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# On the Completeness of System of Cosines in Weighted Morrey Spaces

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**Abstract.** In this work the problem of the completeness of the classical system of cosines is considered in a weighted Morrey spaces with a power weight. These spaces, generally speaking, are not separable. Therefore, classical trigonometric systems are not complete in these spaces. Starting from the shift operator, a subspace of Morrey space in which continuous functions are dense is defined. A sufficient condition on the weight function is found, under which the cosine system is complete in this subspace.

**Key Words and Phrases**: Morrey space, completeness, system of cosines
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## 1. Introduction

Morrey spaces were introduced by Morrey, see [1], in the setting of partial differential equations, and presented in various books, see [2, 3, 4, 5], survey papers [6, 7, 8] and the references therein. The splash of interest to Morrey-type spaces during the last decade has advances in many areas, which allow to consider the basis properties of systems in such spaces in order to fill the gaps in the theory of Morrey spaces. These problems arise naturally in the solution of many partial differential equations by the Fourier method.

Several authors have studied the basis properties of trigonometric systems in Banach function spaces. Well-known results concerning the basis properties of the systems of exponentials in the case of the Lebesgue space $L_p$, $(1 < p < \infty)$, can be found in [9, 10, 11]. Babenko [12] has proved that the degenerate system of exponentials $\left\{|t|^{\alpha}\, e^{int}\right\}_{n\in\mathbb{Z}}$ with $|\alpha| < \frac{1}{2}$ forms a basis for $L_2\left(-\pi, \pi\right)$ but does not form a Riesz basis when $\alpha \neq 0$, where $\mathbb{Z}$ is the set of integers. This result has been generalized by Gaposhkin [13]. In [14], the conditions on the weight function $\rho$, for which the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ forms an unconditional basis for the weighted Besov space have been obtained. Similar problems have been studied in [15, 16, 17, 18, 38, 39]. The basicity of the systems of sines and cosines with degenerate coefficients have been widely analyzed. Amongst the Banach spaces where the basicity are known we mention the Lebesgue space $L_p$, $(1 < p < \infty)$, [19, 20]. Basis properties of the systems of sines, cosines and exponentials with the linear phase in weighted Lebesgue space have been studied in [21, 22, 23]; see also [24, 25, 26].

 

The basis properties of the systems of sines, cosines and exponentials in Morrey spaces are much less studied. In the paper [27], there were studied the basis properties of the system of exponentials in Morrey space. Also, in [28, 37] the basis properties of the perturbed systems of exponentials in Morrey space have been investigated. On the other hand, some approximation problems have been investigated in Morrey-Smirnov classes in [29].

We will use the standard notation. Denote the set of natural numbers by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$. We denote by $L[M]$ the linear span of the set $M$. $\overline{M}$ will stand for the closure of the set $M$. $\|\cdot\|_\infty$ means sup-norm.

Our goal in this paper is the study of completeness of the system $\{\cos nt\}_{n\in\mathbb{N}_0}$ in weighted Morrey space $\mathcal{L}_\nu^{p,\lambda}(0,\pi)$ defined by a product of power weights of the form

$$\nu(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k}\,,\quad t\in[0,\pi]\,, \tag{1}$$

where $t_0=0, t_r=\pi$, and $t_k$ are arbitrary finite points in the interval $(0,\pi)$ for all $k=1,2,...,r-1$, and $\alpha_k\in\mathrm{R}$ for all $k=0,1,...,r$. Also, we will consider the weighted Morrey space $\mathcal{L}_\nu^{p,\lambda}(-\pi,\pi)$, where

$$\nu(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k}\,,\quad t\in[-\pi,\pi]\,, \tag{2}$$

and $t_k$ are arbitrary finite points in the interval $[-\pi,\pi]$ and $\alpha_k\in\mathrm{R}$ for all $k=0,1,...,r$.

Although the same properties of trigonometric systems, as well as their pertubations, are well studied in weighted Lebesgue spaces, the situation changes cardinally in Morrey spaces. For instance, since the functional characterization of dual spaces of Morrey spaces is not known, it avoids working with dual spaces. Another difficulty, that frustrates the "usual" attempts is that, the infinitely differentiable functions(even continuous functions) are not dense in Morrey spaces, but we still seek to prove "density" property of trigonometric functions, which are infinily differentiable. For these reasons, unlike the $L_p$ case, here will be used another methods to study the basis properties(especially, completeness and basisness) in weighted Morrey spaces.

