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Characterization of Parabolic Fractional Integral and Its Commutators in Orlicz Spaces

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Abstract. In this paper, we characterize BMO space in terms of the boundedness of commutators of parabolic maximal operator in Orlicz spaces. As an application of this boundedness, we give necessary and sufficient condition for the boundedness of parabolic fractional integral and its commutators in Orlicz spaces.

Key Words and Phrases: Orlicz space, parabolic fractional integral, commutator, BMO.
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1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \quad (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$\begin{aligned}
&(a) \quad \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every} \quad x \in \mathbb{R}^n; \\
&(b) \quad \rho(0) = 0, \quad \rho(x-y) = \rho(y-x) \geq 0 \\
&\qquad \text{and} \quad \rho(x-y) \leq k(\rho(x-z) + \rho(y-z)); \\
&(c) \quad dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad \text{where} \quad \rho = \rho(x), w = A_{\rho^{-1}} x
\end{aligned}$$

and $d\sigma(w)$is a C^∞measure on the ellipsoid $\{w : \rho(w) = 1\}$.

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Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([2, 3]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)| = v_\rho r^\gamma$, where v_ρ is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x,r) = B(x,r)$. Note that in the standard parabolic case $P = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$. The parabolic maximal function $M^P f$ and the parabolic fractional integral $I_\alpha^P f$, $0 < \alpha < \gamma$, of a function $f \in L_1^{\rm loc}(\mathbb{R}^n)$ are defined by

$$M^P f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy,$$

$$I_\alpha^P f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

If $P = I$, then $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator. It is well known that, the parabolic maximal function and the parabolic fractional integral operators play an important role in harmonic analysis (see [4, 15]).

In this work we present the characterization for parabolic fractional integral operator I_α^P (Theorem 6) and its commutators $[b, I_\alpha^P]$ (Theorem 7) in Orlicz spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. On Young Functions and Orlicz Spaces

Orlicz space was first introduced by Orlicz in [12, 13] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

Definition 1. *A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} & , \quad r \in [0, \infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.

Definition 2. *(Orlicz Space). For a Young function Φ, the set*

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. The space $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L^{\Phi}(\mathbb{R}^n)$ for all parabolic balls $\mathcal{E} \subset \mathbb{R}^n$.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 3. *The weak Orlicz space*

$$WL^{\Phi}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WL^{\Phi}} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t)m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1\right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{t>0} \Phi(t) m(\Omega,\ f,\ t) = \sup_{t>0} t\, m(\Omega,\ f,\ \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega,\ \Phi(|f|),\ t)$$

and

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\Big) dx \le 1, \qquad \sup_{t>0} \Phi(t) m\Big(\Omega,\ \frac{f}{\|f\|_{WL^\Phi(\Omega)}},\ t\Big) \le 1, \tag{2}$$

where $\|f\|_{L^\Phi(\Omega)} = \|f\chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f\chi_\Omega\|_{WL^\Phi}$.

The following analogue of the Hölder's inequality is well known (see, for example, [14]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f and g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)| dx \le 2\|f\|_{L^\Phi(\Omega)} \|g\|_{L^{\tilde\Phi}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let Φ be a Young function and $\mathcal{E}$ be a parabolic balls in $\mathbb{R}^n$. Then*

$$\|\chi_{\mathcal{E}}\|_{L^\Phi} = \|\chi_{\mathcal{E}}\|_{WL^\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 1, Lemma 1 and (1) we get the following estimate.

Lemma 2. *For a Young function Φ and for the parabolic balls $\mathcal{E} = \mathcal{E}(x,r)$ the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)| dy \le 2|\mathcal{E}| \Phi^{-1}\left(|\mathcal{E}|^{-1}\right) \|f\|_{L^\Phi(\mathcal{E})}.$$

In [1] the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L^\Phi(\mathbb{R}^n)$ was obtained.

Theorem 2. *[1] Let Φ any Young function. Then the parabolic maximal operator M^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L^\Phi(\mathbb{R}^n)$.*

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L^1_{\rm loc}(\mathbb{R}^n)\ :\ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}| dy < \infty,$$

where $b_{\mathcal{E}(x,r)} = \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} b(y) dy$. We will need the following properties of BMO-functions:

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|^p dy \right)^{\frac{1}{p}}, \tag{3}$$

where $1 \le p < \infty$, and

$$\left|b_{\mathcal{E}(x,r)} - b_{\mathcal{E}(x,t)}\right| \le C\|b\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{4}$$

where C does not depend on b, x, r and t. We refer for instance to [9] and [10] for details on this space and properties.

