Caspian Journal of Applied Mathematics, Ecology and Economics
V. 3, No 2, 2015, December
ISSN 1560-4055

On Completeness of Eigenfunctions of the Spectral Problem

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Abstract. In this paper is studied the spectral problem for a discontinuous second order differential operator with a spectral parameter in conjugation conditions, that arises by solving the problem on vibrations of a loaded string with the fixed ends. Asymptotic formulas for the eigenvalues and eigenfunctions of the spectral problem are obtained, theorems on the completeness of eigenfunctions in spaces $L_p \oplus C$ and L_p are proved.

Key Words and Phrases: eigenvalues, eigenfunctions, asymptotic formulae, biorthogonal system, completeness, minimality.

2010 Mathematics Subject Classifications: 34B05, 34B24, 34L10, 34L20

1. Introduction

Consider a spectral problem

$$y''(x) + \lambda y(x) = 0 \quad , x \in \left(0, \frac{1}{3}\right) \bigcup \left(\frac{1}{3}, 1\right), \tag{1}$$

$$\left.\begin{array}{l} y(0) = y(1) = 0, \\ y(\frac{1}{3} - 0) = y(\frac{1}{3} + 0), \\ y'(\frac{1}{3} - 0) - y'(\frac{1}{3} + 0) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{2}$$

which arises by solving the problem on vibrations of a loaded string with the fixed ends [1-3]. In the case when a load is placed in the middle of the string, this problem was investigated in [4;5]. Similar questions for the problem on vibrations of a loaded string when the load is fixed in one or two ends of a string, are investigated by other methods in [6-9].

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2. The asymptotic of eigenvalues and eigenfunctions

Let $\lambda = \rho^2$ and introduce the following notation for boundary forms (2).

$$U_\nu(y) = U_{\nu 1}(y) + U_{\nu 2}(y), \nu = \overline{1,4}, \tag{3}$$

where

$$U_{11}(y) = y(0), U_{12}(y) \equiv 0,$$
$$U_{21}(y) \equiv 0, U_{22}(y) = y(1),$$
$$U_{31}(y) = y(\frac{1}{3} - 0), U_{32}(y) = -y(\frac{1}{3} + 0),$$
$$U_{41}(y) = y'(\frac{1}{3} - 0), U_{42}(y) = -y'(\frac{1}{3} + 0) - \rho^2 m y(\frac{1}{3} + 0).$$

Firstly, let us prove the following theorem.

Theorem 1. *Spectral problem* (1), (2) *has two series of simple eigenvalues:* $\lambda_{1,n} = (\rho_{1,n})^2$, $n = 1, 2, ...,$ $\lambda_{2,n} = (\rho_{2,n})^2$, $n = 0, 1, 2, ...,$ *where*

$$\left.\begin{array}{l} \rho_{1,n} = 3\pi n, \\ \rho_{2,n} = \frac{3\pi n}{2} + \frac{2+(-1)^n}{\pi m n} + O(\frac{1}{n^2}). \end{array}\right\} \tag{4}$$

The eigenfunctions $u(x)$, *n=0,1,2,...,* prescribed by formula

$$u_{2n-1}(x) = \sin 3\pi n x, x \in [0,1], n = 1, 2, ...$$

$$u_{2n}(x) = \begin{cases} \{\sin \rho_{2,n}(x - \frac{1}{3}) + \sin \rho_{2,n}(x + \frac{1}{3}), x \in \left[0, \frac{1}{3}\right], \\ \sin \rho_{2,n}(1 - x), x \in \left[\frac{1}{3}, 1\right], n = 0, 1, 2, ... \end{cases} \tag{5}$$

correspond to them.

