Caspian Journal of Applied Mathematics, Ecology and Economics
V. 6, No 2, 2018, December
ISSN 1560-4055

Weak Solvability of the First Boundary Value Problem for a Class of Parabolic Equations with Discontinuous Coefficients in Paraboloid Type Domains

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Abstract. In the paper, weak solvability of the first boundary value problem is proved for a class of parabolic equations with discontinuous coefficients and given in parabolic type domains in Sobolev's weight spaces. The coefficients of these equation bear discontinuity at the vertex of $P-$ domain. At the vertex $P-$ domain touches the characteristics of the equation.

Key Words and Phrases: boundary value problem, weak solvability, parabolic operator.
2010 Mathematics Subject Classifications: 35K05, 35K08, 34B05

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1. Introduction

Let E_n and R_{n+1} be $-$ n- dimensional and $(n+1)$ dimensional Euclidean spaces of the points $x = (x_1, \ldots, x_n)$ and $(x,t) = (x_1, \ldots, x_n, t)$ respectively. D be a bounded domain E_n with a boundary $\partial\, D$, $\quad 0 \in D$, $\quad R^-_{n+1} = R_{n+1} \cap \{(x,t) : t < 0\}$.

The domain $Q \subset \ R^-_{n+1}$ is said to be a paraboloid type domain (or $P-$domain) if its cross section with each hyperplane $t = \tau\ (\tau < 0)$ has the form:

$$\left\{ x : \frac{x}{2\sqrt{-\tau}} \in D \right\}.$$

The domain D is called a foot of the $P-$ domain Q.

Let further $Q_T = Q \cap \{(x,t)\ :\ -T < t < 0\ \}$, $\ S_T = \ \partial Q \cap \{(x,t)\ :\ T < t < 0\ \}$, $D_T = Q \cap \{(x,t)\ :\ t = -T\ \}$, $\Gamma(Q_T)$ be a parabolic boundary of the domain Q_T.

Consider in Q_T the following operator

$$L = \Delta + \lambda \sum_{i,j=1}^{n} \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial U}{\partial t},$$

where Δ is the Laplace operator and the number parameter λ satisfies the condition

$$\frac{1}{d^2} < \lambda < \infty. \tag{1}$$

Here $d = \sup_{y\epsilon D}|y|$. It is easy to see that subject to condition (1) the operator L uniformly parabolic in the domain Q_T. By analogy with the elliptic case, we call the operator L the Gilbarg-Serrin parabolic operator.

Let us agree in the following denotation u_i and u_{ij} are the derivatives of $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$, respectively,

$$u_{xx} = (u_{ij}), \qquad u_x^2 = \sum_{i=1}^n u_i^2, \quad u_{xx}^2 = \sum_{i,j=1}^n u_{ij}^2 \ ; \qquad i,j = \overline{1,n}.$$

Let the number parameter γ satisfy the condition

$$\gamma\epsilon \left(\frac{n^2\left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{2}, \infty \right). \tag{2}$$

$A_0^\infty(Q_T)$ be a space of infinitely differentiable and finite in Q_T functions for which the following integral is finite $\int_{Q_T}(-t)^\gamma u^2 dxdt$, $\qquad L_{2,\gamma}(Q_T)$ be Banach space of measurable functions $u(x,t)$ given on, Q_T with finite norm

$$\|u\|_{L_{2,\gamma}(Q_T)} = \left(\int_{Q_T} (-t)^2 u^2 dxdt \right)^{\frac{1}{2}},$$

$\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$ and $\overset{0}{W}{}_{2,\gamma}^{1,1}(Q_T)$ be Banach spaces of measurable functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{W_{2,\gamma}^{1,0}(Q_T)} = \left(\int_{Q_T} (-t)^\gamma (u^2 + u_x^2) dxdt \right)^{\frac{1}{2}},$$

$$\|u\|_{W_{2,\gamma}^{1,1}(Q_T)} = \left(\int_{Q_T} (-t)^\gamma (u^2 + u_x^2 + u_t^2) dxdt \right)^{\frac{1}{2}},$$

respectively.

$\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$ and$\overset{0}{W}{}_{2,\gamma}^{1,1}(Q_T)$ be subspaces of $W_{2,\gamma}^{1,0}(Q_T)$ and $W_{2,\gamma}^{1,1}(Q_T)$, respectively, in which $A_0^\infty(Q_T)$ is a dense set.

