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On Generalized Fractional Integration of I-function

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Abstract. In the present paper, we study and develop the generalized fractional integral operators given by Saigo [7,8,9]. We establish two theorems that give the images of the product of I-function and a general class of polynomials in Saigo operators. On account of the general nature of the Saigo operators, I-function and a general class of polynomials, a large number of new and known images involving Riemann-Liouville and Erdelyi-Kober fractional integral operators and several special functions follow as special cases of our main findings.

Key Words and Phrases: I- Function, Generalized Polynomials, Fractional integral operators by Saigo, Riemann-Liouville and Erdelyi-Kober.

2010 Mathematics Subject Classifications: 33C45, 33C60

1. Introduction

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [7,8,9], has been introduced by Marichev [6] [see details in Samko et al.[10] and also see Kilbas and Saigo [see [4], p.258] as follows: Let α, β, η be complex numbers and $x > 0$, than the generalized fractional integral operators {The Saigo operators [7]} involving Gaussian hypergeometric function are defined by the following equations:

$$(J_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt; \quad Re(\alpha) > 0 \tag{1.1}$$

and

$$(J_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{(\alpha-1)} t^{-\alpha-\beta} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt; \quad Re(\alpha) > 0 \tag{1.2}$$

where ${}_2F_1\,(.)$ is the Gaussian hypergeometric function defined by

$${}_2F_1\,(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{1.3}$$

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When $\beta = -\alpha$, the above equations (1.1) and (1.2) reduce to the following classical Riemann-Liouville fractional integral operator (see Samko et al., [10], p.94, Eqns. (5.1), (5.3)):

$$(J_{0+}^{\alpha,-\alpha,\eta} f)(x) = (J_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)}\ f(t)dt; \quad Re(\alpha) > 0 \tag{1.4}$$

and

$$(J_{-}^{\alpha,-\alpha,\eta} f)(x) = (J_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{(\alpha-1)} f(t)dt; \quad Re(\alpha) > 0 \tag{1.5}$$

Again, if $\beta = 0$, the equations (1) and (2) reduce to the following Erdelyi-Kober fractional integral operator (see Samko et al., [10], p.322, Eqns. (18.5), (18.6)):

$$(J_{0+}^{\alpha,0,\eta} f)(x) = (J_{\eta,\alpha}^{+} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)}\ t^{\eta}\ f(t)dt; \quad Re(\alpha) > 0 \tag{1.6}$$

and

$$(J_{-}^{\alpha,0,\eta} f)(x) = (K_{\eta,\alpha}^{-} f)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{(\alpha-1)} t^{-\alpha-\eta}\ f(t)dt; \quad Re(\alpha) > 0 \tag{1.7}$$

The General class of Polynomials $S_n^m[x]$ occurring in the present paper was introduced by Srivastava [12] and defined as:

$$S_n^m[x] = \sum_{l=0}^{[n/m]} \frac{(-n_h)_{ml}}{l!} A_{n,l} x^l \tag{1.8}$$

Where n & m are arbitrary positive integers and the coefficients $A_{n,l}(n, l = 0)$ are arbitrary constants, real or complex. For the present study, we have used the I-function studied by Saxena [11] and defined as:

$$I_{P_i,Q_i;R}^{M,N}[z] = I_{P_i,Q_i;R}^{M,N}\left[\text{z} \left| \begin{matrix} (a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i} \\ (b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \phi(\xi)\, z^{\xi} d\xi \tag{1.9}$$

where

$$\phi(\xi) = \frac{\prod_{j'=1}^{M} \Gamma\left(b_{j'} - \beta_{j'}\xi\right) \prod_{j'=1}^{N} \Gamma\left(1 - a_{j'} - \alpha_{j'}\xi\right)}{\sum_{i'=1}^{R} \left[\prod_{j'=M+1}^{Q_{i'}} \Gamma\left(1 - b_{j'i'} - \beta_{j'i'}\xi\right) \prod_{j'=N+1}^{P_{i'}} \Gamma\left(a_{j'i'} - \alpha_{j'i'}\xi\right)\right]} \tag{1.10}$$

and $\omega = \sqrt{-1}$. For the conditions on the several parameters of the I-function, one can refer to [11].

2. Preliminary Lemmas

The following lemmas will be required to establish our main results.