In this work the problem of the completeness of the classical system of cosines is considered in a weighted Morrey spaces with a power weight. These spaces, generally speaking, are not separable. Therefore, classical trigonometric systems are not complete in these spaces. Starting from the shift operator, a subspace of Morrey space in which continuous functions are dense is defined. A sufficient condition on the weight function is found, under which the cosine system is complete in this subspace.

## 2. Preliminaries

### 2.1. (Weighted) Morrey space on an interval

For $1 < p < \infty$ and $0 \leq \lambda < 1$ we define the Morrey space $\mathcal{L}^{p,\lambda}(a,b)$ as the set of functions $f$ on $(a,b)$ such that

$$\|f\|_{p,\lambda} := \|f\|_{\mathcal{L}^{p,\lambda}(a,b)} = \sup_{I\subset(a,b)} \left( \frac{1}{|I|^{\lambda}} \int_I |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty,$$

where $I \subset (a,b)$ is any interval. It is clear that $\mathcal{L}^{p,\lambda}(a,b)$ are Banach spaces. Morrey spaces can be defined in a more general way (see e.g. [5, 8, 29]) but this is enough for our purposes. The $L_p(a,b)$ spaces with the Lebesgue measure correspond with the case $\lambda = 0$. The weighted Morrey space $\mathcal{L}^{p,\lambda}_{\nu}(a,b)$ is defined in the usual way

$$\mathcal{L}^{p,\lambda}_{\nu}(a,b) := \left\{ f : \nu f \in \mathcal{L}^{p,\lambda}(a,b) \right\},$$

with $\|f\|_{p,\lambda;\nu} := \|\nu f\|_{p,\lambda}$. The function $\nu$ is called the weight or weight function of this space.

It is evident that the space $\mathcal{L}^{p,\lambda}_{\nu}(a,b)$ contains constant functions if and only if $\nu \in \mathcal{L}^{p,\lambda}(a,b)$. Throughout the paper, unless otherwise stated, we will assume that $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$ and $0 < \lambda < 1$. Also, the letter "$c$" denotes a positive constant, which is not necessarily the same at each occurance but is independent of the essential variable and quantities. The expression $f\sim g$, $t \to a$ means that in suffeciently small neighborhood $O_\delta$ of the point $t = a$, the inequalities $0 < \delta \leq \left|\frac{f(t)}{g(t)}\right| \leq \delta^{-1} < \infty$ hold in $O_\delta$. If the last inequalities hold on an interval $I$, we write $f\sim g$ on $I$. For example $\sin t \sim t(\pi - t)$ on $[0,\pi]$.

We assume here some familiarity with basic concepts of basis theory and we refer to the books of Heil [30], Christensen [31], Singer [32, 33] and Bilalov B.T. [39] for basic definitions such as complete and minimal systems and basis in Banach spaces.

The following lemma has been proved by Samko [34] in the case of Morrey space on a bounded rectifiable curve. In our case it reads

**Lemma 1.** *The power function $|t - t_0|^{\alpha}$, $t_0 \in [a,b]$, belongs to the Morrey space $\mathcal{L}^{p,\lambda}(a,b)$ if and only if $\alpha \in \left[\frac{\lambda-1}{p}, \infty\right)$.*

Direct application of the above lemma implies the following

**Proposition 1.** *Let $\nu$ be given as in* (1)*. Then*
$\{\cos nt\}_{n\in \mathbb{N}_0} \subseteq \mathcal{L}^{p,\lambda}_{\nu}(0,\pi), 0 < \lambda < 1,$ *if and only if*

$$\alpha_k \in \left[\frac{\lambda-1}{p}, \infty\right), \quad for \ all \ \ k = 0,1,2,...,r. \tag{3}$$