In the domain Q_T consider the first boundary value problem

$$Lu = \Delta \mathrm{u} + \lambda \sum_{i,j=1}^n \frac{x_i x_j}{4(-t)} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f + \sum_{k=1}^n \frac{\partial f^k}{\partial x^k} \tag{3}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{4}$$

where $f\epsilon L_{2,\gamma}(Q_T)$, $f^k \epsilon L_{2,\gamma}(Q_T)$; $\quad k = \overline{1,n}$.

Therewith, it is assumed that with regard to number parameters λ and γ, conditions (1) and (2) are fulfilled. At first give definition of the weak solution of the first boundary value problem (3)-(4).

The function $u(x,t)\epsilon W_{2,\gamma}^{1,0}(Q_T)$ is said to be a weak solution of equation (3) in the domain Q_T if for any function $v(x,t)\epsilon \overset{0}{W}{}_{2,\gamma}^{1,1}(Q_T)$the following integral identity is fulfilled.

$$B_{Q_T}(u,v)=\int_{Q_T}(-t)^\gamma u\ v_t dxdt-\int_{Q_T}(-t)^\gamma\sum_{i,j=1}^n\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)v_iu_jdxdt+$$

$$+\lambda(n+1)\int_{Q_T}(-t)^\gamma\sum_{i=1}^n\frac{x_i}{4(-t)}uv_idxdt+\frac{\lambda n(n+1)}{4}\int_{Q_T}(-t)^\gamma uvdxdt-$$

$$-\gamma\int_{Q_T}(-t)^\gamma uvdxdt\ =\int_{Q_T}(-t)^\gamma fvdxdt-\int_{Q_T}(-t)^\gamma\sum_{k=1}^n f^kv_kdxdt, \tag{5}$$

where δ_{ij} is the Kronecker symbol.

The function $u(x,t)\epsilon \overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T)$ being a weak solution of equation (3) in Q_T is called a weak solution of boundary value problem (3)-(5). Now we show the relation between equation (3) and integral identity (5). At first we represent the Hilbarg-Serrin parabolic operator in the form of a divergent operator with unbounded minor coefficients. We have

$$Lu=\Delta u+\lambda\sum_{i,j=1}^n\left(\frac{x_ix_j}{4(-t)}u_i\right)_j-\lambda(n+1)\sum_{i=1}^n\frac{x_i}{4(-t)}u_i-u_t\ .$$

Consider the domain $Q_{T,\delta}=Q_T\backslash\overline{Q_\delta}$ multiply the both parts of equation (3) by the function $v(x,t)\epsilon A_0^\infty(Q_T)$ and integrate the obtained equality with respect to $Q_{T,\delta}$. We get

$$\int_{Q_{T,\delta}}(-t)^\gamma v\ \Delta u\ dx\ dt+$$

$$+\lambda\int_{Q_{T,\delta}}(-t)^\gamma\sum_{i,j=1}^n\left(\frac{x_{??}x_j}{4(-t)}u_i\right)_j v\ dx\ dt-\lambda(n+1)\int_{Q_{T,\delta}}(-t)^\gamma\sum_{i=1}^n\frac{x_j}{4(-t)}u_jv\ dx\ dt-$$

$$-\int_{Q_{T,\delta}}(-t)^\gamma u_tv\ dx\ dt=\int_{Q_{T,\delta}}(-t)^\gamma f\cdot v\ dx\ dt+\int_{Q_{T,\delta}}(-t)^\gamma\sum_{i=1}^n\frac{\partial f^i}{\partial x_i}v\ dx\ dt. \tag{6}$$

By Ostrogradskii's formula

$$\int_{Q_{T,\delta}}(-t)^\gamma v\ \Delta u\ dx\ dt+\lambda\int_{Q_{T,\delta}}(-t)^\gamma\sum_{i,j=1}^n\left(\frac{x_ix_j}{4(-t)}u_i\right)_j v\ dx\ dt=$$

$$=-\int_{Q_{T,\delta}}(-t)^\gamma\sum_{i,j=1}^n\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_iv_j\ dx\ dt. \tag{7}$$

