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On Boundedness of Hardy Type Integral Operator in Weighted Lebesgue Spaces

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Abstract. In this paper we proved sufficient conditions for boundedness of Hardy type integral operator in weighted Lebesgue spaces.

Key Words and Phrases: weighted Lebesgue spaces, Hardy operator.

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1. Introduction

Let ϕ be a fixed kernel defined on $(0,\infty)$, i.e. $\phi \in L_1^{loc}(0,\infty)$, then the Hardy type integral operator is defined in the following way

$$H_\phi(f)(x) = \int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\,dy. \tag{1}$$

This integral operator (1) is deeply rooted in the study of one-dimensional Fourier analysis and has become an essential part of modern harmonic analysis.In particular, it is closely related to the summability of the classical Fourier series (see [8]). Many important operators in analysis are special cases of the integral operator (1), by taking suitable choice of ϕ.

The considered integral operator (1) has been extensively studied in recent years, particularly its boundedness on the Lebesgue space as well as on the Hardy space(see [2, 3, 4]). We also refer to [5, 6, 7] for some recent work in this vein. Moreover the generalized version of the considered operators on multidimensional Euclidean spaces have been studied (see [2], [8]). About boundedness of Hausdorff operator in different Lebesgue spaces we refer to [1].

In this paper we proved sufficient conditions for boundedness of integral operator (1) in weighted Lebesgue spaces.

2. Main Results

We recall some notation and basic facts about function spaces.

Let ω be a weight function on R_+, i.e $\omega \in L_1^{loc}(R_+)$ and almost everywhere is a positive function. The weighted Lebesgue space $L_{p,\omega}(R_+)$ is the class of all measurable functions f defined on R_+ such that

$$\|f\|_{L_{p,\omega}(R_+)} = \left(\int_0^\infty |f(x)|^p \omega(x)\,dx\right)^{\frac{1}{p}} < \infty.$$

Theorem 1. *Let $1 < p < q < \infty$ and H_ϕ is a Hausdorff operator. Let u be positive non-decreasing weighted function on $(0,\infty)$. Suppose that satisfying the following conditions:*

1) $\int_0^{\frac{1}{2}} \frac{\phi(y)}{y} y^{\frac{1}{p}} dy < +\infty$ and there exists a constant C_1 such that for any $t \geq \frac{1}{2}$ the following inequality holds

$$|\phi(t)| \leq \frac{C_1}{t},$$

2)

$$\sup_{t>0}\left(\int_t^\infty \frac{u(x)}{x^p}dx\right)^{\frac{1}{p}}\left(\int_0^t u(x)^{1-p'}dx\right)^{\frac{1}{p'}} < \infty.$$

Then there exists $C > 0$ for all $f \in L_{p,u}(0,\infty)$ the following inequality holds

$$\left(\int_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty |f(x)|^p u(x)\,dx\right)^{\frac{1}{p}}. \tag{2}$$

Proof: Without loss of generality we may assume that the function u has the form

$$u(t) = u(0) + \int_0^t \psi(\tau)\,d\tau,$$

where $u(0) = \lim_{t\to+0} u(t)$ and ψ is a positive function on $(0,\infty)$. Indeed, for increasing functions on $(0,\infty)$ there exists a sequence of absolutely continuous functions $\varphi_n(t)$ such that $\lim_{n\to\infty}\varphi_n(t) = u(t)$, $0 \leq \varphi_n(t) \leq u(t)$ a.e. $t > 0$ and $\varphi_n(0) = u(0)$. Furthermore the functions $\varphi_n(t)$ are increasing, and besides

$$\varphi_n(t) = \varphi_n(0) + \int_0^t \varphi_n'(\tau)\,d\tau.$$

Where $\lim\limits_{n\to\infty} \varphi_n^{'}(t) = \psi(t)$. Hence, using Fatou's theorem , we obtain estimate (2) for any increasing functions on $(0,\infty)$.