**Remark 1.** *The case $\lambda > 0$ differs from the case $\lambda = 0$: when $\lambda = 0$, conditions* (3) *must be replaced by the conditions*

$$\alpha_k \in \left(-\frac{1}{p}, \infty\right), \quad for \ \ all \ \ k = 0,1,2,...,r.$$

## 2.2. Auxiliary propositions

Let us start by considering the space

$$\left(\mathcal{L}^{p,\lambda}\right)' = \left\{ g : \sup_{\|f\|_{p,\lambda}=1} \|f\, g\|_{L_1} < +\infty \right\},$$

with the norm

$$\|g\|_{\left(\mathcal{L}^{p,\lambda}\right)'} = \sup_{f\in\mathcal{L}^{p,\lambda},\|f\|_{p,\lambda}=1} \|fg\|_{L^1}.$$

It can be proved that $\left(\mathcal{L}^{p,\lambda}\right)'$ is a normed space and the following inequality is satisfied

$$\|fg\|_{L^1} \le \|f\|_{p,\lambda} \|g\|_{\left(\mathcal{L}^{p,\lambda}\right)'}, \tag{4}$$

for all $f \in \mathcal{L}^{p,\lambda}$ and $g \in \left(\mathcal{L}^{p,\lambda}\right)'$.

The following lemma is true.

**Lemma 2.** $|t|^{\beta} \in \left(\mathcal{L}^{p,\lambda}(-\pi,\pi)\right)' \Leftarrow \beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right), 0 \le \lambda < 1, 1 < p < +\infty.$

The following lemma is also true.

**Lemma 3.** $|t|^{\beta} \in \left(\mathcal{L}^{p,\lambda}(0,\pi)\right)' \Leftarrow \beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right), 0 \le \lambda < 1, 1 < p < +\infty.$

*Proof.* Firstly, suppose $\beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. Then, for all $f \in \mathcal{L}^{p,\lambda}(0,\pi)$, we have

$$\int_{-\pi}^{\pi} |t|^{\beta} |f(t)|\, dt = \sum_{k=1}^{\infty} \int_{t\in\left[2^{-k-1}\pi, 2^{-k}\pi\right]} |t|^{\beta} |f(t)|\, dt$$

$$\le c \sum_{k=1}^{\infty} 2^{-k\beta} \int_{t\in\left[2^{-k-1}\pi, 2^{-k}\pi\right]} |f(t)|\, dt$$

$$\le c \sum_{k=1}^{\infty} 2^{-k\beta} 2^{-k\left(1-\frac{1}{p}\right)} \left( \int_{t\in\left[2^{-k-1}\pi, 2^{-k}\pi\right]} |f(t)|^{p}\, dt \right)^{\frac{1}{p}}$$

$$= c \sum_{k=1}^{\infty} 2^{-k\left(\beta+1-\frac{1}{p}+\frac{\lambda}{p}\right)} \|f\|_{p,\lambda} \le c \|f\|_{p,\lambda}.$$

Then, $|t|^{\beta} \in \left(\mathcal{L}^{p,\lambda}(0,\pi)\right)'$.

Conversely, suppose that $\beta \notin \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. That is $\beta + \frac{\lambda-1}{p} \le -1$.

Then, $|t|^{\frac{\lambda-1}{p}} \in \mathcal{L}^{p,\lambda}(0,\pi)$ and

$$\int_{0}^{\pi} |t|^{\beta} |t|^{\frac{\lambda-1}{p}}\, dt = \int_{0}^{\pi} |t|^{\beta+\frac{\lambda-1}{p}}\, dt = \infty.$$

Thus $|t|^{\beta} \notin \left(\mathcal{L}^{p,\lambda}\right)'$. This completes the proof. ◄

