z} >_X \otimes y_n^{(2)}. \tag{25}$$

Let us establish *t-frame* estimates. We have

$$\left\| < y_n^{(2)};\, z >_X \right\|_X = \left\| \vartheta_{\left(y_n^{(2)};\, z\right)} \right\| = \sup_{\|x\|_X = 1} \left| \vartheta_{\left(y_n^{(2)};\, z\right)}(x) \right| =$$

$$= \sup_{\|x\|_X = 1} \left(x \otimes y_n^{(2)};\, z \right)_Z = \sup_{\|x\|_X = 1} \left((I_X \otimes T) \left(x \otimes y_n^{(1)} \right);\, z \right)_Z =$$

$$= \sup_{\|x\|_X = 1} \left(x \otimes y_n^{(1)};\, (I_X \otimes T)^* z \right)_Z = \left\| < y_n^{(1)};\, (I_X \otimes T)^* z >_X \right\|_X .$$

As the system $\vec{y}^{(1)}$ forms a *t-frame* for Z, we have

$$\sum_{n=1}^{\infty} \left\| < y_n^{(2)};\, z >_X \right\|_X^2 = \sum_{n=1}^{\infty} \left\| < y_n^{(1)};\, (I_X \otimes T)^* z >_X \right\|_X^2 \leq$$

$$\leq B \left\| (I_X \otimes T)^* z \right\|_Z^2 \leq B \left\| I_X \otimes T^* \right\|^2 \|z\|_Z^2 =$$

$$= B \|T^*\|^2 \|z\|_Z^2 = B \|T\|^2 \|z\|_Z^2 , \ \ \forall z \in \tilde{Z}.$$

Let us establish the opposite inequality. Take $\forall z \in \tilde{Z}$. Then z has a decomposition (25):

$$z = \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1} \tilde{T}^{-1} z >_X \otimes y_n^{(2)} =$$

$$= \sum_{n=1}^{\infty} \tilde{x}_n \otimes y_n^{(2)}, \ \text{ where } \tilde{x}_n = < y_n^{(1)};\, S^{-1} \tilde{T}^{-1} z >_X , \ \ \forall n \in N.$$

Consequently

$$\|z\|_Z^2 = \left(\sum_{n=1}^{\infty} \tilde{x}_n \otimes y_n^{(2)};\, z \right)_Z = \sum_{n=1}^{\infty} \left(\tilde{x}_n;\, < y_n^{(2)};\, z >_X \right)_X \leq$$

$$\leq \left(\sum_{n=1}^{\infty} \|\tilde{x}_n\|_X^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \left\| < y_n^{(2)};\, z >_X \right\|_X^2 \right)^{1/2} . \tag{26}$$

As the system $\vec{y}^{(1)}$ forms a *t-frame*, we have

$$\sum_{n=1}^{\infty} \|\tilde{x}_n\|_X^2 = \sum_{n=1}^{\infty} \left\| < y_n^{(1)};\, S^{-1}\tilde{T}^{-1}z >_X \right\|_X^2 \leq$$

$$\leq B \left\| S^{-1}\tilde{T}^{-1}z \right\|_Z^2 \leq B \left\| S^{-1} \right\|^2 \left\| \tilde{T}^{-1} \right\|^2 \|z\|_Z^2 .$$

Taking into account this relation in (26), we obtain

$$\|z\|_Z^2 \leq B \left\| S^{-1} \right\|^2 \left\| \tilde{T}^{-1} \right\|^2 \sum_{n=1}^{\infty} \left\| < y_n^{(2)};\, z >_X \right\|_X^2 .$$

Thus we have proved our main

Theorem 9.1. *Let $X;\, Y_k,\ k = 1, 2$ be H-spaces and $T \in L\,(Y_1;\, Y_2)$ be a Noetherian operator. If $\vec{y}^{(1)}$ forms a t-frame for $X \otimes Y_1$, then the system $T\,\vec{y}^{(1)}$ is a t-frame sequence in $X \otimes Y_2$.*

In particular, this theorem has the following

Corollary 9.2. *Suppose that all the conditions of Theorem 8 hold and T is a Fredholm operator. If $\vec{y}^{(1)}$ forms a t-frame for $X \otimes Y_1$ and $T\,\vec{y}^{(1)}$ is complete in Y_2, then $T\,\vec{y}^{(1)}$ forms a t-frame for $X \otimes Y_2$.*

In fact, the operator T is boundedly invertible in this case. The rest obviously follows from the definition of the frame.