The commutators generated by $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the parabolic maximal operator M^P is defined by

$$M_b^P(f)(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(x) - b(y)||f(y)|dy.$$

Next, we recall the notion of weights. Let w be a locally integrable and positive function on $\mathbb{R}^n$. The function w is said to be a Muckenhoupt A_1 weight if there exists a positive constant C such that for any ellipsoid $\mathcal{E}$

$$\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)dx \le C \text{ess}\inf_{x\in\mathcal{E}} w(x).$$

Lemma 3. *[6, Chapter 1] Let $\omega \in A_1$, then the reverse Hölder inequality holds, that is, there exist $q > 1$ such that*

$$\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)^q dx\right)^{\frac{1}{q}} \lesssim \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} w(x)dx$$

for all ellipsoids $\mathcal{E}$.

Lemma 4. *Let Φ be a Young function with $\Phi \in \Delta_2$. Then we have*

$$\frac{1}{2|\mathcal{E}|}\int_{\mathcal{E}} |f(x)|dx \le \Phi^{-1}\left(|\mathcal{E}|^{-1}\right) \|f\|_{L^\Phi(\mathcal{E})} \lesssim \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(x)|^p dx\right)^{\frac{1}{p}}$$

for some $1 < p < \infty$.

Proof. The left-hand side inequality is just Lemma 2.

Next we prove the right-hand side inequality. Our idea is based on [8]. Take $g \in L_{\widetilde{\Phi}}$ with $\|g\|_{L_{\widetilde{\Phi}}} \le 1$. Note that $\widetilde{\Phi} \in \nabla_2$ since $\Phi \in \Delta_2$, therefore M is bounded on $L_{\widetilde{\Phi}}(\mathbb{R}^n)$ from Theorem 2. Let $Q := \|M\|_{L_{\widetilde{\Phi}} \to L_{\widetilde{\Phi}}}$ and define a function

$$Rg(x) := \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2Q)^k},$$

where

$$M^k g := \begin{cases} |g| & k = 0, \\ Mg & k = 1, \\ M(M^{k-1}g) & k \ge 2. \end{cases}$$

For every $g \in L_{\widetilde{\Phi}}$ with $\|g\|_{L_{\widetilde{\Phi}}} \le 1$, the function Rg satisfies the following properties:

- $|g(x)| \le Rg(x)$ for almost every $x \in \mathbb{R}^n$;
- $\|Rg\|_{L_{\widetilde{\Phi}}} \le 2\|g\|_{L_{\widetilde{\Phi}}}$
- $M(Rg)(x) \le 2QRg(x)$, that is, Rg is a Muckenhoupt A_1 weight with the A_1 constant less than or equal to $2Q$.

By Lemma 3, there exist positive constants $q > 1$ and C independent of g such that for all ellipsoids $\mathcal{E}$,

$$\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x)^q dx\right)^{\frac{1}{q}} \le \frac{C}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x)d\mu(x).$$

By Lemmas 2 and 3, we obtain

$$\|Rg\|_{L^q(\mathcal{E})} = |\mathcal{E}|^{1/q}\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x)^q dx\right)^{\frac{1}{q}} \lesssim |\mathcal{E}|^{1/q}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} Rg(x)dx$$
$$\lesssim |\mathcal{E}|^{-1/q'}\frac{\|Rg\|_{L_{\widetilde{\Phi}}}}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)} \lesssim \frac{|\mathcal{E}|^{-1/q'}}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)}.$$

Thus we have

$$\int_{\mathcal{E}} |f(x)g(x)|dx \le \int_{\mathcal{E}} |f(x)|Rg(x)dx \le \|f\|_{L_{q'}(\mathcal{E})}\|Rg\|_{L_q(\mathcal{E})}$$
$$\lesssim \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(x)|^{q'}dx\right)^{\frac{1}{q'}}\frac{1}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)}.$$

Since the Luxemburg-Nakano norm is equivalent to the Orlicz norm (see, for example [14, p. 61]) we get

$$\|f\|_{L^\Phi(\mathcal{E})} \le \sup\left\{\left|\int_{\mathcal{E}} f(x)g(x)dx\right| : g \in L_{\widetilde{\Phi}},\quad \|g\|_{L_{\widetilde{\Phi}}} \le 1\right\}$$
$$\lesssim \left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(x)|^{q'}dx\right)^{\frac{1}{q'}}\frac{1}{\Phi^{-1}\big(|\mathcal{E}|^{-1}\big)}.$$

Consequently, the right-hand side inequality follows with $p = q'$.