## 2.3. Zorko subspace of weighted Morrey space

Denote by $C_0^{\infty}[-\pi,\pi]$ the set of all infinitely differentiable functions with compact support in $(-\pi,\pi)$. We observe that functions in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ can not be approximated by functions in $C_0^{\infty}[-\pi,\pi]$, nor even by continuous functions. That is the set $C_0^{\infty}[-\pi,\pi]$ is not dense in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ (*c.f.* [5,35]). This fact still valid in the weighted setting of Morrey space. For example, let $\nu$ be given as in (2) under conditions (3). Let $\tau_0 \neq t_k$, $\forall k = \overline{0,r}$, $\tau_0 \in (-\pi,\pi)$ be any points. Then, there exists sufficianly small $\delta_0 > 0$, so that

$$t_k \notin O_{\delta_0} \subset (-\pi,\pi)\,,\ \forall k = \overline{0,r}\,,$$

where $O_{\delta_0} = [\tau_0, \tau_0 + \delta_0]$. Then it's clear that $g^{\pm}_{\delta_0}(t) = \chi_{O_{\delta_0}}(t)\,\nu^{\pm 1}(t)$ is a continuous function on $[-\pi,\pi]$. Consider the function

$$f(t) = |t-\tau_0|^{\frac{\lambda-1}{p}}\,\nu^{-1}(t)\,.$$

It's obvious that $f \in L^{p,\lambda}_{\nu}(-\pi,\pi)$. Let $g \in C[-\pi,\pi]$ be any function. From (3) it follows that $g \in L^{p,\lambda}_{\nu}(-\pi,\pi)$. We have

$$\|f-g\|_{L^{p,\lambda}_{\nu}(-\pi,\pi)} \geq \|f-g\|_{L^{p,\lambda}_{\nu}(O_{\delta_0})} =$$

$$= \|f\nu - g\nu\|_{L^{p,\lambda}(O_{\delta_0})} = \|F-G\|_{L^{p,\lambda}(O_{\delta_0})}\,,$$

where $F(t) = |t-\tau_0|^{\frac{\lambda-1}{p}} \in L^{p,\lambda}(O_{\delta_0})$, $G = g\nu \in C(O_{\delta_0})$. For the rest one needs to follow the corresponding example of Zorko [5, 35].

Let $f(\cdot)$ be the given function on $[a,b]$. In determining the Zorko type subspace we will assume that the function $f(\cdot)$ is continued to $[2a-b, 2b-a]$ with the following expression (and this function is also denoted by $f(\cdot)$ )

$$f(x) = \begin{cases} f(2a-x)\,,\ x \in [2a-b,a)\,, \\ f(2b-x)\,,\ x \in (b,2b-a]\,. \end{cases}$$

So, following Zorko [35], we consider the subspace

$$\widetilde{\mathcal{L}^{p,\lambda}_{\nu}}(a,b) := \left\{ f \in \mathcal{L}^{p,\lambda}_{\nu}(a,b) : \|f(.+\delta) - f(.)\|_{p,\lambda;\nu} \to 0\ as \delta \to 0 \right\},$$

where $\nu$ is given as in (2) under conditions (3). We will refer to this subspace as the Zorko subspace of $\mathcal{L}^{p,\lambda}_{\nu}(a,b)$. Also, we consider the $\mathcal{L}^{p,\lambda}_{\nu}$-closure of $\widetilde{\mathcal{L}^{p,\lambda}_{\nu}}(a,b)$ and denote it by $M^{p,\lambda}_{\nu}(a,b)$. It is easy to see that if $\nu \in \mathcal{L}^{p,\lambda}(a,b)$, then $C[-a,b] \subset M^{p,\lambda}_{\nu}(a,b)$. In fact, let $f \in C[a,b]$ be an arbitrary function and $\delta$ be an arbitrary number (with $|\delta|$sufficiently small). It is obvious that the extended function $f(\cdot)$ is continuous on $[2a-b, 2b-a]$. We have

$$\|f(\cdot+\delta)-f(\cdot)\|_{p,\lambda,\nu}=\sup_{I\subset(a,b)}\left(\frac{1}{|I|^{\lambda}}\int_I|(f(t+\delta)-f(t))\,\nu(t)|^p\,dt\right)^{1/p}\leq$$

$$\leq\sup_{t\in[a,b]}|f(t+\delta)-f(t)|\;\|\nu\|_{p,\lambda}\to 0,\;\;\delta\to 0.$$

Thus we have the following

**Lemma 4.** *If $\nu\in L^{p,\lambda}(a,b)$, then $C[a,b]\subset M_{\nu}^{p,\lambda}(a,b)$.*

Since $M_{\nu}^{p,\lambda}(a,b)$ is a closed subspace of $\mathcal{L}_{\nu}^{p,\lambda}(a,b)$, it also contains the $\mathcal{L}_{\nu}^{p,\lambda}$-closure of $C_0^{\infty}[a,b]$; in fact, $M_{\nu}^{p,\lambda}(a,b)$ is precisely that closure.