9.2. Quadratically close t-frames.

Let $X;\, Y$ be H-spaces and $Z = X \otimes Y$.

Systems $\vec{y}^{(1)}; \vec{y}^{(2)} \subset Y$ are called quadratically close in Y, if $\sum_{n=1}^{\infty} \left\| y_n^{(1)} - y_n^{(2)} \right\|_Y^2 < +\infty$.

The following theorem is true.

Theorem 9.3. *Let the system $\vec{y}^{(1)} \subset Y$ form a t-frame for Z and $\vec{y}^{(2)} \subset Y$ be quadratically close to it. Then $\vec{y}^{(2)}$ is a t-frame sequence.*

Proof. Let us introduce a new system

$$y_n^{(3)} = \begin{cases} y_n^{(1)}, & 1 \leq n \leq n_0 - 1\,, \\ y_n^{(2)}, & n \geq n_0\,, \end{cases}$$

where $n_0 \in N$ is a number to be determined. Define the operator T:

$$Tz = \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1}z >_X \otimes y_n^{(3)},\ \ \forall z \in Z.$$

We have

$$\|(I_Z - T)\, z\|_Z = \left\| \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1}z >_X \otimes y_n^{(1)} - \right.$$

$$\left. - \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1}z >_X \otimes y_n^{(3)} \right\|_Z = \left\| \sum_{n=n_0}^{\infty} < y_n^{(1)};\, S^{-1}z >_X \otimes \left(y_n^{(1)} - y_n^{(2)} \right) \right\|_Z \leq$$

$$\leq \left(\sum_{n=n_0}^{\infty} \left\| < y_n^{(1)};\, S^{-1}z >_X \right\|_X^2 \right)^{1/2} \left(\sum_{n=n_0}^{\infty} \left\| y_n^{(1)} - y_n^{(2)} \right\|_Y^2 \right)^{1/2} \leq$$

$$\leq B^{1/2} \left\| S^{-1} \right\| \left(\sum_{n=n_0}^{\infty} \left\| y_n^{(1)} - y_n^{(2)} \right\|_Y^2 \right)^{1/2} \|z\|_Z .$$

If

$$\sum_{n=n_0}^{\infty} \left\| y_n^{(1)} - y_n^{(2)} \right\|_Y^2 < B^{-1} \left\| S^{-1} \right\|^{-2},$$

then it is clear that $\|I_Z - T\| < 1$, and, as a result, the operator $T \in L(Z)$ is boundedly invertible. We have

$$z = TT^{-1}z = \sum_{n=1}^{\infty} < y_n^{(1)};\, S^{-1}T^{-1}z >_X \otimes y_n^{(3)},\ \forall z \in Z.$$

Using this representation, it is easy to prove the *t-frameness* of the system $\vec{y}^{(3)}$ in Z. In fact, let $x_n =< y_n^{(1)};\, S^{-1}T^{-1}z >_X,\ \forall n \in Z$. Consequently

$$\|z\|_Z^2 = (z;\, z)_Z = \left(\sum_{n=1}^{\infty} x_n \otimes y_n^{(3)};\, z \right)_Z =$$

$$= \sum_{n=1}^{\infty} \left(x_n \otimes y_n^{(3)};\, z \right)_Z = \sum_{n=1}^{\infty} \left(x_n;\, < y_n^{(3)};\, z >_X \right)_X \leq$$

$$\leq \left(\sum_{n=1}^{\infty} \|x_n\|_X^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \left\| < y_n^{(3)};\, z >_X \right\|_X^2 \right)^{1/2} . \tag{27}$$

As $\vec{y}^{(1)}$ forms a *t-frame*, we have

$$\sum_{n=1}^{\infty} \|x_n\|_X^2 \leq B \left\| S^{-1} \right\|^2 \left\| T^{-1} \right\| \|z\|_Z^2 .$$

Using this relation in (27), we get the lower *t-frame* estimate for $\vec{y}^{(3)}$.

