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# On approximation by Shift Operators in Morrey Type Spaces

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**Abstract.** Morrey space $M^{p,\alpha}$ and its subspace $MC^{p,\alpha}$ where the continuous functions are dense are considered. Basic properties of convolution are extended to these spaces. It is proved that the convolution in $MC^{p,\alpha}$ can be approximated by finite linear combinations of shifts. Approximate identity in $MC^{p,\alpha}$ is also considered.

**Key Words and Phrases**: Morrey type space, convolution, an approximate identity.

**2010 Mathematics Subject Classifications**: 46E30, 30E25

## 1. Introduction

The concept of Morrey space was introduced by C. Morrey in 1938. Since then, various problems related to this space have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [1; 2]), this space also provides a large class of examples of mild solutions to the Navier-Stokes system [3]. In the context of fluid dynamics, Morrey spaces have been used to model fluid flow when vorticity is a singular measure supported on certain sets in $R^n$ [4]. There appeared lately a large number of research works which considered fundamental problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in these spaces (see, for example, [5] and the references above). More details about Morrey spaces can be found in [6,7].

In view of the aforesaid, there has recently been a growing interest in the study of various problems in Morrey-type spaces. For example, some problems of harmonic analysis and approximation theory have been considered in [6-13].

In the present paper Morrey space $M^{p,\alpha}$ and its subspace $MC^{p,\alpha}$ where the continuous functions are dense are considered. Basic properties of convolution are extended to these spaces. It is proved that the convolution in $MC^{p,\alpha}$ can be approximated by finite linear combinations of shifts. Approximate identity in $MC^{p,\alpha}$ is also considered. The validity of the classical facts concerning the approximate identities is proved in Morrey type spaces.

 

## 2. Needful Information

We will need some facts about the theory of Morrey-type spaces. Let $\Gamma$ be some rectifiable Jordan curve on the complex plane $C$. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

By the Morrey-Lebesgue space $M^{p,\alpha}(\Gamma)$, $0 \le \alpha \le 1$, $p \ge 1$, we mean a normed space of all functions$f(\xi)$ measurable on $\Gamma$ equipped with a finite norm $\|\cdot\|_{M^{p,\alpha}(\Gamma)}$:

$$\|f\|_{M^{p,\alpha}(\Gamma)} = \sup_B \left( \left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty.$$

$M^{p,\alpha}(\Gamma)$ is a Banach space and $M^{p,1}(\Gamma) = L_p(\Gamma)$, $M^{p,0}(\Gamma) = L_\infty(\Gamma)$. The embedding $M^{p,\alpha_1}(\Gamma) \subset M^{p,\alpha_2}(\Gamma)$ is valid for $0 \le \alpha_1 \le \alpha_2 \le 1$. Thus, $M^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in [0,1]$, $\forall p \ge 1$. The case of $\Gamma \equiv [-\pi, \pi]$ will be denoted by $M^{p,\alpha}(-\pi,\pi) \equiv M^{p,\alpha}$.

Denote by $\tilde{M}^{p,\alpha}$ the linear subspace of $M^{p,\alpha}$ consisting of functions whose shifts are continuous in $M^{p,\alpha}$, i.e. $\|f(\cdot + \delta) - f(\cdot)\|_{M^{p,\alpha}} \to 0$ as $\delta \to 0$. The closure of $\tilde{M}^{p,\alpha}$ in $M^{p,\alpha}$ will be denoted by $MC^{p,\alpha}$. In [13] we have proved the following

***Theorem 1.*** *Infinitely differentiable functions on* $[0, 2\pi]$ *are dense in the space* $MC^{p,\alpha}$.

The following Hölder inequality is also valid.