Proof. Take $y_{11}(x) = \sin \rho x, y_{12}(x) = \cos \rho x$ for $x \in \left[0, \frac{1}{3}\right]$ and $y_{21}(x) = \sin \rho(x - \frac{1}{3}), y_{22}(x) = \cos \rho(x - \frac{1}{3})$ for $x \in \left[\frac{1}{3}, 1\right]$ as linear - independent solutions of the equation (1). Eigenfunctions of the problem (1), (2) will be sought in the form

$$y(x,\rho) = \begin{cases} c_{11} y_{11}(x) + c_{12} y_{12}(x), x \in \left[0, \frac{1}{3}\right], \\ c_{21} y_{21}(x) + c_{22} y_{22}(x), x \in \left[\frac{1}{3}, 1\right], \end{cases} =$$

$$= \begin{cases} c_{11} \sin \rho x + c_{12} \cos \rho x, x \in \left[0, \frac{1}{3}\right], \\ c_{21} \sin \rho(x - \frac{1}{3}) + c_{22} \cos \rho(x - \frac{1}{3}), x \in \left[\frac{1}{3}, 1\right]. \end{cases} \tag{6}$$

Let us require that the function $y(x,\rho)$ satisfies the boundary conditions (2) . Then for definition of numbers $c_{j.k}$ we obtain the system of the linear homogeneous equations

$$\begin{cases} C_{11} U_{\nu 1}(y_{11}) + C_{12} U_{\nu 1}(y_{12}) + C_{21} U_{\nu 2}(y_{21}) + C_{22} U_{\nu 2}(y_{22}) = 0, \\ \nu = \overline{1,4}, (or\ briefly\ \sum_{j,k=1}^{2} C_{kj} U_{\nu j}(y_{jk}) = 0), \end{cases} \tag{7}$$

whose determinant is

$$\Delta(\rho) = \det \left\| U_{\nu j}(y_{jk}) \right\|, j, k = 1, 2, \nu = \overline{1,4}$$

Taking into consideration (3), for values of forms $U_{\nu j}(y_{jk})$ we have

$$\left\{\begin{array}{l} U_{11}(y_{11})=0, U_{11}(y_{12})=1, U_{12}(y_{21})=0, U_{12}(y_{22})=0, \\ U_{21}(y_{11})=0, U_{21}(y_{12})=0, U_{22}(y_{21})=\sin\frac{2\rho}{3}, U_{22}(y_{22})=\cos\frac{2\rho}{3}, \\ U_{31}(y_{11})=\sin\frac{\rho}{3}, U_{31}(y_{12})=\cos\frac{\rho}{3}, U_{32}(y_{21})=0, U_{32}(y_{22})=-1, \\ U_{41}(y_{11})=\rho\cos\frac{\rho}{3}, U_{41}(y_{12})=-\rho\sin\frac{\rho}{3}, U_{42}(y_{21})=-\rho, U_{42}(y_{22})=-\rho^2 m \end{array}\right. \tag{8}$$

Opening determinant $\Delta(\rho)$, with the account (8) we obtain

$$\Delta(\rho)=\rho\sin\frac{\rho}{3}\left(-\rho m\sin\frac{2\rho}{3}+2\cos\frac{2\rho}{3}+1\right). \tag{9}$$

From the formula (9) it is obvious, that function $\Delta(\rho)$ has two series of zeroes, the first of which consists of zeros of function $\sin\frac{\rho}{3}$ i.e. $\rho_{1,n}=3\pi n$ and the second series $\rho_{2,n}$ consists of zeros of function $(-\rho m\sin\frac{2\rho}{3}+2\cos\frac{2\rho}{3}+1)$.

Reasoning as in [10, p.20], we obtain that for $\rho_{2,n}$ the following asymptotic formula is true

$$\rho_{2,n}=\frac{3\pi n}{2}+\delta_n,$$

where δ_n satisfies the relation $\sin\frac{2\delta_n}{3}=\frac{(-1)^n}{m\rho_{2,n}}+\frac{2}{m\rho_{2,n}}\cos\frac{2\delta_n}{3}$. From the last relation we have

$$\delta_n=\frac{1}{\pi mn}((-1)^n+2)+O(\frac{1}{n^2})),$$

that proves the validity of (4).