Let us estimate the left –hand side of inequality (2). We have

$$\left(\int\limits_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}} = \left(\int\limits_0^\infty |H_\phi f(x)|^p \left(u(0) + \int\limits_0^x \psi(t)\,dt\right) dx\right)^{\frac{1}{p}}.$$

If $u(0) = 0$, then $\left(\int\limits_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}} = \left(\int\limits_0^\infty |H_\phi f(x)|^p \left(\int_0^x \psi(t)\,dt\right) dx\right)^{\frac{1}{p}}$.

However, if $u(0) > 0$, then

$$\left(\int\limits_0^\infty |H_\phi f(x)|^p u(x)\,dx\right)^{\frac{1}{p}} \le \left(\int\limits_0^\infty |H_\phi f(x)|^p u(0)\,dx\right)^{\frac{1}{p}}$$

$$+\left(\int\limits_0^\infty |H_\phi f(x)|^p \left(\int\limits_0^x \psi(t)\,dt\right) dx\right)^{\frac{1}{p}} = E_1 + E_2.$$

First estimate E_1. By boundedness of integral operator (1) in Lebesgue spaces (see [2, 8]), we get

$$E_1 = \left(\int\limits_0^\infty |H_\phi f(x)|^p u(0)\,dx\right)^{\frac{1}{p}} = (u(0))^{\frac{1}{p}} \left(\int\limits_0^\infty |H_\phi f(x)|^p\,dx\right)^{\frac{1}{p}}$$

$$\le C\,(u(0))^{\frac{1}{p}} \left(\int\limits_0^\infty |f(x)|^p\,dx\right)^{\frac{1}{p}} \le C\left(\int\limits_0^\infty |f(x)|^p u(x)\,dx\right)^{\frac{1}{p}} = C\|f\|_{L_{p,u}(0,\infty)}.$$

Let us estimate the integral E_2. We have

$$E_2 = \left(\int\limits_0^\infty |H_\phi f(x)|^p \left(\int\limits_0^x \psi(t)\,dt\right) dx\right)^{\frac{1}{p}}$$

$$= \left(\int\limits_0^\infty |H_\phi f(x)|^p \left(\int\limits_0^\infty \psi(t)\,\chi_{\{x>t\}}(x)\,dt\right) dx\right)^{\frac{1}{p}}$$

$$= \left(\int\limits_0^\infty \psi(t) \left(\int\limits_t^\infty |H_\phi f(x)|^p\,dx\right) dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi(t)\left(\int\limits_t^\infty\left|\int\limits_{2t}^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\,dy\right|^p dx\right)dt\right)^{\frac{1}{p}}$$

$$+2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi(t)\left(\int\limits_t^\infty\left|\int\limits_0^{2t} \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)dy\right|^p dx\right)dt\right)^{\frac{1}{p}} = E_{21}+E_{22}.$$

We estimate E_{21}. Using Theorem on boundedness of integral operator (1) in Lebesgue space, (see [1,7]) we get

$$E_{21} = 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi(t)\left(\int\limits_0^\infty\left|\int\limits_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\,\chi_{\{y>2t\}}(y)\,dy\right|^p \chi_{\{x>t\}}(x)\,dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int_0^\infty \psi(t)\left(\int\limits_0^\infty\left|\int\limits_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\,\chi_{\{y>2t\}}(y)\,dy\right|^p dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int\limits_0^\infty \psi(t)\left(\int_0^\infty |f(x)|^p\,\chi_{\{y>2t\}}(x)\,dx\right)dt\right)^{\frac{1}{p}}$$

$$= C_2\left(\int\limits_0^\infty |f(x)|^p\left(\int\limits_0^{\frac{x}{2}} \psi(t)\,dt\right)dx\right)^{\frac{1}{p}} \leq C_2\left(\int\limits_0^\infty |f(x)|^p\,u\left(\frac{x}{2}\right)dx\right)^{\frac{1}{p}}$$

$$\leq C_2\left(\int\limits_0^\infty |f(x)|^p\,u(x)\,dx\right)^{\frac{1}{p}} = C_2\,\|f\|_{L_{p,u}(0,\infty)}\,.$$