Lemma 1*(Kilbas and Sebastian [5], p. 871, Eq. (15) to (18)). Let $\alpha, \beta, \eta \in C$ be such that $Re(\alpha) > 0$ and $Re(\mu) > max\{0, Re(\beta - \eta)\}$ then the following relation holds:*

$$(J_{0^+}^{\alpha,\beta,\eta} t^{\mu-1})(x) = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)} x^{\mu-\beta-1} \tag{2.1}$$

In particular, if $\beta = -\alpha$ and $\beta = 0$ in (2.1), we have:

$$\left(J_{0^+}^{\alpha} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)} x^{\mu+\alpha-1}, \quad Re(\alpha) > 0 \text{ and } Re(\mu) > 0 \tag{2.2}$$

and

$$\left(J_{\eta,\alpha}^{+} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)} x^{\mu-1}, \quad Re(\alpha) > 0 \text{ and } Re(\mu) > -Re(\eta) \tag{2.3}$$

Lemma 2 *(Kilbas and Sebastian [5], p. 872, Eq. (21) to (24)).*
Let $\alpha, \beta, \eta \in C$ be such that $Re(\alpha) > 0$ *and* $Re(\mu) > \max\{Re(\beta), Re(\eta)\}$ then the following relation holds:

$$\left(J_{-}^{\alpha,\beta,\eta} t^{\mu-1}\right)(x) = \frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)} x^{\mu-\beta-1} \tag{2.4}$$

In particular, if $\beta = -\alpha$ *and* $\beta = 0$ in (2.4), we have:

$$\left(J_{-}^{\alpha} t^{\mu-1}\right)(x) = \frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)} x^{\mu+\alpha-1}, \quad 1 - Re(\mu) > Re(\alpha) > 0 \tag{2.5}$$

and

$$\left(K_{\eta,\alpha}^{-} t^{\mu-1}\right)(x) = \frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)} x^{\mu-1}, \quad Re(\mu) < 1 + Re(\eta) \tag{2.6}$$

3. Main Results

Theorem 1

$$\left[J_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\Pi_{j=1}^{k} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right] I_{P_i,Q_i;R}^{M,N}\left[\mathrm{zt}^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i}\end{matrix}\right.\right]\right)\right](x)$$

$$= x^{\mu-\beta-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\left\{\frac{(-n_1)_{m_1 l_1}\ldots(-n_h)_{m_h l_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_j l_j}\right\}$$

$$\times I_{P_i+2,Q_i+2;R}^{M,N+2}\left[\mathrm{zx}^{\nu}\left|\begin{matrix}(1-\mu-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(1-\mu-\eta+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i},(1-\mu+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(1-\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1)\end{matrix}\right.\right] \tag{3.1}$$

The conditions of validity of (3.1) are as follows:
$(i)\alpha, \beta, \eta, a, b, z \in C$ and $\lambda_j, \upsilon > 0 \ \forall \ j\epsilon \ \overline{1,k}$
(ii) $|\arg z \ | < \frac{1}{2}\Omega_i\pi$; $\Omega_i > 0$ where

$$\Omega_i = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P_i} \alpha_{ji} + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q_i} \beta_{ji} \qquad \forall \ i\epsilon \ \overline{1,R}$$

$(iii) Re\,(\alpha) > 0$ and $Re\,(\mu) + \upsilon \min_{1\leq j\leq M} Re\left(\frac{b_j}{\beta_j}\right) > max\{0, Re(\beta-\eta)\}$

Proof. In order to prove (3.1), we first express the product of the general class of polynomials occurring on its left-hand side in the series form given by (1.8), and also express the I-function in terms of the MellinBarnes contour integral given by (1.9). Next, we interchange the order of summations and the integral and then on taking the fractional integral operator inside (which is permissible under the conditions stated), it takes the following form (say $\triangle$) after a little simplification:

$$\triangle = \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \cdots \sum_{l_k=0}^{[n_k/m_k]} \{\frac{(-n_1)_{m_1 l_1} \ldots (-n_h)_{m_h l_h}}{l_1! \ldots l_h!} A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h} c_1^{l_1} \ldots c_h^{l_h} x^{\Sigma_{j=1}^{k}\lambda_j l_j}\}$$

$$\times \frac{1}{2\pi\omega} \int_L \phi\,(\xi)\, z^{\xi} (J_{0^+}^{\alpha,\beta,\eta} t^{\mu+\Sigma_{j=1}^{k}\lambda_j l_j+\nu\xi-1})(x) d\xi$$

Finally, applying Lemma 1 and then re-interpreting the Mellin-Barnes contour integral thus obtained in terms of the I-function, we arrive at the RHS of (3.1) after a little simplification.