We have the following result from (3) and Lemma 4.

Lemma 5. *Let $b \in BMO(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-\gamma}\big)\left\|b(\cdot) - b_{\mathcal{E}(x,r)}\right\|_{L^\Phi(\mathcal{E}(x,r))}.$$

The known boundedness statements for the commutator operator M_b^P on Orlicz spaces run as follows, see [5, Corollary 2.3].

Theorem 3. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{R}^n)$. Then M_b^P is bounded on $L^\Phi(\mathbb{R}^n)$ and the inequality*

$$\|M_b^P f\|_{L^\Phi} \le C_0\|b\|_*\|f\|_{L^\Phi} \tag{5}$$

holds with constant C_0 independent of f.

3. Parabolic fractional integral and its commutators in Orlicz spaces

For proving our main results, we need the following estimate.

Lemma 6. *If $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then for every $x \in \mathcal{E}_0$*

$$c_0 r_0^\alpha < I_\alpha^P \chi_{\mathcal{E}_0}(x),$$

where $c_0 = (2k)^{\alpha-\gamma}|\mathcal{E}(0,1)|$.

Proof. If $x, y \in \mathcal{E}_0$, then $\rho(x-y) \le k(\rho(x-x_0)+\rho(y-x_0)) < 2kr_0$. Since $0<\alpha<\gamma$, we get $(2kr_0)^{\alpha-\gamma} < \rho(x-y)^{\alpha-\gamma}$. Therefore

$$I_\alpha^P \chi_{\mathcal{E}_0}(x) = \int_{\mathcal{E}_0} \rho(x-y)^{\alpha-\gamma} dy > (2kr_0)^{\alpha-\gamma}|\mathcal{E}_0| = c_0 r_0^\alpha.$$

The known boundedness statement for I_α^P in Orlicz spaces on spaces of homogeneous type runs as follows.

Theorem 4. *[11] Let $\Phi, \Psi \in \mathcal{Y}$ and*

$$\int_r^\infty t^{\alpha-1}\Phi^{-1}\left(t^{-\gamma}\right) dt \lesssim r^\alpha \Phi^{-1}\left(r^{-\gamma}\right) \qquad \text{for } 0<r<\infty, \tag{6}$$

$$r^\alpha \Phi^{-1}\left(r^{-\gamma}\right) \lesssim \Psi^{-1}\left(r^{-\gamma}\right) \qquad \text{for } 0<r<\infty. \tag{7}$$

Then I_α^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_α^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 5. *Let $\Phi, \Psi \in \mathcal{Y}$ and I_α^P is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ then condition* (7) *holds.*

Proof. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemmas 6 and 1, we have

$$\begin{aligned} r_0^\alpha &\lesssim \Psi^{-1}(r_0^{-\gamma})\|I_\alpha^P \chi_{\mathcal{E}_0}\|_{WL^\Psi(\mathcal{E}_0)} \lesssim \Psi^{-1}(r_0^{-\gamma})\|I_\alpha^P \chi_{\mathcal{E}_0}\|_{WL^\Psi} \\ &\lesssim \Psi^{-1}(r_0^{-\gamma})\|\chi_{\mathcal{E}_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-\gamma})}{\Phi^{-1}(r_0^{-\gamma})}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

Combining Theorems 4 and 5 we have the following result.

Theorem 6. *Let $\Phi, \Psi \in \mathcal{Y}$. If* (6) *holds, then the condition* (7) *is necessary and sufficient for the boundedness of I_α^P from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition* (7) *is necessary and sufficient for the boundedness of I_α^P from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Remark 1. *Note that Theorem 6 in the isotropic case $P = I$ were proved in [7].*

The commutators $[b, I_\alpha^P]$, $|b, I_\alpha^P|$ generated by $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and the operator I_α^P are defined by

$$[b, I_\alpha^P]f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{\rho(x-y)^{\gamma-\alpha}} f(y)dy,$$

$$|b, I_\alpha^P|f(x) = \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{\rho(x-y)^{\gamma-\alpha}} f(y)dy, \qquad 0<\alpha<\gamma,$$

respectively.

The following lemma is the analogue of the Hedberg's trick for $[b, I_\alpha]$.