In what follows we have

$$-\lambda\,(n+1)\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}\;u_iv\;dx\;dt=\frac{\lambda n\,(n+1)}{4}\int_{Q_{T,\delta}}(-t)^{\gamma-1}uv\;dx\;dt+$$

$$+\lambda\,(n+1)\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{x_i}{4\,(-t)}v_iu\;dx\;dt \tag{8}$$

Furthermore

$$\int_{Q_{T,\delta}}(-t)^{\gamma}f\cdot v\;dx\;dt+\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}\frac{\partial f^i}{\partial x_i}v\;dx\;dt=$$

$$=\int_{Q_{T,\delta}}(-t)^{\gamma}f\;v\;dx\;dt-\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i=1}^{n}f^i\;v_i\;dx\;dt \tag{9}$$

Let $\Pi_R=\left\{x:\;|x_i|<R,\;\;i=\overline{1,n}\right\}$, $\mathrm{K}_\delta=\Pi_R\times(-T,-\delta)$. For simplicity we will consider that we can continue the function $u(x,t)$ in $\mathrm{K}_\delta\backslash \mathrm{Q}_{T,\delta}$ so that the obtained continuation $\tilde{u}(x,t)$ be the element of the space $W_{2,\gamma}^{1,0}(K_\delta)$. We continue the function $v(x,t)$ by a zero to$\mathrm{K}_\delta\backslash \mathrm{Q}_{T,\delta}$ and denote the obtained continuation again by $v(x,t)$.We have

$$J_\delta=-\int_{K_\delta}(-t)^{\gamma}\tilde{u}_tv\;dx\;dt=-\delta\int_{\Pi_R}?\widetilde{D}\,(x,-\delta)\;\;v\;\;(x,-\delta)\,dx+$$

$$+\int_{K_\delta}(-t)^{\gamma}v_t\tilde{u}\;dx\;dt-\gamma\int_{K_\delta}(-t)^{\gamma-1}\widetilde{u}\;\;v\;dx\;dt=-\delta^{\gamma}\int_{K_\delta}u\,(x,-\delta)\;\;v\;\;(x,-\delta)\;\;dx+$$

$$+\int_{Q_{T,\delta}}(-t)^{\gamma}v_tudxdt-\gamma\int_{Q_{T,\delta}}(-t)^{\gamma-1}uvdxdt.$$

Hence it follows that

$$\lim_{\delta\to0+}\;J_\delta\;=\int_{Q_T}(-t)^{\gamma}v_t\;u\;dx\;dt-\gamma\int_{Q_T}(-t)^{\gamma-1}u\;v\;dx\;dt. \tag{10}$$

Now, taking into account (7)-(10) in (6), and tending δ to zero we arrive at integral identity (5).

Theorem 1. *If with respect to number parameters λ and γ conditions* (1) *and* (2) *are fulfilled, then the first boundary value problem* (3)-(4) *is uniquely weakly solvable in the space $\overset{0}{W}{}_{2,\gamma}^{1,0}\,(Q_T)$ for any $f(x,t)\in L_{2,\gamma}(Q_T)$ and $f^k\,(x,t)\in L_{2,\gamma}\,(Q_T)$; $k=\overline{1,n}$.*

Proof. At first prove the existence of the solution. To this end we consider the extending sequence of domains $\{D_m\}$, $m=1,2,\ldots$; approximating from within the domain D, i.e. $\overline{D_m}\subset D_{m+1}$, $\overline{D_m}\subset D,\lim_{m\to\infty}D_m=D$. Therewith we choose D_m so that for any natural m $\partial D_m\in C^2$. Let further Q^m be a $P-$domain whose foot is the domain D_m,

$$Q_T^m=Q^m\cap\{(x,t):t>-T\}\,,\quad Q_{T,\delta}^m=Q_T^m\backslash\overline{Q}_\delta^m\;,\quad \delta\in(0,T)\,.$$