Now, substituting in (7) $\rho=3\pi n$, considering (8), we get $c_{12}=c_{22}=0, c_{21}=(-1)^n c_{11}$. Therefore, we choose $c_{11}=1, c_{21}=(-1)^n$, from (6) we obtain the eigenfunction $u_{2n-1}(x)$, corresponding to eigenvalue $\lambda_{1,n}=(3\pi n)^2$ in the form of

$$u_{2n-1}(x)=\sin 3\pi nx, x\in[0,1], n=1,2,...$$

Similarly, when $\rho=\rho_{2,n}$ from (7) and (8) we conclude that

$$c_{12}=0,$$

$$c_{21}\sin\frac{2\rho_{2,n}}{3}+c_{22}\cos\frac{2\rho_{2,n}}{3}=0,$$

$$c_{22}=c_{11}\sin\frac{\rho}{3}.$$

Taking into consideration that $\sin\frac{2\rho_{2,n}}{3}\neq 0$ and choosing $\sin c_{11}=2\cos\frac{2\rho_{2,n}}{3}$, we obtain the eigenfunction $u_{2n}(x)$ from (6), corresponding to eigenvalue $\lambda_{2,n}=(\rho_{2,n})^2$ in the following form

$$u_{2n}(x)=\left\{\begin{array}{l} \sin\rho_{2,n}(x-\frac{1}{3})+\sin\rho_{2,n}(x+\frac{1}{3}), x\in\left[0,\frac{1}{3}\right] \\ \sin\rho_{2,n}(1-x), x\in\left[\frac{1}{3},1\right], n=0,1,2,... \end{array}\right.$$

Theorem is proved.

3. Construction of the Green's function and the resolvents of the linearized operator

Now let's pass to construction of the Green's function of problem (1) , (2). It is defined as a kernel of integral representation for solution of the corresponding non-homogeneous problem

$$y''(x)+\rho^2 y(x)=f(x), \tag{10}$$

satisfying boundary conditions (2). The solution of problem (10) ,(2) will be sought in the form

$$y(x)=\begin{cases} y_1(x),\ for\ x\in\left[0,\frac{1}{3}\right], \\ y_2(x), for\, x\in\left[\frac{1}{3},1\right], \end{cases} \tag{11}$$

where

$$\begin{cases} y_1(x)=c_{11}y_{11}(x)+c_{12}y_{12}(x)+\int_0^{\frac{1}{3}} g(x,\xi,\rho)f(\xi)d\xi, x\in\left[0,\frac{1}{3}\right], \\ y_2(x)=c_{21}y_{21}(x)+c_{22}y_{22}(x)+\int_{\frac{1}{3}}^{1} g(x,\xi,\rho)f(\xi)d\xi, x\in\left[\frac{1}{3},1\right]. \end{cases} \tag{12}$$

$$g(x,\xi,\rho)=\begin{cases} -\frac{1}{2\rho}\sin\rho(x-\xi), \text{ for } \xi<x, \\ \frac{1}{2\rho}\sin\rho(x-\xi), \text{ for } \xi>x. \end{cases} \tag{13}$$

Let us require that the function (11) satisfies the boundary conditions (2). Then for definition of numbers $C_{j,k}$ we obtain the system of algebraic equations

$$U_\nu(y)=\sum_{j,k=1}^{2} C_{j,k}U_{\nu j}(y_{jk})+\int_0^{\frac{1}{3}} U_{\nu 1}(g)f(\xi)d\xi+\int_{\frac{1}{3}}^{1} U_{\nu 2}(g)f(\xi)d\xi=0, \nu=\overline{1,4}. \tag{14}$$