Now let us prove the validity of upper estimate. It is quite obvious that the inequality $\sum_{n=1}^{\infty}\left\|y_n^{(3)}-y_n^{(1)}\right\|_Y^2<+\infty$ is true. We have

$$\left\|< y_n^{(3)};\, z >_X\right\|_X^2 \le 2\left(\left\|< y_n^{(3)}-y_n^{(1)};\, z >_X\right\|_X^2+\left\|< y_n^{(1)};\, z >_X\right\|_X^2\right)\le$$

$$\le 2\left(\left\|y_n^{(3)}-y_n^{(1)}\right\|_Y^2\|z\|_Z^2+\left\|< y_n^{(1)};\, z >_X\right\|_X^2\right).$$

Consequently

$$\sum_{n=1}^{\infty}\left\|< y_n^{(3)};\, z >_X\right\|_X^2\le 2\sum_{n=1}^{\infty}\left\|y_n^{(3)}-y_n^{(1)}\right\|_Y^2\|z\|_Z^2+$$

$$+B\|z\|_Z^2=B_1\|z\|_Z^2\,,\ \ \forall z\in Z,$$

where $B_1=B+2\sum_{n=1}^{\infty}\left\|y_n^{(3)}-y_n^{(1)}\right\|_Y^2$. As a result, we obtain that the system $\vec{y}^{(3)}$ forms a *t-frame* for Z. Applying Statement 2, we get the validity of theorem. So the theorem is proved.

Example 2. Denote by $\mathscr{L}_2(C^m)\equiv L_2((a,\,b);\,C^m)$ the C^m-valued Bochner measurable functions, squared C^m-norms of which are summable on $I\equiv(a,\,b)$, where $-\infty\le a<b\le+\infty$. Consider the Hilbert structure in $\mathscr{L}_2(C^m)$, and let $L_2(I)$ be a usual Lebesgue space. It is known (see, e.g., [41]) that $\mathscr{L}_2(C^m)$ is isometrically isomorphic to $C^m\otimes L_2(I)$. Take $\forall\vec{x}=(x_1,\,\ldots;\,x_m)\in C^m$ and $\forall y\in L_2(I)$. Tensor product $\vec{x}\otimes y$ is realized as the product $y(t)\,\vec{x}$: $\vec{x}\otimes y=y(t)\,\vec{x}$. Let $\vec{z}\in\mathscr{L}_2(C^m)$, $\vec{z}(t)=(z_1(t);\,..;\,z_m(t))$. We have

$$\vartheta_{(y;\,\vec{z})}(\vec{x})=(\vec{x}\otimes y;\,\vec{z})_{\mathscr{L}_2(I)}=(\vec{x};\,< y;\,z >)_{C^m}\,,\ \ \forall\vec{x}\in C^m.$$

Consequently

$$(\vec{x}\otimes y;\,\vec{z})_{\mathscr{L}_2(I)}=\int_a^b(y(t)\,\vec{x};\,\vec{z}(t))_{C^m}\,dt=$$

$$=\int_a^b\sum_{k=1}^m x_k\overline{z_k(t)}y(t)\,dt=\sum_{k=1}^m x_k\int_a^b\overline{z_k(t)}y(t)\,dt.$$

From these relations we immediately obtain that

$$< y;\,z >_{C^m}=\left(\int_a^b\overline{z_1(t)}y(t)\,dt;\,\ldots;\,\int_a^b\overline{z_m(t)}y(t)\,dt\right)\in C^m,$$

and, as a result

$$\|< y;\,\vec{z} >_{C^m}\|_{C^m}^2=\sum_{k=1}^m\left|\int_a^b\overline{z_m(t)}y(t)\,dt\right|^2.$$

From this relation it follows directly that if the system $\{y_n\}_{n\in N}\subset L_2(I)$ forms an (ordinary) frame for $L_2(I)$, then it forms a *t-frame* for $\mathscr{L}_2(C^m)$.

In particular, taking into account (16), we now obtain that the arbitrary element $\vec{z}(t) = (z_1(t); ...; z_m(t)) \in \mathrm{L}_2(C^m)$ has an expansion

$$\vec{z}(t) = \sum_{n=1}^{\infty} \vec{\lambda}_n y_n(t),$$

where $\vec{\lambda}_n = \left(\lambda_1^{(n)}; ...; \lambda_m^{(n)}\right) \in C^m$, $\forall n \in N$. This fact easily follows from the definition of an ordinary frame. Similarly we can consider the space $\mathrm{L}_2(H) \equiv L_2((a,b); H)$ of H-valued Bochner measurable functions, squared H-norms of which are summable on (a,b), where H is some Hilbert space.

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