Lemma 7. *If $0<\alpha<\gamma$ and $f, b \in L^1_{\rm loc}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ and $r>0$ we get*

$$|b, I_\alpha^P|(\chi_{\mathcal{E}(x,r)}|f|)(x) \lesssim r^\alpha M_b^P f(x).$$

Proof.

$$|b, I_\alpha^P|(\chi_{\mathcal{E}(x,r)}|f|)(x) = \int_{\mathcal{E}(x,r)} \frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha}} |b(x)-b(y)|dy$$

$$= \sum_{j=0}^{\infty} \int_{\mathcal{E}(x,2^{-j}r)\setminus\mathcal{E}(x,2^{-j-1}r)} \frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha}} |b(x)-b(y)|dy$$

$$\lesssim \sum_{j=0}^{\infty} (2^{-j}r)^\alpha (2^{-j}r)^{-\gamma} \int_{\mathcal{E}(x,2^{-j}r)} |f(y)||b(x)-b(y)|dy \lesssim r^\alpha M_b^P f(x).$$

Lemma 8. *If $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \le C|b, I_\alpha^P|\chi_{\mathcal{E}_0}(x)$$

for every $x \in \mathcal{E}_0$.

Proof. The proof is similar to the proof of Theorem 6.

Theorem 7. *Let $0<\alpha<\gamma$, $b \in BMO(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.*

1. *If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}\big(r^{-\gamma}\big) + \int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1}dt \le C\Psi^{-1}\big(r^{-\gamma}\big) \tag{8}$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha^P]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. *If $\Psi \in \Delta_2$, then the condition (7) is necessary for the boundedness of $|b, I_\alpha^P|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

3. *Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1}dt \le Cr^\alpha \Phi^{-1}\big(r^{-\gamma}\big) \tag{9}$$

holds for all $r>0$, where $C>0$ does not depend on r, then the condition (7) is necessary and sufficient for the boundedness of $|b, I_\alpha^P|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Proof. (1) For arbitrary $x_0 \in \mathbb{R}^n$, set $\mathcal{E} = \mathcal{E}(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2k\mathcal{E}}$ and $f_2 = f\chi_{\complement_{(2k\mathcal{E})}}$, where k is the constant from the triangle inequality.

For $x \in \mathcal{E}$ we have

$$\begin{aligned}|[b, I^P_\alpha]f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{\rho(x-y)^{\gamma-\alpha}}|f_2(y)|dy \approx \int_{\complement_{(2k\mathcal{E})}} \frac{|b(y) - b(x)|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy \\ &\lesssim \int_{\complement_{(2k\mathcal{E})}} \frac{|b(y) - b_\mathcal{E}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy + \int_{\complement_{(2k\mathcal{E})}} \frac{|b(x) - b_\mathcal{E}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy \\ &= J_1 + J_2(x),\end{aligned}$$

since $x \in \mathcal{E}$ and $y \in {}^{\complement}(2k\mathcal{E})$ implies

$$\frac{1}{2k}\rho(y-x_0) \le \rho(x-y) \le \Big(k + \frac{1}{2}\Big)\rho(y-x_0).$$

Let us estimate J_1.

$$\begin{aligned}J_1 &= \int_{\complement_{(2k\mathcal{E})}} \frac{|b(y) - b_\mathcal{E}|}{\rho(y-x_0)^{\gamma-\alpha}}|f(y)|dy \approx \int_{\complement_{(2k\mathcal{E})}} |b(y) - b_\mathcal{E}||f(y)| \int_{\rho(y-x_0)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}}dy \\ &\approx \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)\setminus(2k\mathcal{E})} |b(y) - b_\mathcal{E}||f(y)|dy\frac{dt}{t^{\gamma+1-\alpha}} \\ &\lesssim \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)} |b(y) - b_\mathcal{E}||f(y)|dy\frac{dt}{t^{\gamma+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality, by (1), (4), (5) and Lemma 2 we get

$$\begin{aligned}J_1 &\lesssim \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} |b(y) - b_{\mathcal{E}(x_0,t)}||f(y)|dy\frac{dt}{t^{\gamma+1-\alpha}} \\ &\quad + \int_{2r}^{\infty} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}| \int_{\mathcal{E}(x_0,t)} |f(y)|dy\frac{dt}{t^{\gamma+1-\alpha}} \\ &\lesssim \int_{2r}^{\infty} \big\|b(\cdot) - b_{\mathcal{E}(x_0,t)}\big\|_{L_{\widetilde{\Phi}}(\mathcal{E}(x_0,t))} \|f\|_{L_\Phi(\mathcal{E}(x_0,t))}\frac{dt}{t^{\gamma+1-\alpha}} \\ &\quad + \int_{2r}^{\infty} |b_{\mathcal{E}(x_0,r)} - b_{\mathcal{E}(x_0,t)}|\|f\|_{L_\Phi(\mathcal{E}(x_0,t))}\Phi^{-1}\big(|\mathcal{E}(x_0,t)|^{-1}\big)\frac{dt}{t^{1-\alpha}} \\ &\lesssim \|b\|_* \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big)\|f\|_{L_\Phi(\mathcal{E}(x_0,t))}\Phi^{-1}\big(|\mathcal{E}(x_0,t)|^{-1}\big)\frac{dt}{t^{1-\alpha}}. \\ &\lesssim \|b\|_* \|f\|_{L^\Phi} \int_{2r}^{\infty} \Big(1 + \ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1}dt.\end{aligned}$$