Let's define numbers $C_{j,k}$ from (14) and substituting their values in (12), for solving the problem (10), (12) we obtain the formula

$$\begin{aligned} y_1(x)&=\int_0^{\frac{1}{3}} G_{11}(x,\xi,\rho)f(\xi)d\xi+\int_{\frac{1}{3}}^{1} G_{12}(x,\xi,\rho)f(\xi)d\xi, x\in\left[0,\tfrac{1}{3}\right], \\ y_2(x)&=\int_0^{\frac{1}{3}} G_{21}(x,\xi,\rho)f(\xi)d\xi+\int_{\frac{1}{3}}^{1} G_{22}(x,\xi,\rho)f(\xi)d\xi, x\in\left[\tfrac{1}{3},1\right], \end{aligned} \tag{14'}$$

where

$$G_{11}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} g & y_{11} & y_{12} & 0 & 0 \\ U_{\nu 1}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \dots & \dots & \dots & \dots & \dots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$G_{12}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} 0 & y_{11} & y_{12} & 0 & 0 \\ U_{\nu 2}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \dots & \dots & \dots & \dots & \dots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$G_{21}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} 0 & 0 & 0 & y_{21} & y_{22} \\ U_{\nu 1}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

$$G_{22}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\begin{vmatrix} g & 0 & 0 & y_{21} & y_{22} \\ U_{\nu 2}(g) & U_{\nu 1}(y_{11}) & U_{\nu 1}(y_{12}) & U_{\nu 2}(y_{21}) & U_{\nu 2}(y_{22}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \nu=\overline{1,4} & & & & \end{vmatrix}$$

and $\Delta(\rho)$ is a determinant from (7).

$$\begin{array}{l} U_{11}(g)=-\frac{1}{2\rho}\sin\rho\xi, U_{12}(g)=0 \\ U_{21}(g)=0, U_{22}(g)=-\frac{1}{2\rho}\sin\rho(1-\xi) \\ U_{31}(g)=-\frac{1}{2\rho}\sin\rho(\frac{1}{3}-\xi), U_{32}(g)=-\frac{1}{2\rho}\sin\rho(\frac{1}{3}-\xi) \\ U_{41}(g)=-\frac{1}{2}\cos\rho(\frac{1}{3}-\xi), U_{42}(g)=-\frac{\rho m}{2}\sin\rho(\frac{1}{3}-\xi)-\frac{1}{2}\cos\rho(\frac{1}{3}-\xi) \end{array} \tag{15}$$

Let's substitute (15), (8) and (9) n determinants of formulae for $G_{kj}(x,\xi,\rho)$. Transforming the received determinants similar to [11, p. 95], and then opening them, we obtain the formulae for the Green's function components. We'll formulate it as a lemma.

Lemma 2. *For the Green's function components $G_{kj}(x,\xi,\rho)$ of the problem* (1), (2) *the following expressions are true:*

$$G_{11}(x,\xi,\rho)=\begin{cases} -\frac{1}{\rho}\sin\rho(x-\xi)+\frac{\sin\rho x\sin\rho(1-\xi)}{\Delta(\rho)}+\frac{\rho m\sin\frac{2\rho}{3}\sin\rho x\sin\rho(\xi-\frac{1}{3})}{\Delta(\rho)}, \xi<x, \\ \frac{1}{\rho}\sin\rho(x-\xi)+\frac{\sin\rho\xi\sin\rho(1-x)}{\Delta(\rho)}+\frac{\rho m\sin\frac{2\rho}{3}\sin\rho\xi\sin\rho(x-\frac{1}{3})}{\Delta(\rho)}, \xi>x, \end{cases} \tag{16}$$

$$G_{12}(x,\xi,\rho)=-\frac{1}{\Delta(\rho)}\sin\rho x\sin\rho(1-\xi), \tag{17}$$

$$G_{21}(x,\xi,\rho)=\frac{1}{\Delta(\rho)}\sin\rho\xi\sin\rho(1-x), \tag{18}$$

$$G_{22}(x,\xi,\rho)=\begin{cases} -\frac{1}{\rho}\sin\rho(x-\xi)+\frac{\sin\rho x\sin\rho(1-\xi)}{\Delta(\rho)}-\frac{\rho m\sin\frac{\rho}{3}\sin\rho(1-\xi)\sin\rho(x-\frac{1}{3})}{\Delta(\rho)}, \xi<x, \\ \frac{1}{\rho}\sin\rho(x-\xi)+\frac{\sin\rho\xi\sin\rho(1-x)}{\Delta(\rho)}-\frac{\rho m\sin\frac{\rho}{3}\sin\rho(1-x)\sin\rho(\xi-\frac{1}{3})}{\Delta(\rho)}, \xi>x. \end{cases} \tag{19}$$