***Lemma 1.*** *Let* $f \in L^{p,\alpha}(I) \wedge g \in L^{q,\alpha}(I), \frac{1}{p} + \frac{1}{q} = 1$, $p \in [1, +\infty)$. *Then the following Hölder inequality holds:*

$$\|fg\|_{L_1} \le |I|^{1-\alpha} \|fg\|_{1,\alpha} \le |I|^{1-\alpha} \|f\|_{p,\alpha} \|g\|_{q,\alpha}.$$

## 3. Main results

Consider the Morrey-Lebesgue space $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$.
Let $\frac{1}{p} + \frac{1}{q} = 1$. Take $f \in M^{p,\alpha}$;$g \in M^{q,\alpha}$. Consider the convolution

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy, x \in [-\pi, \pi].$$

Functions $f(\cdot)$ and $g(\cdot)$ are extended out of the segment $[-\pi, \pi]$ be zero.
As $M^{p,\alpha} \subset L_1$, then the existence of the convolution follows from the classical facts.

The following theorem is evident.

***Theorem 2.*** *Let* $1 < p < +\infty$, $0 < \alpha < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in M^{p,\alpha} \wedge g \in M^{q,\alpha}$, *then the convolution* $f * g$ *is defined everywhere on* $[-\pi, \pi]$ *and the following inequality is true:*

$$\|f * g\|_\infty \le c_\alpha \|f\|_{p,\alpha} \|g\|_{q,\alpha}.$$

*Moreover, if* $f \in MC^{p,\alpha}$ *or* $g \in MC^{q,\alpha}$, *then the convolution* $f * g$ *is continuous on* $[-\pi, \pi]$.

Using the definition of Morrey type space this theorem is proved absolutely analoguos to the classical case.

The following theorem is also true.

***Theorem 3.*** *Let* $f \in L_1 \wedge g \in M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$. *Then* $f * g \in M^{p,\alpha}$, *and moreover the following inequality is true*

$$\|f * g\|_{p,\alpha} \leq \|f\|_{L_1} \|g\|_{p,\alpha} .$$

Let us prove the following theorem.

***Theorem 4.*** *Let* $f \in L_1$ *and* $g \in E$, *where* $E$ *denotes any of the spaces* $C[-\pi, \pi]$ *or* $MC^{p,\alpha}$, $1 \leq p < +\infty$, $0 < \alpha < 1$. *Then the convolution* $f * g$ *in* $E$ *can be approximated by finite linear combinations of shifts* $g$, *i.e.* $\forall \varepsilon > 0$, $\exists \{a_k\}_1^n \subset [-\pi, \pi] \wedge \{\lambda_k\}_1^n \subset R$:

$$\left\| f * g - \sum_{k=1}^{n} \lambda_k T_{a_k} g \right\|_E < \varepsilon.$$

**Proof.** The case of $E = C[-\pi, \pi]$ is known (see e.g. [14]). Consider the case of $E = MC^{p,\alpha}$. Following the classical scheme, as a subset $S_0$ , such that the finite linear combinations of elements from $S_0$ are dense in $L_1$, we take a set of functions $f$, each of which on $[-\pi, \pi]$ coincides with the characteristic function of some interval $M = [a, b]$, $-\pi < a < b < \pi$, and further on continues periodically.

Let $\varepsilon > 0$ be arbitrary. Let us divide $M$ into a finite subintervals $I_k$ which length is $|I_k| < \delta$. Take $\forall a_k \in I_k$. Let $f(x) = \chi_M(x)$. We have

$$(f * g)(x) - \sum_k |I_k| g(x - a_k) = \int_{\bigcup_k I_k} g(x - y)\, dy -$$

$$- \sum_k \int_{I_k} g(x - a_k)\, dy = \sum_k \int_{I_k} [g(x - y) - g(x - a_k)]\, dy = \sum_k h_k(x),$$

where

$$h_k(x) = \int_k [g(x - y) - g(x - a_k)]\, dy.$$

Consequently

$$\left\| (f * g)(x) - \sum_k |I_k| g(x - a_k) \right\|_{p,\alpha} \leq \sum_k \|h_k\|_{p,\alpha} .$$

Let $I \subset [-\pi, \pi]$ be an arbitrary interval. We have

$$\|h_k\|_{L_p(I)}^p = \int_I |h_k|^p\, dx = \int_I \left| \int_{I_k} [g(x - y) - g(x - a_k)]\, dy \right|^p dx \leq$$

$$\leq \int_I \left( |I_k|^{p/q} \int_{I_k} |g(x - y) - g(x - a_k)|^p\, dy \right) dx =$$

$$= |I_k|^{p/q} \int_{I_k} \left( \int_I |g(x-y) - g(x-a_k)|^p \, dx \right) dy.$$