**Proposition 2.** *Let $\nu$ be given as in* (2) *and the following condition holds*

$$\alpha_k\in\left[-\frac{1-\lambda}{p},-\frac{1-\lambda}{p}+1\right),\;k=\overline{0,r}.\tag{5}$$

*Then the set $C^{\infty}[-\pi,\pi]$ is dense in $M_{\nu}^{p,\lambda}(-\pi,\pi)$.*

We need the following lemma.

**Lemma 5.** *[Minkowski inequality for integrals in weighted Morrey spaces] Let $(X;X_{\sigma};\mu)$ be a measurable space with an $\sigma$-additive measure $\mu(\cdot)$ on a set $X$, $\nu=\nu(t)$ a weight function, $dy$ a linear Lebesgue measure on an interval $(a,b)$ and $F(x,y)$ is $\mu\times dy$-measurable. If $1\leq p<\infty$, then*

$$\left\|\int_X F(x,y)d\mu(x)\right\|_{p,\lambda;\nu}\leq\int_X\|F(x,y)\|_{p,\lambda;\nu}\,d\mu(x).$$

*Proof.* By using the Minkowski inequality for integrals in $L_p(a,b)$,

$$\left\|\int_X F(x,y)\nu(y)d\mu(x)\right\|_{L_p}\leq\int_X\|F(x,y)\nu(y)\|_{L_p}\,d\mu(x),$$

we have

$$\left(\int_{B_r(x)}\left|\int_X F(x,y)\nu(y)d\mu(x)\right|^p dy\right)^{\frac{1}{p}}\leq\int_X\left(\int_{B_r(x)}|F(x,y)\nu(y)|^p\,dy\right)^{\frac{1}{p}}d\mu(x),$$

where $B_r(x)$ is a ball with a radius $r>0$ and the center at $x\in X$. Then

$$\left(\frac{1}{r^{\lambda}}\int_{B_r(x)}\left|\int_X F(x,y)\nu(y)d\mu(x)\right|^p dy\right)^{\frac{1}{p}}$$

$$\leq \int_X \left( \frac{1}{r^\lambda} \int_{B_r(x)} |F(x,y)\nu(y)|^p \, dy \right)^{\frac{1}{p}} d\mu(x).$$

The required result follows by taking the supremum over all $x \in (a,b)$ and $r > 0$ in the last inequality. ◄

It is now easy to provide the

**Proof of Proposition 2.** Let $f \in M_\nu^{p,\lambda}(-\pi,\pi)$, and $\varepsilon > 0$, be a sufficiently small number. Consider the function

$$w_\varepsilon(t) = \begin{cases} c_\varepsilon e^{\left(\frac{-\varepsilon^2}{\varepsilon^2 - t^2}\right)}, & |t| < \varepsilon, \\ 0, & |t| \geq \varepsilon, \end{cases}$$

where $c_\varepsilon$ is chosen such that $\int_{-\infty}^{\infty} w_\varepsilon(t)dt = 1$. Define the function $f_\varepsilon(t)$ as

$$f_\varepsilon(t) = \int_{-\infty}^{\infty} w_\varepsilon(s) f(t-s) ds.$$

As $\varepsilon > 0$ is sufficiently small, this definition is correct. Indeed, it is enough to prove that $f \in L_1(-\pi,\pi)$. From $f \in M_\nu^{p,\lambda}(-\pi,\pi)$ it follows that $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$. Let (5) holds. By using Lemma 2 it is easy to prove that $\nu^{-1} \in \left(L^{p,\lambda}(-\pi,\pi)\right)'$. Since $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$, we have $f = (f\nu)\,\nu^{-1} \in L_1(-\pi,\pi)$.