Let us now proceed to the construction of linearizing operator. By $W_p^k\left(0,\frac{1}{3}\right)\oplus W_p^k\left(\frac{1}{3},1\right)$ we denote a space functions whose contractions on segments $\left[0,\frac{1}{3}\right]$ and $\left[\frac{1}{3},1\right]$ belong correspondingly to Sobolev spaces $W_p^k\left(0,\frac{1}{3}\right)$ and $W_p^k\left(\frac{1}{3},1\right)$. Let's define the operator $\mathscr{L}$ in $L_p(0,1)\oplus C$ as follows:

$$D\left(\mathscr{L}\right)=\left\{\begin{array}{l} \widehat{u}\in L_p(0,1)\oplus C:\widehat{u}=\left(u,mu\left(\frac{1}{3}\right)\right), u\in W_p^2\left(0,\frac{1}{3}\right)\bigcup\left(\frac{1}{3},1\right), \\ u(0)=u(1)=0, u\left(-\frac{1}{3}\right)=u\left(\frac{1}{3}\right), \end{array}\right\} \tag{20}$$

and for $\widehat{u}\in D(\mathscr{L})$

$$\mathscr{L}\,\widehat{u}=\left(-u'';u'\left(\frac{1}{3}-0\right)-u'\left(\frac{1}{3}+0\right),\widehat{u}\in D(\mathscr{L})\right). \tag{21}$$

Lemma 3. *Operator defined by the formulae* (20), (21) *is a linear closed operator with dense definitional domain in* $L_p(0,1)\oplus C$. *Eigenvalues of the operator* $\mathscr{L}$ *and problem* (1), (2) *coincide, and* $\left\{\widehat{u}_k\right\}_{k\in N_0}$ *are eigenvectors of the operator* $\mathscr{L}$, *where* $N_0=N\bigcup\{0\}$, $\widehat{u}_{2n-1}=(u_{2n-1}(x);0)$, $\widehat{u}_{2n}=(u_{2n}(x);m\sin\frac{2\rho_{2,n}}{3})$.

Proof. To prove the first part of the lemma we take $\widehat{u}\ (u,\alpha)\in L_p(0,1)\oplus C$ and we define the functional $F(\widehat{u})$ as follows:

$$F(\widehat{u})=mu\left(\frac{1}{3}\right)-\alpha.$$

Let us assume

$$U_\nu(\widehat{u})=U_\nu(u),\nu=1,2,3.$$

Then $F,U_\nu,\nu=1,2,3$ are bounded linear functionals on $W_p^2\left(0,\frac{1}{3}\right)\bigcup\left(\frac{1}{3},1\right)\oplus C$, but unbounded on $L_p(0,1)\oplus C$. Therefore (see, for example, [12, pp. 27-29]) the set

$$D(\mathscr{L})=\left\{\widehat{u}=(u,\alpha),u\in W_p^2\left(0,\frac{1}{3}\right)\bigcup\left(\frac{1}{3},1\right),F(\widehat{u})=U_\nu(\widehat{u})=0,\nu=1,2,3\right\}.$$

is everywhere dense in $L_p(0,1)\oplus C$, and $\mathscr{L}$ is a closed operator as contraction of corresponding closed maximal operator.
The second part of the lemma is verified directly.
The lemma is proved.