Denote by f^h and $f^{k,h}$ the Friedrichs averaged functions, respectively, $k=\overline{1,n}$ with a parameter $h>0$. Consider for $h>0$ and natural m the family of the first boundary value problems

$$\Delta u^{m,h}+\lambda\sum_{i,j=1}^{n}\left(\frac{x_ix_j}{4\left(-t\right)}u_i^{m,h}\right)_j-\lambda\left(n+1\right)\sum_{i=1}^{n}\frac{x_i}{4\left(-t\right)}u_i^{m,h}-u_t^{m,h}=$$

$$=f^h+\sum_{k=1}^{n}\frac{\partial f^{k,h}}{\partial x_k}\ ;\qquad (x,t)\in Q_{T,\delta}^m, \tag{11}$$

$$u^{m,h}\Big|_{\Gamma\left(Q_{T,\delta}^m\right)}=0. \tag{12}$$

As for any natural m and positive h and δ the coefficients and the right side of equation (11) are infinitely differentiable in $\overline{Q}_{T,\delta}^m$ functions, problem (11)-(12) has a unique classic solution $u^{m,h}(x,t)$. Indeed,$u^{m,h}(x,t)$ depends on δ as well, but for brevity of notation we write $u^{m,h}(x,t)$ instead of $u_\delta^{m,h}(x,t)$. Multiply the both sides of equation (11) by the function $(-t)^\gamma u^{m,h}(x,t)$ and integrate the obtained equality with respect to the domain $Q_{T,\delta}^m$.

We get

$$\int_{Q_{T,\delta}^m}(-t)^\gamma\Delta u^{m,h}\cdot u^{m,h}dx\ dt+\int_{Q_{T,\delta}^m}(-t)^\gamma u^{m,h}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{x_ix_j}{4\left(-t\right)}u_j^{m,h}\right)dx\ dt-$$

$$-\lambda\left(n+1\right)\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{i=1}^{n}\frac{x_i}{4\left(-t\right)}u_i^{m,h}u^{m,h}dx\ dt-\int_{Q_{T,\delta}^m}(-t)^\gamma u_i^{m,h}u_t^{m,h}dxdt=$$

$$=\int_{Q_{T,\delta}^m}(-t)^\gamma f^h\cdot u^{m,h}dx\ dt+\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{k=1}^{n}\frac{\partial f^{k,h}}{\partial x_k}u^{m,h}dx\ dt. \tag{13}$$

Further we have

$$\int_{Q_{T,\delta}^m}(-t)^\gamma\Delta u^{m,h}\cdot u^{m,h}dx\ dt+\lambda\int_{Q_{T,\delta}^m}(-t)^\gamma u^{m,h}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{x_ix_j}{4\left(-t\right)}u_j^{m,h}\right)dx\ dt=$$

$$=-\int_{Q_{T,\delta}^m}(-t)^\gamma\left(u_x^{m,h}\right)^2dx\ dt-\lambda\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{i,j=1}^{n}\frac{x_ix_j}{4\left(-t\right)}u_i^{m,h}u_j^{m,h}dxdt. \tag{14}$$

Furthermore

$$-\lambda\left(n+1\right)\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{i=1}^{n}\frac{x_i}{4\left(-t\right)}u_iu\ dxdt=\frac{\lambda\left(n+1\right)\cdot n}{2}\int_{Q_{T,\delta}^m}(-t)^\gamma\frac{u^2}{4\left(-t\right)}dx\ dt, \tag{15}$$

$$\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{k=1}^{n}\frac{\partial f^{k,h}}{\partial x_k}u^{m,k}dx\ dt=-\int_{Q_{T,\delta}^m}(-t)^\gamma\sum_{k=1}^{n}f^{k,h}u_k^{m,h}dx\ dt. \tag{16}$$

Finally, by means of arguments similar to ones that were used by deriving integral identity (5), we get

$$-\int_{Q^m_{T,\delta}}(-t)^{\gamma}u\ u_t dxdt=-\frac{\gamma}{2}\int_{Q^m_{T,\delta}}(-t)^{\gamma}u^2dxdt+i_1(\delta)\,, \tag{17}$$

where $\lim\limits_{\delta\to\infty} i_1(\delta)=0$.

Taking into account (13)-(16) in (12) and tending δ to zero, we conclude

$$\int_{Q^m_{T,\delta}}(-t)^{\gamma}\left(\left(u_x^{m,h}\right)^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_i^{m,h}\cdot u_j^{m,h}\right)\ dx\ dt-$$

$$-\frac{\lambda(n+1)-4\gamma}{2}\cdot\int_{Q^m_T}(-t)^{\gamma}\frac{\left(u^{m,h}\right)^2}{4(-t)}\ dx\ dt=$$

$$=\int_{Q^m_T}(-t)^{\gamma}\sum_{k=1}^{n}f^k u_k^{m,h}dx\ dt-\int_{Q^m_T}(-t)^{\gamma}f\ \ u^{m,h}dx\ dt. \tag{18}$$

Here $u^{m,h}(x,t)=\lim_{\delta\to 0+}u_{\delta}^{m,h}(x,t)$.