It is clear that $f_\varepsilon(t)$ is infinitely differentiable function on $[-\pi,\pi]$, and

$$\|f_\varepsilon - f\|_{p,\lambda;\nu} = \left\| \int_{-\infty}^{\infty} w_\varepsilon(s) f(t-s) ds - f(t) \right\|_{p,\lambda;\nu}$$

$$= \left\| \int_{-\infty}^{\infty} w_\varepsilon(s) \left[ f(t-s) - f(t) \right] ds \right\|_{p,\lambda;\nu}$$

Applying Lemma 5, we get

$$\|f_\varepsilon - f\|_{p,\lambda;\nu} \leq \int_{-\infty}^{\infty} \|w_\varepsilon(s) \left[ f(.-s) - f(.) \right]\|_{p,\lambda;\nu} \, ds$$

$$\leq \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(s) ds$$

$$= \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \to 0 \;\; as \;\; \varepsilon \to 0.$$

This completes the proof.

By similar way we can define $M_\nu^{p,\lambda}(0,\pi)$ and prove the following

**Proposition 3.** *Let $\nu$ be given as in* (1) *and the conditions* (5) *be satisfied. Then the set $C^\infty[0,\pi]$, of all infinitely differentiable functions with compact support in $(0,\pi)$, is dense in $M_\nu^{p,\lambda}(0,\pi)$.*

## 3. Main result

In this section we will establish the completeness of system of cosines in weighted Morrey spaces.

**Theorem 1.** *The system* $\{\cos nt\}_{n\in N_0}$ *is complete in* $M_\nu^{p,\lambda}(0,\pi), 0<\lambda<1, 1<p<+\infty$, *if conditions*

$$\alpha_0;\alpha_r \in \left(-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right), \ \alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p}+1\right), \ k=\overline{1,r-1}, \qquad (6)$$

*are satisfied.*

*Proof.* First, note that $\{\cos nt\}_{n\in N_0} \subset M_\nu^{p,\lambda}(0,\pi)$. Indeed, by Lemma 1 under (5) we have $\nu \in L^{p,\lambda}(0,\pi)$. Then from Lemma 4 we have $C[0,\pi] \subset M_\nu^{p,\lambda}(0,\pi)$, and as a result $\{\cos nt\}_{n\in N_0} \subset M_\nu^{p,\lambda}(0,\pi)$. Show that under (6) the set $C_0^\infty[0,\pi]$ is also dense in $M_\nu^{p,\lambda}(0,\pi)$. Indeed, from Proposition 3, we have that the set $C^\infty[0,\pi]$ is dense in $M_\nu^{p,\lambda}(0,\pi)$. Let$f \in M_\nu^{p,\lambda}(0,\pi)$ be any function and $\varepsilon>0$ be any number. Then $\exists g \in C^\infty[0,\pi]$:

$$\|f-g\|_{p,\lambda;\nu} < \frac{\varepsilon}{2}.$$

Set $E_\delta^+ = (0,\delta)$, $E_\delta^- = (\pi-\delta,\pi)$. We have

$$\left\|g\chi_{E_\delta^\pm}\right\|_{L_\nu^{p,\lambda}(0,\pi)} = \|g\|_{L_\nu^{p,\lambda}(E_\delta^\pm)} \le \|g\|_\infty \|\nu\|_{L^{p,\lambda}(E_\delta^\pm)}.$$

For sufficiently small $\delta>0$ we get

$$\|\nu\|_{L^{p,\lambda}(E_\delta^+)} \le C\|t^{\alpha_0}\|_{L^{p,\lambda}(E_\delta^+)} \to 0, \ \delta\to 0.$$

Analogously we have

$$\|\nu\|_{L^{p,\lambda}(E_\delta^-)} \le C\|(\pi-t)^{\alpha_r}\|_{L^{p,\lambda}(E_\delta^-)} \to 0, \ \delta\to 0.$$

Let $\delta_0 < \frac{1}{2}\min\{t_1;\pi-t_{r-1}\}$ is so that

$$\|\nu\|_{L^{p,\lambda}(E_\delta^+)} + \|\nu\|_{L^{p,\lambda}(E_\delta^-)} < \frac{\varepsilon}{4\|g\|_\infty}, \ \forall\delta\in(0,\delta_0).$$