For construction resolvent of operator $\mathscr{L}$, consider the equation

$$\mathscr{L}\ \widehat{u}\ -\lambda\ \widehat{u}=\widehat{f}, \tag{22}$$

where $\widehat{u}\in D(\mathscr{L})$, $\widehat{f}=(f,\beta)\in L_p(0,1)\oplus C$. We can rewrite equation (22) in the form of

$$\begin{cases} -u''=\lambda u+f,\\ u'\left(-\frac{1}{3}\right)-u'\left(\frac{1}{3}\right)-\lambda mu\left(\frac{1}{3}\right)=\beta,\\ U_\nu(u)=0,\nu=1,2,3.\end{cases} \tag{23}$$

Lemma 4. *For solution* $\widehat{u}=\left(u,mu\left(\frac{1}{3}\right)\right)$ *of the equation* (22) *it holds the following representations*

$$u(x,\rho)=\frac{\beta\cos\frac{2\rho}{3}\cdot\sin\rho x}{\rho\sin\frac{\rho}{3}(1+2\cos\frac{2\rho}{3}-\rho m\sin\frac{2\rho}{3})}-\int_0^x f(\xi)\frac{\sin\rho(x-\xi)}{\rho}d\xi+$$

$$\frac{1}{\Delta(\rho)}\int_0^{\frac{1}{3}} f(\xi)\cdot\sin\rho x\cdot\sin\rho(1--\xi)d\xi-\frac{1}{\Delta(\rho)}\int_0^{\frac{1}{3}}\rho m\sin\rho x\cdot\sin\frac{2\rho}{3}\cdot\sin\rho\left(\frac{1}{3}-\xi\right)f(\xi)d\xi-$$

$$-\frac{1}{\Delta(\rho)}\int_{\frac{1}{3}}^1\sin\rho x\cdot\sin\rho(1-\xi)f(\xi)d\xi, \tag{24}$$

if $x\in\left[0,\frac{1}{3}\right]$.

$$u(x,\rho)=\frac{\beta\cos\frac{2\rho}{3}\cdot\sin\rho x}{\rho\sin\frac{\rho}{3}(1+2\cos\frac{2\rho}{3}-\rho m\sin\frac{2\rho}{3})}-\frac{1}{\Delta(\rho)}\int_0^{\frac{1}{3}} f(\xi)\sin\rho\xi\cdot\sin\rho(x-1)d\xi+$$

$$\int_x^1\frac{\sin\rho(x-\xi)}{\rho}f(\xi)d\xi+\frac{1}{\Delta(\rho)}\int_{\frac{1}{3}}^1\sin\rho x\cdot\sin\rho(1-\xi)f(\xi)d\xi-$$

$$-\frac{1}{\Delta(\rho)}\int_{\frac{1}{3}}^1\rho m\sin\frac{\rho}{3}\cdot\sin\rho\left(x-\frac{1}{3}\right)\cdot\sin\rho(1-\xi)f(\xi)d\xi. \tag{25}$$

if $x\in\left[\frac{1}{3},1\right]$.

$$u\left(\frac{1}{3},\rho\right)=\frac{1}{\rho(1+2\cos\frac{2\rho}{3}-\rho m\sin\frac{2\rho}{3})}\times$$

$$\times\left(\beta\cos\frac{2\rho}{3}+2\cos\frac{\rho}{3}\int_0^{\frac{1}{3}} f(\xi)\sin\rho\xi d\xi+\int_{\frac{1}{3}}^1\sin\rho(1-\xi)f(\xi)d\xi\right). \tag{26}$$