The existence of the pointwise limit is proved in the same as in [1].

If now $\frac{\lambda n(n+1)-4\gamma}{2}\le 0$, i.e. $\gamma\ge\frac{\lambda n(n+1)}{2}$, then from (17) it follows

$$\int_{Q^m_T}(-t)^{\gamma}\left(\left(u_x^{m,h}\right)^2+\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_i^{m,h}\cdot u_j^{m,h}\right)dx\ dt\le$$

$$\le\int_{Q^m_T}(-t)^{\gamma}\sum_{k=1}^{n}f^{k,h}u_k^{m,h}dx\ dt-\int_{Q^m_T}(-t)^{\gamma}f^h u^{m,h}dx\ dt. \tag{19}$$

Note that for ≥ 0 , $\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_i^{m,h}\cdot u_j^{m,h}\ge 0$. But if $-\frac{1}{d^2}<\lambda<0$, then

$$\lambda\sum_{i,j=1}^{n}\frac{x_ix_j}{4(-t)}u_i^{m,h}\cdot u_j^{m,h}\ge\lambda\ d^2\left(u_x^{m,h}\right)^2.$$

Thus, if $\gamma\ge\frac{\lambda n(n+1)}{2}$, then from (18) we get

$$\int_{Q^m_T}(-t)^{\gamma}\left(u_x^{m,h}\right)^2\ dx\ dt\le\ \ C_1(\lambda,n,d,\gamma)\,.$$

$$\left(\int_{Q^m_T}(-t)^{\gamma}\sum_{k=1}^{n}f^{k,h}u_k^{m,h}dx\ dt-\int_{Q^m_T}(-t)^{\gamma}f^h u^{m,h}dx\ dt\right). \tag{20}$$

Now consider the case

$$\gamma \in \left(\frac{n^2 \left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}, \frac{\lambda n \left(n+1\right)}{2} \right). \tag{21}$$

According to inequality (17)

$$\frac{\lambda n \left(n+1\right) - 4\gamma}{2} \int_{Q_T^m} (-t)^{\gamma} \frac{\left(u^{m,h}\right)^2}{4(-t)} \; dx \; dt \le$$

$$\le \frac{2 \; \lambda \; n \left(n+1\right) - 8\gamma}{n^2} \int_{Q_T^m} (-t)^{\gamma} \sum_{i,j=1}^{n} \frac{x_i x_j}{4\left(-t\right)} u_i^{m,h} \cdot u_j^{m,h} dx \; dt. \tag{22}$$

But on the other hand, from (20) it follows that there exists $\mu \in (0,1)$ for which

$$\frac{2 \; \lambda \; n \left(n+1\right) - 8\gamma}{n^2} < \frac{1}{d^2} + \lambda - \frac{\mu}{d^2}.$$

So, from (18) and (21) we conclude

$$\int_{Q_T^m} (-t)^{\gamma} \left(\left(u_x^{m,h}\right)^2 + \frac{\mu - 1}{d^2} \sum_{i,j=1}^{n} \frac{x_i x_j}{4\left(-t\right)} u_i^{m,h} \cdot u_j^{m,h} dx \; dt \right) \le$$

$$\le \int_{Q_T^m} (-t)^{\gamma} \left(\sum_{i=1}^{n} f^{i,h} u_i^{m,h} - f^h u^{m,h} \right) dx \; dt. \tag{23}$$

As $\mu < 1$, then

$$\frac{\mu - 1}{d^2} \sum_{i,j=1}^{n} \frac{x_i x_j}{4\left(-t\right)} u_i^{m,h} \cdot u_j^{m,h} \ge (\mu - 1) \left(u_x^{m,h}\right)^2. \tag{24}$$

From (22)-(23) it follows that

$$\mu \int_{Q_T^m} (-t)^{\gamma} \left(u_x^{m,h}\right)^2 dx \; dt \le \int_{Q_T^m} (-t)^{\gamma} \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h} \right) dx \; dt.$$