Now we estimate E_{22}. Note that if $x>t, y\leq 2t$, then $\frac{x}{y}\geq\frac{1}{2}$. By virtue of condition 1) of Theorem 1, one has

$$E_{22} = 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi(t)\left(\int\limits_t^\infty\left|\int\limits_0^{2t} \frac{\varphi\left(\frac{x}{y}\right)}{y} f(y)\,dy\right|^p dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int\limits_0^\infty \psi(t)\left(\int\limits_t^\infty\left(\int\limits_0^{2t} \frac{\left|\varphi\left(\frac{x}{y}\right)\right|}{y}\,|f(y)|\,dy\right)^p dx\right)dt\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p'}}\left(\int_0^\infty \psi(t)\left(\int_t^\infty\left(\int_0^{2t}\frac{|f(y)|}{x}dy\right)^p dx\right)dt\right)^{\frac{1}{p}}$$

$$= 2^{\frac{1}{p'}}\left(\int_0^\infty \psi(t)\left(\int_t^\infty\frac{dx}{x^p}\right)\left(\int_0^{2t}|f(y)|\,dy\right)^p dt\right)^{\frac{1}{p}}.$$

We get following formula in a way that made use of change of variables $\left(t=\frac{z}{2}, dt=\frac{1}{2}dz, 0<z<\infty\right)$

$$E_{22}=2^{\frac{1}{p'}-\frac{1}{p}}\left(\int_0^\infty \psi\left(\frac{t}{2}\right)\left(\int_{\frac{t}{2}}^\infty\frac{dx}{x^p}\right)\left(\int_0^{t}|f(y)|\,dy\right)^p dt\right)^{\frac{1}{p}}.$$

As is well-known, the classical Hardy operator of function $|f|$ is determined by

$$\int_0^t |f(y)|\,dy.$$

We have

$$\int_{2t}^\infty \psi\left(\frac{s}{2}\right)\left(\int_{\frac{s}{2}}^\infty\frac{dx}{x^p}\right)ds = 2\int_t^\infty \psi(s)\left(\int_s^\infty\frac{dx}{x^p}\right)ds$$

$$=2\int_t^\infty \psi(s)\left(\int_0^\infty \chi_{(s,\infty)}(x)\,x^{-p}dx\right)ds = 2\int_0^\infty \psi(s)\,\chi_{(t,\infty)}(s)$$

$$\times\left(\int_0^\infty \chi_{(s,\infty)}(x)\,x^{-p}dx\right)ds = 2\int_0^\infty\int_0^\infty \psi(s)\,x^{-p}\chi_{(t,\infty)}(s)\,\chi_{(s,\infty)}(x)\,dxds$$

$$=2\int_t^\infty x^{-p}\left(\int_t^x \psi(s)\,ds\right)dx \leq 2\int_t^\infty x^{-p}\left(\int_0^x \psi(s)\,ds\right)dx \leq 2\int_t^\infty x^{-p}u(x)\,dx.$$

From this, we get

$$\int_t^\infty \psi(s)\left(\int_s^\infty\frac{dx}{x^p}\right)ds \leq \int_t^\infty\frac{u(x)}{x^p}dx.$$

Let v and ω is weight functions defined on $(0,\infty)$. Follows by the theory of boundedness of two-weighted Hardy operators, (see [9]) we have

$$\left(Hf\left(x\right) = \int\limits_0^x f\left(t\right)dt: \; H: L_{p,v}(0,\infty) \to L_{p,\omega}(0,\infty) \right) \Leftrightarrow$$
$$\Leftrightarrow A = \sup_{t>0} \left(\int\limits_t^\infty \omega\left(x\right)dx \right)^{\frac{1}{p}} \left(\int\limits_0^t v(x)^{1-p'}dx \right)^{\frac{1}{p'}} < \infty. \tag{3}$$