As a result, we obtain

$$\frac{1}{|I|^{1-\alpha}} \int_I |h_k|^p \, dx \le |I_k|^{p/q} \int_{I_k} \left( \frac{1}{|I|^{1-\alpha}} \int_I |g(x-y) - g(x-a_k)|^p \, dx \right) dy \le$$

$$\le |I_k|^{p/q} \int_{I_k} \|T_y g - T_{a_k} g\|_{p,\alpha}^p \, dy \Rightarrow$$

$$\|h_k\|_{p,\alpha}^p \le |I_k|^{p/q} \int_{I_k} \|T_y g - T_{a_k} g\|_{p,\alpha}^p \, dy. \tag{1}$$

By Theorem 4, we have $\exists \delta > 0$ :

$$\|T_y g - T_{a_k} g\|_{p,\alpha} < \varepsilon, \, \forall y \in I_k.$$

From (1) it follows that

$$\|h_k\|_{p,\alpha}^p \le |I_k|^{p/q} |I_k| \, \varepsilon^p = |I_k|^p \, \varepsilon^p \Rightarrow \|h_k\|_{p,\alpha} \le |I_k| \, \varepsilon.$$

As a result, we have

$$\left\| (f * g)(x) - \sum_k |I_k| \, T_{a_k} g \right\|_{p,\alpha} \le \sum_k |I_k| \, \varepsilon = |M| \, \varepsilon \le 2\pi\varepsilon.$$

Since, $\sum_k |I_k| \, T_{a_k} g$ is a finite linear combination of shifts $g$, then it is clear that $f * g \in \overline{V_g}$, where $\overline{V_g}$ is a closed linear subspace in $E$, generated by shifts $T_a g$ of the function $g$, i.e. closure of in $E$ is a set of all linear combination of elements $T_a g$.

If $f \in L_1$ is an arbitrary element, then for $\forall \varepsilon > 0$, there exist a partition of $[-\pi, \pi]$ into a finite number of intervals $M_k$, and a number $\lambda_k$, such that, the inequality

$$\left\| f(\cdot) - \sum_k \lambda_k \chi_{M_k}(\cdot) \right\|_{L_1} < \varepsilon, \tag{2}$$

holds. It follows directly from the previous result that $\tilde{f} * g \in \overline{V_g}$, where $\tilde{f}(\cdot) = \sum_k \lambda_k \chi_{M_k}(\cdot)$. Then from (2) we obtain that $f * g \in \overline{V_g}$.

The theorem is proved.

Let us consider the approximate identities for convolutions in the space $M^{p,\alpha}$. By the approximate identity (for convolution) we mean $\left\{ K_n^{(\cdot)} \right\}_{n \in N} \subset L_1(-\pi, \pi)$, satisfying the following conditions:

$\alpha)$ $\sup_n \|K_n\|_{L_1} < +\infty$;

$\beta)$ $\lim_n \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) \, dx = 1$;

$\gamma)$ $\lim_{n \to \infty} \int_{\delta \le |x| \le \pi} |K_n(x) \, dx| = 0$ , $\forall \delta \in (0, \pi)$.

The following theorem is true.

***Theorem 5.*** *Let $\{K_n\}_{n \in N}$ be an approximate identity. Then the following properties are true:*

1. $\lim\limits_{n\to\infty} \|K_n * f - f\|_\infty = 0\,,\ \forall f \in C\,[-\pi,\pi]$;

2. $\lim\limits_{n\to\infty} \left\|\frac{d^m}{dx^m}\left(K_n * f\right) - \frac{d^m}{dx^m} f\right\|_\infty = 0\,,\ \forall f \in C\,[-\pi,\pi]$;

3. $\lim\limits_{n\to\infty} \|K_n * f - f\|_{p,\alpha} = 0\,,\ \forall f \in MC^{p,\alpha}$, $1 \le p < +\infty\,,\ 0 < \alpha < 1.$

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