A geometric observation shows $2k\mathcal{E} \subset \mathcal{E}(x, \delta)$ for all $x \in \mathcal{E}$, where $\delta = (2k+1)kr$. Using Lemma 7, we get

$$J_0(x) := |[b, I^P_\alpha]f_1(x)| \lesssim \int_{2k\mathcal{E}} \frac{|b(y) - b(x)|}{\rho(x-y)^{\gamma-\alpha}}|f(y)|dy$$

$$\lesssim \int_{\mathcal{E}(x,\delta)} \frac{|b(y)-b(x)|}{\rho(x-y)^{\gamma-\alpha}} |f(y)|dy \lesssim r^{\alpha} M_b^P f(x).$$

Consequently, we have

$$J_0(x) + J_1 \lesssim \|b\|_* r^{\alpha} M_b^P f(x) + \|b\|_* \|f\|_{L^\Phi} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big)\ t^{\alpha-1} dt.$$

Thus, by (8) we obtain

$$J_0(x) + J_1 \lesssim \|b\|_* \left(M_b^P f(x) \frac{\Psi^{-1}(r^{-\gamma})}{\Phi^{-1}(r^{-\gamma})} + \Psi^{-1}(r^{-\gamma}) \|f\|_{L^\Phi} \right).$$

Choose $r>0$ so that $\Phi^{-1}(r^{-\gamma}) = \frac{M_b^P f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}$. Then

$$\frac{\Psi^{-1}(r^{-\gamma})}{\Phi^{-1}(r^{-\gamma})} = \frac{(\Psi^{-1}\circ\Phi)\big(\frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}\big)}{\frac{M_b f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}}.$$

Therefore, we get

$$J_0(x) + J_1 \le C_1 \|b\|_* \|f\|_{L^\Phi} (\Psi^{-1}\circ\Phi)\Big(\frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}\Big).$$

Let C_0 be as in (5). Consequently by Theorem 3 we have

$$\int_{\mathcal{E}} \Psi\left(\frac{J_0(x)+J_1}{C_1\|b\|_*\|f\|_{L^\Phi}}\right) dx \le \int_{\mathcal{E}} \Phi\left(\frac{M_b^P f(x)}{C_0\|b\|_*\|f\|_{L^\Phi}}\right) dx$$
$$\le \int_{\mathbb{R}^n} \Phi\left(\frac{M_b^P f(x)}{\|M_b^P f\|_{L^\Phi}}\right) dx \le 1,$$

i.e.

$$\|J_0(\cdot) + J_1\|_{L^\Psi(\mathcal{E})} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{10}$$

In order to estimate J_2, by (5), Lemma 2 and condition (8), we also get

$$\|J_2\|_{L^\Psi(\mathcal{E})} = \left\| \int_{\complement(2k\mathcal{E})} \frac{|b(\cdot)-b_{\mathcal{E}}|}{\rho(y-x_0)^{\gamma-\alpha}} |f(y)|dy \right\|_{L^\Psi(\mathcal{E})}$$
$$\approx \|b(\cdot)-b_{\mathcal{E}}\|_{L^\Psi(\mathcal{E})} \int_{\complement(2k\mathcal{E})} \frac{|f(y)|}{\rho(y-x_0)^{\gamma-\alpha}} dy$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement(2k\mathcal{E})} \frac{|f(y)|}{\rho(y-x_0)^{\gamma-\alpha}} dy$$
$$\approx \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{\complement(2k\mathcal{E})} |f(y)| \int_{\rho(y-x_0)}^{\infty} \frac{dt}{t^{\gamma+1-\alpha}} dy$$

$$\approx \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2kr}^{\infty} \int_{\mathcal{E}(x_0,t)\setminus(2k\mathcal{E})} |f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}}$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)|dy \frac{dt}{t^{\gamma+1-\alpha}}$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(\mathcal{E}(x_0,t))} \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt$$
$$\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \|f\|_{L^{\Phi}} \int_{2r}^{\infty} \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt$$
$$\lesssim \|b\|_* \, \|f\|_{L^{\Phi}}.$$