Thus, from inequality (3), we have

$$\sup_{t>0} \left(\int\limits_t^\infty \psi(s) \left(\int\limits_s^\infty \frac{dx}{x^p} \right) ds \right)^{\frac{1}{p}} \left(\int\limits_0^t u\left(x\right)^{1-p'} dx \right)^{\frac{1}{p'}}$$
$$\leq \sup_{t>0} \left(\int\limits_t^\infty \frac{u(x)}{x^p} dx \right)^{\frac{1}{p}} \left(\int\limits_0^t u\left(x\right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty. \tag{4}$$

Taking $\omega(x) = \psi\left(\frac{x}{2}\right)x^{1-p}$ and $v(x) = u(x)$ and applying (3) and (4), we have

$$E_{22} \leq C_6 \left(\int\limits_0^\infty \psi\left(\frac{t}{2}\right) \left(\int\limits_{\frac{t}{2}}^\infty \frac{dx}{x^p} \right) \left(\int\limits_0^t |f\left(y\right)| dy \right)^p dt \right)^{\frac{1}{p}}$$
$$= C_7 \left(\int\limits_0^\infty \omega(t) \left(\int\limits_0^t |f\left(y\right)| dy \right)^p dt \right)^{\frac{1}{p}} \leq C_8 \left(\int\limits_0^\infty |f\left(t\right)|^p u\left(t\right) dt \right)^{\frac{1}{p}}.$$

The proof is completed.

Corollary 1. Let $1 < p < \infty$ and H_ϕ - is the classical Hardy operator or Riemann-Liouville operator.

Then these operators satisfy all terms of theorem 1 and these operators are bounded on $L_{p,u}\left(0,\infty\right)$.

Theorem 2. Let $1 < p < \infty$ and $H_\phi-$ Hausdorff operator. Let u be positive non-increasing weighted function on $(0,\infty)$. Suppose that satisfying the following conditions:
1) $\int\limits_0^{\frac{1}{2}} \frac{\phi(y)}{y} y^{\frac{1}{p}} dy < +\infty$ and there exists a constant $C_1 > 0$ such that for any $\forall t \in (0,2)$ the following inequality holds

$$|\phi\left(t\right)| \leq C_1;$$

$$2) \sup_{t>0} \left(\int\limits_0^t \frac{u\left(x\right)}{x^p} dx \right)^{\frac{1}{p}} \left(\int\limits_t^\infty u\left(x\right)^{1-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

Then there exists $C > 0$ for all $f \in L_{p,u}(0,\infty)$ the following inequality holds

$$\left(\int_0^\infty |H_\phi f(x)|^p u(x)\, dx \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty |f(x)|^p u(x)\, dx \right)^{\frac{1}{p}}.$$

The proof of Theorem 2 is also similar to the proof of the corresponding Theorem 1.

References

[1] R.A. Bandaliyev, P. Gorka, *Hausdorff operator in Lebesgue spaces,* Math. Ineq. Appl., 2018 (accepted).

[2] G. Brown, F. Moricz, *Multivariate Hausdorff operators on spaces,* $L_p(R^n)$, J. Math. Anal. Appl., **271**, 2002, 443-454.

[3] J. Chen, D. Fan, J. Li, *Hausdorff operators on function spaces,* Chin. Ann. Math. Ser. B, **33**, 2013, 537-556

[4] J. Chen, X. Wu, *Best constant for Hausdorff operators on n-dimensional product spaces*, Sci. China Math., **57**, 2014, 569-578.

[5] A. Lerner, E. Liflyand, Multidimensional Hausdorff operators on The real Hardy spaces, *J. Austral. Math. Soc.*, **83**, 2007, 79-86

[6] E. Liflyand and F. Moricz, *The Hausdorff operator is bounded on real* H^1 *Space*, Proc. Amer. Math. Soc., **128**, 2000, 1391-1396

[7] E. Liflyand, *Hausdorff operators on Hardy spaces,* Eurasian Math. J., **4**, 2013, 101-144.

[8] X. Lin, *The boundedness of multidimensional Hausdorff operators on function spaces,* Ph.D thesis, Univ. Wisconsin- Milwaukee, 2013.

[9] V.G. Maz'ya, *Sobolev spaces*, Springer-Verlag, Berlin New-York, 1985, 416 pp.

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