The last inequality and estimation (19) allows to conclude that for $\gamma \in \left(\frac{n^2 \left(\lambda - \frac{1}{d^2}\right) + 2\lambda n}{8}, \infty \right)$ the following inequality is valid

$$\int_{Q_T^m} (-t)^{\gamma} \left(u_x^{m,h}\right)^2 dx \; dt \le C_2 \left(\lambda, n, d, \gamma\right) \int_{Q_T^m} (-t)^{\gamma} \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h} \right) dx \; dt. \tag{25}$$

According to Friedrich's inequality we get

$$\int_{Q_T^m} (-t)^\gamma \left(u^{m,h}\right)^2 dx\ dt \le C_3 (\lambda, n, d, \gamma) \int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h}\right) dx\ dt. \quad (26)$$

Thus, from (24)-(25) we conclude

$$\int_{Q_T^m} (-t)^\gamma \left(\left(u^{m,h}\right)^2 + \left(u_x^{m,h}\right)^2\right) dx\ dt \le$$

$$leC_4 (\lambda, n, d, \gamma) \int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h}\right) dx\ dt. \quad (27)$$

Further, for any $\varepsilon > 0$ we have

$$\int_{Q_T^m} (-t)^\gamma \left(\sum_{k=1}^{n} f^{k,h} u_k^{m,h} - f^h u^{m,h}\right) dx\ dt \le$$

$$\le \frac{\varepsilon}{2} \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^{n} \left(u_k^{m,h}\right)^2 dx\ dt + \frac{1}{2\varepsilon} \int_{Q_T^m} (-t)^\gamma \sum_{k=1}^{n} \left(f^{k,h}\right)^2 dx\ dt +$$

$$+ \frac{\varepsilon}{2} \int_{Q_T^m} (-t)^\gamma \left(u^{m,h}\right)^2 dx\ dt + \frac{1}{2\varepsilon} \int_{Q_T^m} (-t)^\gamma \left(u^{m,h}\right)^2 dx\ dt. \quad (28)$$

Now choosing $\varepsilon = \frac{1}{C_4}$ from (26)-(27) we get

$$\int_{Q_T^m} (-t)^\gamma \left(\left(u^{m,h}\right)^2 + \left(u_x^{m,h}\right)^2\right) dx\ dt \le C_5 (\lambda, n, d, \gamma) \times$$

$$\times \left(\int_{Q_T^m} (-t)^\gamma \sum_{k=1}^{n} \left(f^{k,h}\right)^2 dx\ dt + \int_{Q_T^m} (-t)^\gamma \left(f^h\right)^2 dx\ dt\right). \quad (29)$$

Without loss of generality, we can consider that for $f \ne 0;\ f^k \ne 0;\ k = \overline{1,n}$. Therefore from (28) it follows that for rather small $h > 0$

$$\left\|u^{m,k}\right\|_{W_{2,\gamma}^{1,0}(Q_T^m)} \le C_6 (\lambda, n, d, \gamma) \left(\|f\|_{L_{2,\gamma}(Q_T)} + \sum_{k=1}^{n} \left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \quad (30)$$

Fix an arbitrary natural m. From inequality (29) it follows that the family of functions $\left\{u^{m,h}(x,t)\right\}$ is weakly compact (with respect to h) in the space $\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T^m)$. Thus, there exists such a sequence $h_l \to 0$ as $l \to \infty$ and the function $u^m(x,t) \in \overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T^m)$ that the functional sequence $\left\{u^{m,h_l}(x,t)\right\}$ weakly converges to the function $u^m(x,t)$ in $\overset{0}{W}{}_{2,\gamma}^{1,0}(Q_T^m)$

as $l\to\infty$. This means that for any function $u^m(x,t)\in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)$ it holds the limit equality

$$\lim_{l\to\infty} B_{Q_T^m}\left(u^{m,h_l},v\right)=B_{Q_T^m}(u^m,v)\ . \tag{31}$$

But on the other hand

$$B_{Q_T^m}\left(u^{m,h_l},v\right)=\int_{Q_T^m}(-t)^\gamma f^{h_l}\ v\ dx\ dt-\int_{Q_T^m}(-t)^\gamma\sum_{k=1}^n f^{k,h_l}v_k dx\ dt. \tag{32}$$