If we put $\beta = -\alpha$ in (3.1)then in view of (2.2), we get the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by (1.4):

Corollary 1.1

$$\left[J_{0^+}^{\alpha}\left(t^{\mu-1}\Pi_{j=1}^{k} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right] I_{P_i,Q_i;R}^{M,N}\left[\mathrm{z}t^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i}\end{matrix}\right.\right]\right)\right](x)$$

$$= x^{\mu-\beta-1} \sum_{l_1=0}^{[n_1/m_1]} \sum_{l_2=0}^{[n_2/m_2]} \cdots \sum_{l_k=0}^{[n_k/m_k]} \{\frac{(-n_1)_{m_1 l_1} \ldots (-n_h)_{m_h l_h}}{l_1! \ldots l_h!} A'_{n_1,l_1} \ldots A^{(k)}_{n_h,l_h} c_1^{l_1} \ldots c_h^{l_h} x^{\Sigma_{j=1}^{k}\lambda_j l_j}\}$$

$$\times I_{P_i+1,Q_i+1;R}^{M,N+1}\left[\mathrm{z}x^{\nu}\left|\begin{matrix}(1-\mu-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1)\,(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i},(1-\mu-\alpha-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1)\end{matrix}\right.\right] \qquad (3.2)$$

which holds under the conditions easily obtainable from those mentioned with (3.1). Again, if we put $\beta = 0$ in (3.1) then in view of (2.3), we get the following corollary pertaining to Erdelyi-Kober fractional integral operators defined by (1.6).

Corollary 1.2

$$\left[J_{\eta,\alpha}^{+}\left(t^{\mu-1}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]I_{P_i,Q_i;R}^{M,N}\left[\text{zt}^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i}\end{matrix}\right.\right]\right)\right](x)$$

$$=x^{\mu-1-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}...\sum_{l_k=0}^{[n_k/m_k]}\{\frac{(-n_1)_{m_1l_1}...(-n_h)_{m_hl_h}}{l_1!...l_h!}A'_{n_1,l_1}...A^{(k)}_{n_h,l_h}c_1^{l_1}...c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_j l_j}\}$$

$$\times I_{P_i+1,Q_i+1;R}^{M,N+1}\left[\text{zx}^{\nu}\left|\begin{matrix}(1-\mu-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i},(1-\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1)\end{matrix}\right.\right] \qquad (3.3)$$

where
$Re\,(\alpha)>0$ and $Re\,(\mu)+\upsilon\min_{1\le j\le M}Re\left(\frac{b_j}{\beta_j}\right)\ >-Re(\eta)$
and the conditions (i) and (ii) mentioned with Theorem 1 are also satisfied.

Theorem 2

$$\left[J_{-}^{\alpha,\beta,\eta}\left(t^{\mu-1}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]I_{P_i,Q_i;R}^{M,N}\left[\text{zt}^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i}\end{matrix}\right.\right]\right)\right](x)$$

$$=x^{\mu-\beta-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}...\sum_{l_k=0}^{[n_k/m_k]}\{\frac{(-n_1)_{m_1l_1}...(-n_h)_{m_hl_h}}{l_1!...l_h!}A'_{n_1,l_1}...A^{(k)}_{n_h,l_h}c_1^{l_1}...c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_j l_j}\}$$

$$\times I_{P_i+2,Q_i+2;R}^{M,N+2}\left[\text{zx}^{\nu}\left|\begin{matrix}(\mu-\beta+\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(\mu-\eta+\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i},(\mu+\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1),(\mu-\alpha-\beta-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1)\end{matrix}\right.\right] \qquad (3.4)$$