Proof. The solution of (23) will be sought in the form

$$u(x,\rho)=\begin{cases} C_{11}y_{11}(x)+C_{12}y_{12}(x)+y_1(x), \text{ for } x\in\left[0,\frac{1}{3}\right], \\ C_{21}y_{21}(x)+C_{22}y_{22}(x)+y_2(x), \text{ for } x\in\left[\frac{1}{3},1\right], \end{cases} \tag{27}$$

where $y_1(x)$ and $y_2(x)$ are defined by (14). Since $y(x)$, defined by (11) satisfies boundary conditions (2), then

$$U_\nu(y)=0, \nu=\overline{1,4}. \tag{28}$$

Let's demand the function $u(x,\rho)$ satisfy boundary conditions $U_\nu(u)=0,\nu=1,2,3,U_\nu(u)$ $=\beta$. Then taking into account (28) from (27) we obtain

$$\begin{cases} C_{11}U_{\nu1}(y_{11})+C_{12}U_{\nu1}(y_{12})+C_{21}U_{\nu2}(y_{21})+C_{22}U_{\nu2}(y_{22})=0, \nu=1,2,3; \\ C_{11}U_{41}(y_{11})+C_{12}U_{41}(y_{12})+C_{21}U_{42}(y_{21})+C_{22}U_{42}(y_{22})=\beta. \end{cases}$$

Solving this system with respect to unknowns C_{kj}, we get

$$C_{11}=-\frac{\beta}{\Delta(\rho)}U_{11}(y_{12})\left[U_{22}(y_{21})U_{32}(y_{22})-U_{22}(y_{22})U_{32}(y_{21})\right],$$

$$C_{12}=\frac{\beta}{\Delta(\rho)}U_{11}(y_{11})\left[U_{22}(y_{21})U_{32}(y_{22})-U_{22}(y_{22})U_{32}(y_{21})\right],$$

$$C_{21}=\frac{\beta}{\Delta(\rho)}U_{22}(y_{22})\left[U_{11}(y_{11})U_{31}(y_{12})-U_{11}(y_{12})U_{31}(y_{11})\right],$$

$$C_{22}=-\frac{\beta}{\Delta(\rho)}U_{22}(y_{21})\left[U_{11}(y_{11})U_{31}(y_{12})-U_{11}(y_{12})U_{31}(y_{11})\right].$$

Substituting the received values of the coefficients C_{kj} in (27) and taking into account formulae (8) ,(16)-(19) , we obtain the validity of formulas (24) and (25) . And formulae (26) is obtained from (25) (or from (24)) by substitution $x=\frac{1}{3}$.

Lemma is proved.

4. Completeness of the eigenfunctions in spaces $L_p(0,1)\oplus C$ and $L_p\,(0,1)$

Theorem 2. *System $\{\hat{u}_n\}_{n\in N_0}$ of eigenvectors of the operator $\mathscr{L}$ is complete in $L_p\,(0,1)\oplus C$, $1<p<\infty$.*

Proof. To prove the completeness of the system of eigenfunctions of the operator $\mathscr{L}$ in $L_p\,(0,1)\oplus C$ we need to get the estimation of the resolvent of the operator $\mathscr{L}$ at great values of $|\rho|$. We will use the following known inequalities

$$|\sin\rho|\leq ce^{|\rho|\sin\varphi},\ |\cos\rho|\leq ce^{|\rho|\sin\varphi}, \tag{29}$$

where $\rho=re^{i\varphi}, 0\leq\varphi\leq\pi$. Besides, outside of circles of the same radius δ with centres in zero of $\sin\rho$ the following estimation is true

$$|\sin\rho|\geq m_\delta e^{r\sin\varphi}. \tag{30}$$

From estimations (29) , (30) and from the formula (9) it follows that at great values of $|\rho|$ outside circles $K_{j,n}\,(\delta)=\{\rho:|\rho-\rho_{j,n}|<\delta\}$ of radius δ with the centres in zero of $\Delta(\rho)$ the following estimation is true

$$|\Delta(\rho)|\geq M_\delta re^{r\sin\varphi}. \tag{31}$$

Assume $G\,(\delta)=C\backslash\bigcup_{j,n}K_{j,n}\,(\delta)$. From the representations (24), (25) considering the inequalities (29) –(31) we obtain the inequality

$$|u(x,\rho)|\leq\frac{C_\delta}{|\rho|}, \rho\in G\,(\delta)\ ,\ |\rho|\geq r_0,$$

which fairly uniform on $x\in[0,1]$. From the last estimation it follows that for the resolvent $R(\lambda)=(\mathscr{L}-\lambda I)^{-1}$ of the operator $\mathscr{L}$ outside of the above-stated circles the following estimation is true

$$\left\|R(\rho^2)\right\|\leq\frac{C_\delta}{|\rho|}, |\rho|\geq r_0. \tag{32}$$

Having estimation (32), by a standard method (see for example [13]) we obtain that eigenfunctions of operator $\mathscr{L}$ form a complete system in $L_p\,(0,1)\oplus C$.