Furthermore

$$\lim_{l\to\infty}\left(\int_{Q_T^m}(-t)^\gamma f^{h_l}\ v\ dx\ dt-\int_{Q_T^m}(-t)^\gamma\sum_{k=1}^n f^{k,h_l}v_k\ \ dx\ dt\right)=$$

$$=\int_{Q_T^m}(-t)^\gamma f\ v\ dx\ dt-\int_{Q_T^m}(-t)^\gamma\sum_{k=1}^n f^k v_k dx\ dt. \tag{33}$$

From (30)-(32) we conclude that

$$B_{Q_T^m}(u^m,v)=\int_{Q_T^m}(-t)^\gamma f\ v\ dx\ dt-\int_{Q_T^m}(-t)^\gamma\sum_{k=1}^n f^k v_k dx\ dt.$$

The last equality means that the function $u^m(x,t)$ is a weak solution of equation (3) in the domain Q_T^m. Furthermore, for the function $u^m(x,t)$ the following estimation is valid

$$\|u^m\|_{W^{1,0}_{2,\gamma}(Q_T^m)}\le C_7(\lambda,n,d,\gamma)\left(\|f\|_{L_{2,\gamma}(Q_T)}+\sum_{k=1}^n\left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \tag{34}$$

For any natural m we continue the function $u^m(x,t)$ by a zero in $Q_T\backslash Q_T^m$ and denote the obtained continuation again by $u^m(x,t)$. It is easy to see that $u^m(x,t)\in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$. Therewith, according to (33) the following estimation is valid

$$\|u^m\|_{W^{1,0}_{2,\gamma}(Q_T)}\le C_8\left(\|f\|_{L_{2,\gamma}(Q_T)}+\sum_{k=1}^n\left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \tag{35}$$

From (34) it follows that the family of functions $\{u^m(x,t)\},\ldots .m=1,2,\ldots$ is weakly compact in the space $\overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T)$. Thus, there exists such a function $u(x,t)\in \overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)$ and sequence $m_r\to\infty$ as $r\to\infty$ that $u(x,t)$ is a weak limit of $u^{m_r}(x,t)$ as $r\to\infty$in $\overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)$. This means that for any function $v(x,t)\in \overset{0}{W}{}^{1,0}_{2,\gamma}$ the following limit equality is valid:

$$\lim_{r\to\infty} B_{Q_T}(u^{m_r},v)=B_{Q_T}(u,v)\ .$$

Moreover, using the above arguments, we can show that

$$B_{Q_T}(u,v)=\int_{Q_T^m}(-t)^{\gamma}f\ v\ dx\ dt-\int_{Q_T^m}(-t)^{\gamma}\sum_{k=1}^{n}f^k\ v\ dx\ dt.$$

From the last equality it follows that the function $u(x,t)$ is a weak solution of the first boundary value problem (3)-(4). Furthermore, for the functions $u(x,t)$ the following estimation is valid

$$\|u^m\|_{\overset{0}{W}{}^{1,0}_{2,\gamma}(Q_T^m)}\le C_9\left(\|f\|_{L_{2,\gamma}(Q_T)}+\sum_{k=1}^{n}\left\|f^k\right\|_{L_{2,\gamma}(Q_T)}\right). \tag{36}$$

Thereby the existence of the weak solution of the first boundary value problem (3)-(4) is proved. Now prove its uniqueness. It suffices to show that a homogeneous problem has only a trivial solution. Let $u(x,t)$ be the solution of homogeneous problem (3)-(4), i.e. for $f\equiv 0$; $f^k\equiv 0$; $k=\overline{1,n}$. Fix an arbitrary $\delta\in(0,T)$ and consider the function $v(x,t)\in\overset{0}{W}{}^{1,1}_{2,\gamma}(Q_{T+\delta})$ vanishing for $t\le T$ and $t\ge-\delta$. Let further $K=\Pi_R\times(-T-\delta,0)$, $\Pi_R=\{x:\ |x_i|<R,\ i=\overline{1,n}\}$. Continue the function $u(x,t)$ and $v(x,t)$ by zero to $K\backslash Q_T$ and denote the obtained continuations again by $u(x,t)$ and $v(x,t)$, respectively. It is easy to see that $u(x,t)\in\overset{0}{W}{}^{1,1}_{2,\gamma}(K)$, while $v(x,t)\in\overset{0}{W}{}^{1,1}_{2,\gamma}(K)$, Denote for