Consequently, we have

$$\|J_2\|_{L^{\Psi}(\mathcal{E})} \lesssim \|b\|_* \, \|f\|_{L^{\Phi}}. \tag{11}$$

Combining (10) and (11), we get

$$\|[b, I_{\alpha}^P]f\|_{L^{\Psi}(\mathcal{E})} \lesssim \|b\|_* \|f\|_{L^{\Phi}}. \tag{12}$$

By taking supremum over $\mathcal{E}$ in (12), we get

$$\|[b, I_{\alpha}^P]f\|_{L^{\Psi}} \lesssim \|b\|_* \|f\|_{L^{\Phi}},$$

since the constants in (12) don't depend on x_0 and r.

(2) We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemmas 8, 5 and 1 we have

$$r_0^{\alpha} \lesssim \frac{\||b, I_{\alpha}^P|\chi_{\mathcal{E}_0}\|_{L^{\Psi}(\mathcal{E}_0)}}{\|b(\cdot) - b_{\mathcal{E}_0}\|_{L^{\Psi}(\mathcal{E}_0)}} \lesssim \Psi^{-1}(r_0^{-\gamma})\||b, I_{\alpha}^P|\chi_{\mathcal{E}_0}\|_{L^{\Psi}(\mathcal{E}_0)}$$
$$\lesssim \Psi^{-1}(r_0^{-\gamma})\||b, I_{\alpha}^P|\chi_{\mathcal{E}_0}\|_{L^{\Psi}} \lesssim \Psi^{-1}(r_0^{-\gamma})\|\chi_{\mathcal{E}_0}\|_{L^{\Phi}} \lesssim \frac{\Psi^{-1}(r_0^{-\gamma})}{\Phi^{-1}(r_0^{-\gamma})}.$$

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem.

Remark 2. *Note that Theorem 7 in the isotropic case $P = I$ were proved in [7].*

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References

[1] G. A. Abbasova, *Boundedness of the parabolic maximal operator in Orlicz spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **37(4)**, 2017, Mathematics, 5-11.

[2] O.V. Besov, V.P. Il'in, P.I. Lizorkin, *The L_p-estimates of a certain class of non-isotropically singular integrals,* (Russian) Dokl. Akad. Nauk SSSR, **169**, 1966, 1250-1253.

[3] E.B. Fabes, N. Rivère, *Singular integrals with mixed homogeneity,* Studia Math., **27**, 1966, 19-38.

[4] G.B. Folland, E.M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes, 28, Princeton Univ. Press, Princeton, 1982.

[5] X. Fu, D. Yang, W. Yuan, *Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces over non-homogeneous spaces*, Taiwanese J. Math. **16**, 2012, 2203-2238.

[6] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, M. Krbec, *Weight theory for integral transforms on spaces of homogeneous type*, Longman, Harlow, (1998).

[7] V.S. Guliyev, F. Deringoz, S.G. Hasanov, *Riesz potential and its commutators on Orlicz spaces*, J. Inequal. Appl. 2017, Paper No. 75, 18 pp.

[8] M. Izuki, Y. Sawano, *Characterization of BMO via ball Banach function spaces*, Vestn. St.-Peterbg. Univ. Mat. Mekh. Astron. **4(62) (1)**, 2017, 78-86.

[9] F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 1961, 14:415-426.

[10] Ruilin Long, Le Yang, *BMO functions in spaces of homogeneous type*, Sci. Sinica Ser. A, **27(7)** :695-708.

[11] E. Nakai, *On generalized fractional integrals in the Orlicz spaces on spaces of homogeneous type*, Scientiae Mathematicae Japonicae, **54** (2001), 473-487.

[12] W. Orlicz, *Über eine gewisse Klasse von Räumen*, vom Typus B, Bull. Acad. Polon. A, 1932, 207-220. ; reprinted in: Collected Papers, PWN, Warszawa, 1988, 217-230.

[13] W. Orlicz, *Über Räume (L^M)*, Bull. Acad. Polon. A , 1936, 93-107. ; reprinted in: Collected Papers, PWN, Warszawa, 1988, 345-359.

[14] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, M. Dekker, Inc., New York, 1991.

[15] E.M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.

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