Let us note that the system of eigenvectors of the operator $\mathscr{L}$ $\{\hat{u}_n\}_{n\in N_{-0}}$ has a biorthogonal-conjugate system $\{\hat{v}_n\}_{n\in N_0}$, that are the system of eigenvectors of the conjugate operator $\mathscr{L}^*$, which in its turn is the linearized operator of the conjugate spectral problem:

$$v'' + \lambda v = 0, x \in \left(0, \frac{1}{3}\right) \bigcup \left(\frac{1}{3}, 1\right), \tag{1*}$$

$$v(0) = v(1) = 0,$$

$$v\left(\frac{1}{3} - 0\right) = v\left(\frac{1}{3} + 0\right), \tag{2*}$$

$$v'\left(\frac{1}{3} - 0\right) - v'\left(\frac{1}{3} + 0\right) = \lambda \bar{m} v\left(\frac{1}{3}\right).$$

Taking into account this by Theorem 2 we obtain

Corollary. *System $\{\hat{u}_n\}_{n\in N_0}$ of the eigenvectors of the operator $\mathscr{L}$ is complete and minimal in $L_p(0,1) \oplus C$, $1 < p < \infty$.*

Now let us consider the completeness and minimality of a system $\{u_n\}_{n\in N_0}$ of eigenfunctions of the problem (1), (2). It is clear that this system is overflowing in $L_p(0,1)$: One function of this system is unnecessary. Let us clarify the following question: Which function can be excluded from the system while maintaining the properties of completeness and minimality and which can not? The answer to this question is given by the following theorem.

Theorem 3. *Let n_0 be* some number from the set of indexes N_0. *If n_0 is an even number, then the system $\{u_n\}_{n\in N_0\backslash\{n_0\}}$ is complete and minimal in $L_p(0,1)$, $1 < p < \infty$, if n_0 is an odd number, then the system $\{u_n\}_{n\in N_0\backslash\{n_0\}}$ is not complete and minimal in this space.*

Proof. According to the above mentioned, the system $\{\hat{u}_n\}_{n\in N_0}$ has a biortogonal-conjugated system $\{\hat{v}_n\}_{n\in N_0}$, where

$$\hat{v}_n = \left(v_n(x); \bar{m} v_n\left(\frac{1}{3}\right)\right),$$

and $v_n(x)$ are eigenfunctions of the adjoint problem (1^*), (2^*) . Carrying out similar calculations for eigenfunctions, we find that the formulas are valid

$$v_{2n-1}(x) = c_{1n} \sin 3\pi n x, \ n = 1, 2.....$$

$$v_{2n}(x) = \begin{cases} c_{2n}\left(\sin \rho_{2n}\left(x - \frac{1}{3}\right) + \sin \rho_{2n}\left(x + \frac{1}{3}\right)\right) & x \in \left[0, \frac{1}{3}\right], \\ c_{2n} \sin \rho_{2n}(1 - x); & x \in \left[\frac{1}{3}; 1\right], \end{cases}$$

where c_{1n}, $c_{2n}-$ are normalization numbers. From this formula it is clear that if n_0 is an even number, then $v_{n_0}\left(\frac{1}{3}\right) \neq 0$, and if n_0 is an odd number, then $v_{n_0}\left(\frac{1}{3}\right) = 0$. Therefore, all statements of the theorem follow from the results of [14] (see also [15], [16]).

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Received 25 October 2015
Accepted 19 November 2015