$$h\in(0,\delta]\ \frac{1}{h}\int_{t-h}^{t}v(x,\tau)\ d\tau\quad\text{by}\quad v_{\overline{h}}(x,\tau)$$

and put into integral identity (5) instead of the function $u(x,t)$ the function $v_{\overline{h}}(x,\tau)$ and get

$$B_K\left(u,v_{\overline{h}}\right)=0. \tag{37}$$

Taking into account the equalities$\left(v_{\overline{h}}\right)_t=(v_t)_{\overline{h}}$, $\left(v_{\overline{h}}\right)_i=(v_i)_{\overline{h}}$ $\ i=\overline{1,n}$and also

$$-\int_K(-t)^{\gamma}u\ (v_t)_{\overline{h}}\ dx\ dt=-\int_K((-t)^{\gamma}u)_h v_t dx\ dt=\int_K[((-t)^{\gamma}u)_h]_t v dx\ dt,$$

$$\int_K(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_iu_i\,(v_j)_{\overline{h}}\ dx\ dt=\int_K(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}u_i\right)v_j\ dx\ dt,$$

where $u_h(x,t)=\frac{1}{h}\int_t^{t+h}u(x,\tau)\,d\tau$, assuming $v(x,t)=u_h(x,t)$ tending h to zero, from (36) we get

$$\frac{\lambda\ n(n+1)-4\gamma}{8}\int_{Q_{T,\delta}}(-t)^{\gamma-1}u^2\ dx\ dt-\int_{Q_{T,\delta}}(-t)^{\gamma}\sum_{i,j=1}^{n}\left(\delta_{ij}+\lambda\frac{x_ix_j}{4(-t)}\right)u_iu_j\ dx\ dt=0. \tag{38}$$

Now, behaving as in deriving estimation (24), we get that if with respect to number parameters λ and γ conditions (1) and (2) are fulfilled, then $\int_{Q_{T,\delta}} (-t)^{\gamma} u_x^2 \, dx \, dt = 0$. The last equality yields

$$\int_{Q_{T,\delta}} (-t)^{\gamma-1} u^2 \, dx \, dt = 0.$$

With regard to arbitrariness of δ we conclude

$$\int_{Q_T} (-t)^{\gamma-1} u^2 \, dx \, dt = 0.$$

Hence it follows that $u\,(x,t) = 0$ almost everywhere in Q_T. ◄

In fact in the course of proof we established the estimation of the weak solution of the first boundary value problem (3)-(4). We formulate this statement in the form of a separate theorem.

Theorem 2. *If the conditions of the previous theorem are fulfilled then for the weak solution of the first boundary value problem* (3)-(4)*, estimation* (35) *is valid.*

References

[1] O.A. Ladyzhenskaya, V.A. Solonnikov, *Uraltseva N.N. linear and quaslinear equations of parabolic type*, Moscow, Nauka, 1967.

[2] Yu.A. Alkhutov, *Behavior of solutions of second order parabolic equations in noneylindrical domains*, Doklady RAN, **345(5)**, 1995, 583-585.

[3] Yu.A. Alkhutov, I.T. Mamedov, *Some properties of the solution of the first boundary value problem for second order parabolic equations with discontinuous coefficients*, Dokl. AN SSSR, **284(1)**, 1985, 477-500.

[4] V.A. Rukavishnikov, U. Rukavishnikova, *On the isomorphic mapping of weighted spaces by an elliptic operator with degeneration on the domain boundary*, Diff. Urav., **50(3)**, 2014, 345-351.

[5] A.B. Kostin, *Counterexamples in inverse problem for parabolic, elliptic, hyperbolic equations*, Zhurnal matematiki I matematicheskoi fiziki, **54(5)**, 2014, 797-810.

[6] N.J. Jafarov, *First boundary value problemfor class of parabolic equations with discontinuous coefficients, the basic coercive estimation in Sobolev weight space* , Trans. of NAS of Azerb. Ser. of phys. – tech. and math. sci., **XXXV(1)**, 2015, 51-58.

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Received 23 August 2018
Accepted 25 September 2018