Set

$$g_{\delta_0}(t) = \begin{cases} g(t), \ t\in(0,\pi)\setminus\left(E_{\delta_0/2}^+ \bigcup E_{\delta_0/2}^-\right), \\ 0, \ t\in\left(E_{\delta_0/2}^+ \bigcup E_{\delta_0/2}^-\right). \end{cases}$$

Consider

$$G_{\delta_0;\tau}(t) = \int_{-\infty}^{\infty} \omega_\varepsilon(s)\, g_{\delta_0}(t-s)\,ds.$$

It is clear that

$$\|G_{\delta_0;\tau} - g_{\delta_0}\|_{p,\lambda;\nu} \to 0,\ \tau \to 0.$$

Since $g_{\delta_0}(\cdot)$ is finitly supported on $(0,\pi)$, for sufficiently small $\tau > 0$ the function $G_{\delta_0;\tau}$ is also finitly supported on $(0,\pi)$, and as a result $G_{\delta_0;\tau} \in C_0^\infty[0,\pi]$. Let $\tau < \frac{\delta_0}{2}$ be so that

$$\|G_{\delta_0;\tau_0} - g_{\delta_0}\|_{p,\lambda;\nu} < \frac{\varepsilon}{4}.$$

We have

$$\|f - G_{\delta_0;\tau_0}\|_{p,\lambda;\nu} \le \|f-g\|_{p,\lambda;\nu} + \|g - g_{\delta_0}\|_{p,\lambda;\nu} +$$

$$+ \|g_{\delta_0} - G_{\delta_0;\tau_0}\|_{p,\lambda;\nu} \le \frac{\varepsilon}{2} + \|g\|_{L_\nu^{p,\lambda}\left(E^+_{\delta_0/2}\bigcup E^-_{\delta_0/2}\right)} + \frac{\varepsilon}{4} < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, from here we get that $C_0^\infty[0,\pi]$ is dense in $M_\nu^{p,\lambda}(0,\pi)$.

So, for every $f \in M_\nu^{p,\lambda}(0,\pi)$ and $\varepsilon > 0$, there exists $f_\varepsilon \in C_0^\infty[0,\pi]$ such that $\|f - f_\varepsilon\|_{p,\lambda;\nu} < \varepsilon$. It is known that the Fourier sine series of $f_\varepsilon$ converges uniformly to this function on $[0,\pi]$. That is, if

$$S_m(t) = \sum_{n=1}^{m} c_n(f_\varepsilon)\cos nt,\ m \in \mathrm{N},$$

where $c_n(f_\varepsilon) = \frac{2}{\pi}\int_0^\pi f_\varepsilon(t)\cos nt\, dt$, then there exists $m_0 = m_0(\varepsilon) \in \mathrm{N}$, such that

$$\sup_{t\in 0,\pi]} |f_\varepsilon(t) - S_m(t)| < \varepsilon,\ \ for\ \ all\ \ m \ge m_0.$$

Therefore

$$\|f_\varepsilon - S_m\|_{p,\lambda;\nu} = \sup_{I\subset(0,\pi)}\left(\frac{1}{|I|^\lambda}\int_I |f_\varepsilon(t) - S_m(t)|^p\,|\nu(t)|^p\,dt\right)^{\frac{1}{p}}$$

$$\le \varepsilon \sup_{I\subset(0,\pi)}\left(\frac{1}{|I|^\lambda}\int_I |\nu(t)|^p\,dt\right)^{\frac{1}{p}} = \varepsilon\,\|\nu\|_{p,\lambda}.$$

Then

$$\|f - S_m\|_{p,\lambda;\nu} \le \|f - f_\varepsilon\|_{p,\lambda;\nu} + \|f_\varepsilon - S_m\|_{p,\lambda;\nu} < \left(1 + \|\nu\|_{p,\lambda}\right)\varepsilon.$$

Thus, we arrive at the result since $\varepsilon$ was arbitrary. Thus, if the conditions (5) are satisfied, then the system $\{\cos nt\}_{n\in \mathrm{N}_0}$ is complete in $M_\nu^{p,\lambda}(0,\pi)$.

The theorem is proved. ◄

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