The conditions of validity of (3.4) are as follows:
$(i)\alpha,\beta,\eta,a,b,z\in C$ and $\lambda_j,\upsilon>0\ \forall\ j\in\ \overline{1,k}$
$(ii)\ |\arg z\ |<\frac{1}{2}\Omega_i\pi\ \ ;\ \Omega_i>0$ where

$$\Omega_i=\sum_{j=1}^{N}\alpha_j-\sum_{j=N+1}^{P_i}\alpha_{ji}+\sum_{j=1}^{M}\beta_j-\sum_{j=M+1}^{Q_i}\beta_{ji}\qquad\forall\ i\epsilon\ \overline{1,R}$$

$(iii)Re\,(\alpha)>0$ and $Re\,(\mu)-\upsilon\min_{1\le j\le M}Re\left(\frac{b_j}{\beta_j}\right)\ >max\{Re(\beta),Re(\eta)\}$

Proof. Proceeding on the lines similar to those followed for proving the Theorem 1 and also using the Lemma 2, we easily arrive at (3.4).
If we put $\beta=-\alpha$ in (3.4), then in view of (2.5) we arrive at the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by (1.4):

Corollary 2.1

$$\left[J_{-}^{\alpha}\left(t^{\mu-1}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]I_{P_i,Q_i;R}^{M,N}\left[\text{zt}^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i}\end{matrix}\right.\right]\right)\right](x)$$

$$= x^{\mu+\alpha-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\{\frac{(-n_1)_{m_1l_1}\cdots(-n_h)_{m_hl_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_jl_j}\}$$

$$\times I_{P_i+2,Q_i+2;R}^{M,N+2}\left[\text{zx}^{\nu}\left|\begin{matrix}(\mu+\alpha+\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1)(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i},(\mu+\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1)\end{matrix}\right.\right] \quad (3.5)$$

which holds under the conditions easily obtainable from those mentioned with (3.4). Again, if we put $\beta = 0$ in (3.4) then in view of (2.6), we get the following corollary pertaining to Erdelyi-Kober fractional integral operators defined by (1.6).

Corollary 2.2

$$\left[K_{\eta,\alpha}^{-}\left(t^{\mu-1}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]I_{P_i,Q_i;R}^{M,N}\left[\text{zt}^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i}\end{matrix}\right.\right]\right)\right](x)$$

$$= x^{\mu-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\{\frac{(-n_1)_{m_1l_1}\cdots(-n_h)_{m_hl_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_jl_j}\}$$

$$\times I_{P_i+2,Q_i+2;R}^{M,N+1}\left[\text{zx}^{\nu}\left|\begin{matrix}(\mu-\eta+\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1),(a_{j'},\alpha_{j'})_{1,N};(a_{j'i'},\alpha_{j'i'})_{N+1,p_i}\\(b_{j'},\beta_{j'})_{1,M};(b_{j'i'},\beta_{j'i'})_{M+1,Q_i},(\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1)\end{matrix}\right.\right] \quad (3.6)$$

which holds under the conditions easily obtainable from those mentioned with (3.4).

4. Special cases

(i)If we take $P_i = P, Q_i = Q$ *and* $R = 1$in the Theorem 1, the I-Function reduces in Fox's H-Function [1,13] and we get the following result:

$$\left[J_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]H_{P,Q}^{M,N}\left[\text{zt}^{\nu}\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,P}\\(b_{j'},\beta_{j'})_{1,Q}\end{matrix}\right.\right]\right)\right](x)$$

$$= x^{\mu-\beta-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\{\frac{(-n_1)_{m_1l_1}\cdots(-n_h)_{m_hl_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_jl_j}\}$$

$$\times H_{P+2,Q+2}^{M,N+2}\left[\text{zx}^{\nu}\left|\begin{matrix}(1-\mu-\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1),(1-\mu-\eta+\beta-\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1),(a_{j'},\alpha_{j'})_{1,P}\\(b_{j'},\beta_{j'})_{1,Q},(1-\mu+\beta-\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1),(1-\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_jl_j,\nu;1)\end{matrix}\right.\right] \quad (4.1)$$

provided that the conditions easily obtainable from those mentioned with Theorem 1 are satisfied.

(ii)Further on setting $M = 1 = N = P, Q = 2, z = -z, a_{1'} = 1 - a, \alpha_{1'} = 1, b_{1'} = 0, \beta_{1'} = 1, b_{2'} = 1 + q - a\nu', \beta_{2'} = \nu'$ in (4.1) we get the following result in terms of Lorenzo-Hartley G-function:

$$\left[J_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu+\nu'-q-2}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]\ G_{\nu',q,a}\left[\,\mathrm{z,t}\right]\right)\right](x)$$

$$= x^{\mu-\beta-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\left\{\frac{(-n_1)_{m_1 l_1}\ldots(-n_h)_{m_h l_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_j l_j}\right\}$$

$$\times H_{3,4}^{1,3}\left[-\mathrm{zx}^{\nu}\left|\begin{matrix}\left(1-\mu-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-\mu-\eta+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),(1-a,1)\\(0,1),\left(1+q-a\nu',\nu'\right),\left(1-\mu+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right)\end{matrix}\right.\right] \tag{4.2}$$

provided that the conditions easily obtainable from those mentioned with Theorem 1 are satisfied.

(iii) Next, on setting $a = 1$ in (4.2), we get the following result in terms of Lorenzo-Hartley R-function:

$$\left[J_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu+\nu'-q-2}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]\ R_{\nu',q}\left[\,\mathrm{z,t}\right]\right)\right](x)$$

$$= x^{\mu-\beta-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\left\{\frac{(-n_1)_{m_1 l_1}\ldots(-n_h)_{m_h l_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_j l_j}\right\}$$

$$\times H_{3,4}^{1,3}\left[-\mathrm{zx}^{\nu}\left|\begin{matrix}\left(1-\mu-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-\mu-\eta+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),(0,1)\\(0,1),\left(1+q-\nu',\nu'\right),\left(1-\mu+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right)\end{matrix}\right.\right] \tag{4.3}$$

provided that the conditions easily obtainable from those mentioned with Theorem 1 are satisfied.

(iv) On setting $R = 1, M = 1, N = P_i = P, Q_i = Q + 1, b_{1'} = 0, \beta_{1'} = 1, a_{j'} = 1 - a_{j'}, b_{j'i'} = 1 - b_{j'},\ \ \beta_{j'i'} = \beta_{j'}$ in (3.1), we arrive at the following result in terms of Wright's generalized hypergeometric function [3]:

$$\left[J_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\Pi_{j=1}^{k}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}\right]{}_p\psi_q\left[-\mathrm{zt}^{\nu}\left|\begin{matrix}\left(a_{j'},\alpha_{j'}\right)_{1,P}\\\left(b_{j'},\beta_{j'}\right)_{1,Q}\end{matrix}\right.\right]\right)\right](x)$$

$$= x^{\mu-\beta-1}\sum_{l_1=0}^{[n_1/m_1]}\sum_{l_2=0}^{[n_2/m_2]}\cdots\sum_{l_k=0}^{[n_k/m_k]}\left\{\frac{(-n_1)_{m_1 l_1}\ldots(-n_h)_{m_h l_h}}{l_1!\ldots l_h!}A'_{n_1,l_1}\ldots A^{(k)}_{n_h,l_h}c_1^{l_1}\ldots c_h^{l_h}x^{\Sigma_{j=1}^{k}\lambda_j l_j}\right\}$$

$$\times I_{P+2,Q+3;1}^{1,P+2}\left[\mathrm{zx}^{\nu}\left|\begin{matrix}\left(1-\mu-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-\mu-\eta+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-a_{j'},\alpha_{j'}\right)_{1,P}\\(0,1),\left(1-b_{j'},\beta_{j'}\right)_{1,Q},\left(1-\mu-\eta+\beta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\Sigma_{j=1}^{k}\lambda_j l_j,\nu;1\right)\end{matrix}\right.\right] \tag{4.4}$$

provided that the conditions easily obtainable from those mentioned with Theorem 1 are satisfied.

(v) On suitably specifying the parameters in (4.4), we arrive at the results due to Hussain [2], Kilbas [3] and Kilbas and Sebastian [5].
A number of other special cases of Theorem 2 can also be obtained on following the lines similar to those mentioned above for the Theorem 1 but we do not record them here explicitly.

5. Conclusion

In this paper, we have developed the Images of generalized fractional integral operators given by Saigo[7, 8, and 9] in terms of the product of I-function and a general class of polynomials in a compact and elegant form.On account of being general and unified in nature, the results established here yield a large number of known and new results involving simpler functions on suitable specifications of the parameters involved.

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